AN INTRODUCTION TO PROOFS OF DETERMINACY OF LONG GAMES

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Abstract. We present the basic methods used in proofs of determinacy of long games, and apply these methods to games of continuously coded length.

From the dawn of time women and men have aspired upward. The development of determinacy proofs is no exception to this general rule. There has been a steady search for higher forms of determinacy, beginning with the results of Gale–Stewart [2] on closed length ω games and continuing to this day. Notable landmarks in this quest include proofs of Borel determinacy in Martin [5]; analytic determinacy in Martin [4]; projective determinacy in Martin–Steel [8]; and $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ in Woodin [17].[1] Those papers consider length ω games with payoff sets of increasing complexity. One could equivalently fix the complexity of the payoff and consider games of increasing length. Such “long games” form the topic of this paper.

Long games form a natural hierarchy, the hierarchy of increasing length. This hierarchy can be divided into four categories: games of length less than $\omega \cdot \omega$; games of fixed countable length; games of variable countable length; and games of length ω_1.

Games in the first category can be reduced to standard games of length ω, at the price of increasing payoff complexity. The extra complexity only involves finitely many real quantifiers. Thus the determinacy of games of length less than $\omega \cdot \omega$, with analytic payoff say, is the same as projective determinacy.

Games in the second category can be reduced to combinations of standard games of length ω, with increased payoff complexity, and some additional strength assumptions. The first instance of this is given in Blass [1]. The techniques presented there can be used to prove the determinacy of length $\omega \cdot \omega$ games on natural numbers, with analytic payoff say, from $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ + “$\mathbb{R}^{\#}$ exists.” In another, choiceless reduction to standard games, Martin and Woodin independently showed that AD + “all sets of reals admit scales” implies that all games in the second category are determined.

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[1] $\mathsf{AD}^{\mathrm{L}(\mathbb{R})}$ is the statement that all standard length ω games with payoff in $\mathrm{L}(\mathbb{R})$ are determined.

Meeting

It is in the third category that the methods presented here begin to yield new determinacy principles. (The one previously known determinacy proof for games in the third category is a theorem of Steel [16], which applies to games of the kind described in Remark 1.1.)

Neeman [12] concentrates on third category games and reaches to the low end of the fourth category. Our goal here is to provide an introduction to the methods of [12]. We illustrate these methods with one game of the first category, one of the second, and one of the third. The proofs, like the results, form a hierarchy.

The proofs in the first category are closely related to the main construction of Martin–Steel [8].

The proofs in the second category can be viewed as combinations of (1) a construction which reduces one side of a given game to an *iteration game*; and (2) an appeal to a winning strategy for the good player in the iteration game. (Iteration games are described in Section 1.2.) This is a general pattern that continues higher up. Determinacy is thus dependent upon *iterability*—the existence of winning strategies for the good player in iteration games. We say more on this at the end of Section 1.2.

The construction for part (1) above is a matter of breaking the construction of the first category into blocks, and reassembling the blocks spreading them over countably many stages. In some ways this is analogous to the way scale propagation under infinitely many real quantifiers relates to the basic propagation under one quantifier. Readers interested in a side tour may check Sections 6C and 6E of Moschovakis [10], Moschovakis [11], and Martin [6] for results on scale propagation.

Third category proofs use the techniques of the second category, but the reassembling of the blocks is not done at the outset. Instead the decisions on how to spread the blocks of the construction are taken during the game and depend on the players' moves. Similar methods apply to open games of length ω_1 (the low end of the fourth category). Beyond that determinacy is not known.

We try to make this progression of ideas evident through the organization of the paper. In Section 2 we present the basic tools. One of the two lemmas there, Lemma 2.8, draws heavily on the techniques of Martin–Steel [8]. In Section 3 we use the basic tools to prove the determinacy of standard length ω games with Σ^1_2 payoff. In Section 4 we prove the determinacy of games of fixed length $\omega \cdot \omega$, with Σ^1_2 payoff. The proof involves breaking and reassembling the previous construction of Section 3. In Section 6 we prove the determinacy of games of continuously coded length. (These are games of the third category; of variable countable length. We define these games in Section 1.1.) Again the proof involves breaking and reassembling a construction of the kind done in Section 3. But now the break line is not fixed at the outset; it varies depending on the actual moves during the game.

Sections 3 and 4 are included for their role in the development of methods which lead to Section 6. The results stated in those two sections, Theorem 3.1 and Theorem 4.12 are not new. Both are due to Woodin by methods different from ours. Theorem 3.1 in slightly weakened form was first proved by Martin–Steel [8].

