Inner models and ultrafilters in $L(\mathbb{R})$

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Part 1:

1. Preliminaries.
2. The club filter on ω_1.
3. An ultrafilter on $[\omega_1]^{<\omega_1}$.
4. Forcing over $L(\mathbb{R})$ to add a cub subset of ω_1.

Let M have ω Woodin cardinals. Let δ be their supremum. Let $\mathbb{P} \in M$ be the poset $\mathrm{col}(\omega, {<}\delta)$.

Let $G = \langle G_\xi \mid \xi < \delta \rangle$ be $\mathbb{P}$–generic/M.

Define: $R^* = R^*[G] = \bigcup\limits_{\beta<\delta} \mathbb{R}^{M[G\restriction\beta]}$.

R^* is called a **symmetric collapse** of M.

A set of reals B is **realized as a symmetric collapse** of M if there is a generic G so that $R^*[G] = B$.

Note: Let φ be a formula and let $a_1, \ldots, a_k$ be reals or ordinals in M. Let R_1 and R_2 be two symmetric collapses of M. Then

$$\mathsf{L}(R_1) \models \varphi[a_1, \ldots, a_k] \iff \mathsf{L}(R_2) \models \varphi[a_1, \ldots, a_k].$$

(This follows from the homogeneity of $\mathbb{P}$.)

Cor: $\varphi[a_1, \ldots, a_k]$ is true in $\mathsf{L}(R^*[G])$ iff this is forced by the empty condition.

We informally refer to $\mathsf{L}(R^*[G])$ (rather than $R^*[G]$ itself) as a symmetric collapse of M.

We say that $\varphi[a_1, \ldots, a_k]$ is forced to hold in the symmetric collapse of M if it is forced (by the empty condition) to hold in $\mathsf{L}(R^*[G])$.

Suppose now that M is iterable (more on this later) and that $\mathcal{P}(\delta)^M$ is countable in V. Let $g\colon \omega \to \mathbb{R}$ be a generic surjection.

Fact: In V$[g]$ there is an M^* and an elementary $\pi\colon M \to M^*$ so that $\mathbb{R}$ (the true $\mathbb{R}$ of V) is realized as a symmetric collapse of M^*.

Any statement forced to hold in the symmetric collapse of M is also forced to hold in the symmetric collapse of M^*, since π is elementary.

It follows that any statement forced to hold in the symmetric collapse of M, holds in the *true* L$(\mathbb{R})$.

This works for statements with real parameters and parameters bounded in δ. (One can arrange that π does not move such parameters.)

The Fact is used to prove $AD^{L(\mathbb{R})}$ from the following *large cardinal assumption*:

For each $u \in \mathbb{R}$ there is a class model M s.th.
(1) $u \in M$;
(2) M has ω Woodin cardinals, say with sup δ;
(3) $\mathcal{P}(\delta)^M$ is countable in V; and
(4) M is iterable.

We will use the fact and the large cardinal assumption directly, to obtain ultrafilters in $L(\mathbb{R})$.

An ultrafilter on ω_1:

Let M be a countable model of ZFC with (at least) a measurable cardinal. Let $a(M)$ be the first measurable cardinal of M.

The measures in M can be used to form ultrapowers, and the process can be iterated.

$$
\begin{array}{ccccc}
M & \longrightarrow & \mathrm{Ult}(M,\mu) & \longrightarrow & \mathrm{Ult}(M_1,\mu_1) \dashrightarrow \\
 & & \| & & \| \\
 & & M_1 & \scriptstyle \mu_1 \in M_1 & M_2
\end{array}
$$

By a (linear) iterate of M we mean any model P obtained through a countable iteration of this kind.

M is (linearly) **iterable** if all its iterates are wellfounded.

For an iterable M define

$$C_M = \{a(P) \mid P \text{ is an iterate of } M\}.$$

Note then $C_M \subset {\omega_1}^{\mathsf{V}}$.

Let M_1 and M_2 be countable, iterable models with (at least) a measurable cardinal.

Let M^* be a countable, iterable model with a measurable cardinal, and such that both M_1 and M_2 belong to M^*. (Such M^* exists by our large cardinal assumption. Note both M_1 and M_2 are coded by reals.)

It's easy to see then that both $C_{M^*} \subset C_{M_1}$ and $C_{M^*} \subset C_{M_2}$.

It follows that the collection

$$\{C_M \mid \text{M ctbl, iterable, with a measurable}\}$$

has the finite intersection property.

Let $\mathcal{F}$ be the filter generated by this collection.

An argument similar to the above shows that in fact the collection has the countable intersection property. So $\mathcal{F}$ is countably complete.