As chance would have it the methods of Section 6 are also useful for longer games of the third category, specifically games ending at ω_1 in L of the play. These in turn are useful for the determinacy proof for open games of length ω_1. But we shall not reach that far here. Our discussion ends with Theorem 6.15, which establishes determinacy for games of continuously coded length.

§1. Preliminaries. We take this section to define precisely the long games which we intend to prove determined and sketch the large cardinal notions needed for the proofs, mainly iteration trees and iteration games. Our sketch of the large cardinal notions is informal, maybe even superficial, but it suffices for our needs.

1.1. The games. Following standard abuse we let $\mathbb{R}$ denote Baire space, namely the space $\mathbb{N}^\omega$. Let $C \subset \mathbb{R}^{<\omega_1}$ be given. Let $\nu\colon \mathbb{R} \to \mathbb{N}$, a partial function, be given. $G_{\text{cont}-\nu}(C)$ is played according to Diagram 1.

$$\begin{array}{c|cccccc}
\text{I} & \cdots\cdots\cdots & y_\alpha(0) & & y_\alpha(2) & & \\
\hline
\text{II} & & & y_\alpha(1) & & y_\alpha(3) & \cdots
\end{array}$$

DIAGRAM 1. The game $G_{\text{cont}-\nu}(C)$.

In mega-round α, players I and II alternate playing natural numbers $y_\alpha(i)$, $i < \omega$, producing a real y_α. If $\nu(y_\alpha)$ is not defined, the game ends. I wins iff $\langle y_0, y_1, \ldots\ldots, y_\alpha\rangle \in C$. Otherwise we set $n_\alpha = \nu(y_\alpha)$. If there exists $\xi < \alpha$ so that $n_\alpha = n_\xi$, the game ends. Again I wins iff $\langle y_0, y_1, \ldots\ldots, y_\alpha\rangle \in C$. Otherwise the game continues.

The end length of a run of $G_{\text{cont}-\nu}(C)$ may vary depending on the moves played by the two players. But the length is always countable. Indeed, a map witnessing that the length is countable is produced continuously—one extra bit of information at each mega-round—during the play. The game is said to have **continuously coded length**.

REMARK 1.1. Our definition here generalizes the definition of continuously coded games in Steel [16], where ν acted on $\langle y_\xi \mid \xi < \alpha\rangle$, and n_α was set to be $\nu(y_\xi \mid \xi < \alpha)$. (Why is our definition a generalization? One could easily force one of the players to code $\langle y_\xi \mid \xi < \alpha\rangle$ into her moves for y_α. Thus in our settings too ν can refer to $\langle y_\xi \mid \xi < \alpha\rangle$.) The generalization is proper, in the sense that there are games which fall within our definition, but outside the definition of Steel [16].

We make a few simple observations about the game.

Claim 1.2. *Suppose that $\langle y_\xi \mid \xi < \lambda \rangle$ is a position of limit length. Then $n_\xi \to \infty$ as $\xi \to \lambda$.*

Proof. This is immediate. The n_ξ-s are distinct, and so they cannot be forever bounded. ⊣

Suppose $\langle y_\xi \mid \xi < \alpha \rangle$ is a position reached during the game. The map $\xi \mapsto n_\xi = \nu(y_\xi)$ embeds α into $\mathbb{N}$, and can be used to code the position by a real. We let $\ulcorner y_\xi \mid \xi < \alpha \urcorner$ denote this real code. The precise method of coding is not important, so long as it satisfies the following property:

Property 1.3. The real codes $\ulcorner y_\xi \mid \xi < \alpha \urcorner$ and $\ulcorner y_\xi \mid \xi < \alpha + 1 \urcorner$ agree to $n_\alpha = \nu(y_\alpha)$.

Any reasonable use of the map $\xi \mapsto n_\xi$ to code $\langle y_\xi \mid \xi < \alpha \rangle$ will have this property.

Remark 1.4. Combining Claim 1.2 and Property 1.3 we see that the reals $x_\alpha = \ulcorner y_\xi \mid \xi < \alpha \urcorner$ converge to $x_\lambda = \ulcorner y_\xi \mid \xi < \lambda \urcorner$ as $\alpha \to \lambda$.

Remark 1.4 will be crucial later when we reach the determinacy proof in Section 6. Indeed continuity is important throughout this paper, starting already in the arguments of Sections 3.