Claim: $\mathcal{F}$ is an ultrafilter in $\mathsf{L}(\mathbb{R})$.

Proof: Let $X \in \mathsf{L}(\mathbb{R})$ be a subset of ω_1. For simplicity suppose X is definable in $\mathsf{L}(\mathbb{R})$ from a real parameter u. Fix a formula ψ so that $\alpha \in X$ iff $\mathsf{L}(\mathbb{R}) \models \psi[\alpha, u]$.

Using the large cardinal assumption fix a model M, with ω Woodin cardinals etc., and with $u \in M$.

Ask: Is $\psi[a(M), u]$ forced to *hold* in the symmetric collapse of M?

Suppose yes $(*)$.

Let P be an iterate of M. Have then an elementary embedding $j \colon M \to P$ (the **iteration embedding** generated by the various ultrapowers taken).

By $(*)$ and since j is elementary, $\psi[a(P), u]$ is forced to hold in the symmetric collapse of P.

By preliminaries' Fact, it follows that $\psi[a(P), u]$ really holds in L($\mathbb{R}$).

So $a(P) \in X$.

This is true for each iterate P of M.

So $C_M = \{a(P) \mid P \text{ an iterate of } M\}$ is contained in X.

Showed: If $\psi[a(M), u]$ is forced to *hold* in the symmetric collapse of M then $C_M \subset X$.

A similar argument shows that if $\psi[a(M), u]$ is forced to *fail* then $C_M \subset \omega_1 - X$.

So $\mathcal{F}$ is an ultrafilter. $\square$

An ultrafilter on $[\omega_1]^{<\omega_1}$:

Let M be a countable model with (at least) a measurable limit of measurable cardinals. Let $\kappa = \kappa(M)$ be the first such cardinal in M.

Let $\langle \tau_\xi \mid \xi < \gamma \rangle$ list the measurable cardinals of M below κ, in increasing order.

Define $a(M) = \langle \tau_\xi \mid \xi < \gamma \rangle$.

Note $a(M)$ then belongs to $[\omega_1]^{<\omega_1}$.

For an iterable M define:

$$C_M = \{a(P) \mid P \text{ is an iterate of } M\}.$$

Note then $C_M \subset [\omega_1]^{<\omega_1}$.

The sets C_M generate an ultrafilter: simply carry the earlier proof (for ω_1), with the current definitions. Call this ultrafilter $\mathcal{F}$.

Let $\gamma(M) = \text{o.t.}\{\tau < \kappa \mid \tau \text{ is measurable in } M\}$. The length of the seq. $a(M)$ is precisely $\gamma(M)$.

Note: If P is an ultrapower of M by a measure on κ, then $\gamma(P) > \gamma(M)$.

It follows that $C_M = \{a(P) \mid P \text{ is an iterate of } M\}$ has sequences of *arbitrarily large* countable length.

So $\mathcal{F}$ does not concentrate on any particular countable length. (We say that $\mathcal{F}$ "concentrates on long sequences.")

The projection of $\mathcal{F}$ to $[\omega_1]^1$ is simply our previous ultrafilter on ω_1. (This is because the first coordinate in $a(M) = \langle \tau_\xi \mid \xi < \gamma \rangle$ is the first measurable of M.)

Similarly the projection of $\mathcal{F}$ to $[\omega_1]^\alpha$ for each countable α is the α-length iteration of our previous ultrafilter on ω_1.

Say that $X \subset [\omega_1]^{<\omega_1}$ is **nice** if:

(1) X belongs to $\mathcal{F}$;
(2) X is countably closed ($r_0{}^\frown r_1{}^\frown \ldots r_n \in X$ for each n, then $r_0{}^\frown r_1{}^\frown \cdots \in X$); and
(3) For each $s \in X$, $\{r \mid s^\frown r \in X\}$ belongs to $\mathcal{F}$.

Each C_M is nice:

$C_M \in \mathcal{F}$ by definition, and by composing iterations one can check C_M is countably closed.

As for (3): For $s \in C_M$ have some iterate P of M so that $s = a(P)$. Let Q be the ultrapower of P by a measure on $\kappa(P)$. Notice then $s = a(P)$ is a strict initial segment of $a(Q)$. Let Q^* be a generic extension of Q collapsing the ordinals of $a(P)$ to ω. Then $a(Q) = a(P)^\frown a(Q^*)$, and $\{r \mid a(P)^\frown r \in C_M\}$ contains C_{Q^*}.

Note: If X is nice and $s \in X$, then $X^* = \{s^* \mid s^\frown s^* \in X\}$ is also nice.

There is a natural forcing notion suggested by $\mathcal{F}$. Conditions are pairs (t, Y) where:

$t \in [\omega_1]^{<\omega_1}$; Y is a set of extensions of t; and $\{s \mid t^\frown s \in Y\}$ is nice.

More on this forcing later.

Claim: Let $g\colon [\omega_1]^{<\omega_1} \to \omega_2$. Then there is a set $X \in \mathcal{F}$ so that $g \restriction X$ is *bounded* below ω_2.

Proof: Recall that ω_2 is equal to $\boldsymbol{\delta}^1_2$, the sup of $\boldsymbol{\Delta}^1_2$ prewellorderings.

Have a norm $\rho\colon \mathbb{R} \to \omega_2$ (partial, surjective) so that if $E \subset \mathrm{dom}(\rho)$ is $\boldsymbol{\Sigma}^1_2$ then $\rho'' E$ is bounded below ω_2.

Define $g^*(a) = \{x \mid x \in \mathrm{dom}(\rho) \wedge \rho(x) = g(a)\}$. This is g "in the codes."

g^* belongs to $\mathsf{L}(\mathbb{R})$. For simplicity suppose it is definable in $\mathsf{L}(\mathbb{R})$ from a real parameter, u. Fix ψ so that $x \in g^*(a)$ iff $\mathsf{L}(\mathbb{R}) \models \psi[a, x, u]$.

Suppose P satisfies our large cardinal assumption (ω Woodin cardinals, etc.) with $u \in P$.

Then inside every symmetric collapse of P, there is a real x so that $\psi[a(P), x, u]$ holds in the symmetric collapse.

This follows from the preliminaries' Fact:

L($\mathbb{R}$) satisfies $(\exists x)\psi[a(P), x, u]$, just take any x in $g^*(a(P))$.

So the symmetric collapse of P must also satisfy $(\exists x)\psi[a(P), x, u]$.

If $\psi[a(P), x, u]$ holds in a symmetric collapse of P, then (again by the preliminaries' Fact) it holds in L($\mathbb{R}$), meaning that $x \in g^*(a(P))$.

We showed: $\{ \, x \mid \psi[a(P), x, u]$ holds in a symmetric collapse of $P \, \}$ is non-empty and contained in $g^*(a(P))$.

Now let M satisfy our large cardinal assumption with $u \in M$.

Let E be the set of reals x so that:

($\exists$ an iterate P of M)($\psi[a(P), x, u]$ holds in a symmetric collapse of P).

By the previous slide

$$E \subset \bigcup \{ g^*(a(P)) \mid P \text{ an iterate of } M \},$$

and E meets each $g^*(a(P))$.

It follows that $E \subset \text{dom}(\rho)$ and $\rho''E$ is precisely equal to $\{g(a(P)) \mid P \text{ an iterate of } M\}$.

Recall $C_M = \{a(P) \mid P \text{ is an iterate of } M\}$.

We showed: $\text{range}(g \restriction C_M) = \rho''E$.

Now E is $\mathbf{\Sigma}^1_2$: "There exists a (linear) iterate P of M" amounts to saying that there is a linear iteration, of wellfounded countable length, leading from M to P.

It follows that $\rho''E$ is bounded below ω_2. $\square$

Return now to the forcing.

Conditions are pairs (t, Y) so that $t \in [\omega_1]^{<\omega_1}$; Y is a set of extensions of t; and $\{s \mid t^\frown s \in Y\}$ is nice.

(t^*, Y^*) extends (t, Y) if t^* extends t and $Y^* \subset Y$.

Let $\mathbb{P}$ denote this forcing. A generic object adds a cub subset of ω_1.

$\mathbb{P}$ is countably closed. So ω_1 is not collapsed.

Remark: In general forcing with countable conditions over L($\mathbb{R}$) may collapse $\mathbb{R}$ to ω_1 (in particular collapse ω_2 and all cardinals up to Θ).

Claim: $\mathbb{P}$ does not collapse ω_2.

Proof: Let $\dot{f}$ name a function from ω_1 into ω_2. For a stem t let $A(t) = \{\beta \mid$ for some α and some Y, $(t, Y) \Vdash \dot{f}(\check{\alpha}) = \check{\beta}\}$.

Note: for each α, the set $\{\beta \mid$ for some Y, $(t, Y) \Vdash \dot{f}(\check{\alpha}) = \check{\beta}\}$ has at most one element.

So $A(t)$ has size at most ω_1.

Let $g(t) = \sup A(t)$. Then $g\colon [\omega_1]^{<\omega} \to \omega_2$.

Using last claim can find a nice Y so that $g \restriction Y$ is bounded.

$(\emptyset, Y)$ then forces $\dot{f}$ to be bounded in ω_2. $\square$