Let us say that the payoff set C is Γ **in the codes**—where Γ is some pointclass, for example Σ^1_2—if there is a Γ set $A \subset \mathbb{R} \times \mathbb{R}$ so that

$$\langle y_\xi \mid \xi \leq \alpha \rangle \in C \iff \langle x_\alpha, y_\alpha \rangle \in A$$

where $x_\alpha = \ulcorner y_\xi \mid \xi < \alpha \urcorner$.

Our goal is to give a proof of determinacy for the games $G_{\text{cont}-\nu}(C)$ when ν is continuous and C is Σ^1_2 in the codes. As in illustrative case we will first consider games of fixed length. We will handle games of two lengths: games of length ω, and then games of length $\omega \cdot \omega$. We remind the reader of the format of these games:

Let $C \subset \mathbb{R}^\omega = \mathbb{N}^{\omega \cdot \omega}$ be given. In $G_{\omega \cdot \omega}(C)$ players I and II play ω mega-rounds according to Diagram 2.

$$\begin{array}{c|cccccccc}
\mathrm{I} & y_0(0) & & \ldots\ldots & y_1(0) & & \ldots & \\
\hline
\mathrm{II} & & y_0(1) & & & y_1(1) & & \ldots
\end{array}$$

Diagram 2. The game $G_{\omega \cdot \omega}(C)$.

In mega-round k the players alternate playing natural numbers $y_k(i)$, producing together a real y_k. Once ω mega-rounds are completed, I wins if $\langle y_k \mid k < \omega \rangle$ belongs to C. Otherwise II wins.

Let $C \subset \mathbb{R} = \mathbb{N}^\omega$ be given. In $G_\omega(C)$ the players play only one mega-round, alternating natural number moves $y(i)$ as in Diagram 3 to produce together the real y. I wins if $y \in C$. Otherwise II wins.

I	$y(0)$		$y(2)$		$\dots$	
II		$y(1)$		$y(3)$		$\dots$

Diagram 3. The game $G_\omega(C)$.

1.2. Iteration trees. We include here an informal description of iteration trees and the notions of iterability which we shall need. This description is far from complete, and even further from precise. The reader who desires more thorough knowledge should consult Kanamori [3] and Martin–Steel [9].

An **extender** on κ is a directed system of measures on κ. For an exact definition see [3, §26]. We use $\mathrm{dom}(E)$ to denote κ. An extender E allows us to form an **ultrapower** of V, denoted $\mathrm{Ult}(\mathrm{V}, E)$, and an elementary **ultrapower embedding** $\pi\colon \mathrm{V} \to \mathrm{Ult}(\mathrm{V}, E)$. We refer the reader to [3, §26] or [8, §1] for the exact construction.

Let us say that two ZFC models Q^* and Q **agree** to κ if $\mathcal{P}(\kappa) \cap Q^* = \mathcal{P}(\kappa) \cap Q$. Suppose $Q \models$"E is an extender on κ." Suppose Q^* and Q agree to κ. Then E measures all subsets of κ in Q^*, and can thus be used to form an **ultrapower** $\mathrm{Ult}(Q^*, E)$ of Q^*, and an elementary **ultrapower embedding** $\sigma\colon Q^* \to \mathrm{Ult}(Q^*, E)$. $\mathrm{Ult}(Q^*, E)$ needn't always be wellfounded, but if it is then we assume it is transitive.

An **iteration tree** $\mathcal{T}$ of length ω consists of:

- A tree order T on ω;
- A sequence of models $\langle M_k \mid k < \omega\rangle$; and
- Embeddings $j_{k,l}\colon M_k \to M_l$ for $k\ T\ l$.

An iteration tree **on** M is a tree with $M_0 = M$.

A sample iteration tree together with its tree order T is displayed in Diagram 5. A precise definition can be found in [8, §3]. Rather than reproduce this definition let us only explain how to *form* an iteration tree: Suppose $M_0, \dots, M_l$ and the order $T \restriction l+1$ are known. We wish to form M_{l+1} and extend the tree order to $T \restriction l+2$. To do this, we pick some extender E_l in M_l, and pick some $k \le l$ so that M_l and M_k agree to $\mathrm{dom}(E_l)$. (Note that taking $k = l$ gives this agreement for free.) Set $M_{l+1} = \mathrm{Ult}(M_k, E_l)$, and extend T by letting k be the predecessor of $l+1$. The result is presented in Diagram 4. An iteration tree of length ω is any object produced by ω repetitions of this process.

A cofinal **branch** through an iteration tree of length ω is an infinite set $b \subset \omega$ which is linearly ordered by T. The sample iteration tree of Diagram 5 has an **even** branch—the branch consisting of $\{0, 2, 4, 6, \dots\}$. Most of our iteration trees will have an even branch, and some complicated tree structure on the odd models. We use M_{even} to denote the direct limit of the models along

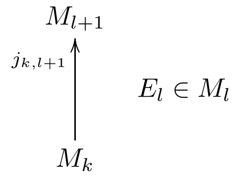

DIAGRAM 4. Forming M_{l+1}.

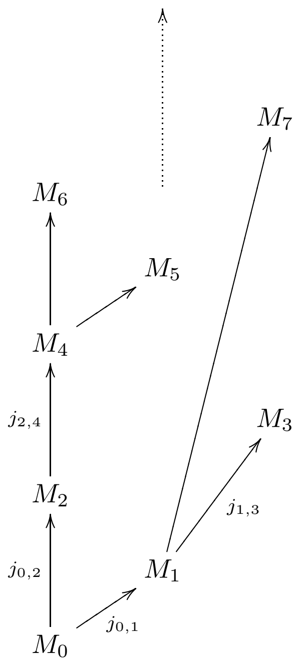

DIAGRAM 5. A sample iteration tree, with the tree order $0\ T\ 1$, $0\ T\ 2$, $1\ T\ 3$, $1\ T\ 7$, ...

the even branch. In general given a cofinal branch b we use M_b to denote the direct limit of the models along b.

We shall need a couple of notions of iteration games. The notions we need are defined below. We call both of them "iteration games" though they correspond more closely to the standard notion of a "weak iteration game." Iteration games were first defined by Martin and Steel. The interested reader can find the general definition in [9].

Let M be a given model. In the first iteration game which we consider, players "good" and "bad" collaborate to produce a sequence of iteration trees as in Diagram 6.

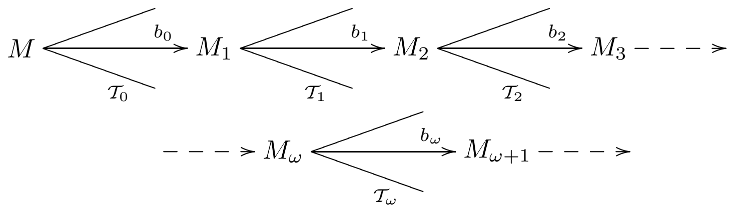

Diagram 6. An iteration game.

In round ξ "bad" plays a length ω iteration tree $\mathcal{T}_\xi$ on M_ξ. "Good" plays a cofinal branch b_ξ through $\mathcal{T}_\xi$. We let $M_{\xi+1}$ be the direct limit model determined by b_ξ and proceed to the next round. For limit λ we let M_λ be the direct limit of the models M_ξ, $\xi < \lambda$. We start with the given model $M = M_0$. The game continues to ω_1.

If ever a model M_ξ, where $\xi < \omega_1$, is reached which is illfounded, "bad" wins. Otherwise "good" wins.

In the second iteration game which we consider, round ξ has the form presented in Diagram 7.

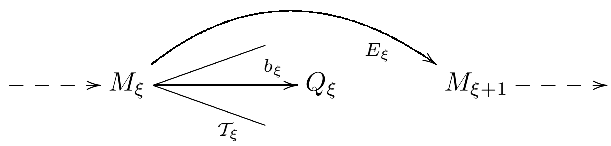

Diagram 7. Round ξ of the second type iteration game.

"Bad" plays a length ω iteration tree $\mathcal{T}_\xi$ on M_ξ. "Good" plays a cofinal branch b_ξ, giving rise to the direct limit Q_ξ. Then "bad" plays an extender E_ξ in Q_ξ, with $\mathrm{dom}(E_\xi)$ within the level of agreement between M_ξ and Q_ξ. We set $M_{\xi+1} = \mathrm{Ult}(M_\xi, E_\xi)$ and continue to the next round.

As before the game continues to ω_1, taking direct limits at limit stages. If ever a model Q_ξ or M_ξ, where $\xi < \omega_1$, is reached which is illfounded, "bad" wins. Otherwise "good" wins.

M is **iterable** if the good player has a winning strategy for each of the iteration games described above and combinations thereof. We refer to such winning strategies as **iteration strategies**.

Typically in our constructions the iteration trees, but not the branches through them, will be produced by some mechanism which is part of the construction. To keep the construction going we will need a method of picking branches through the iteration trees we encounter. It will be important to maintain the wellfoundedness of all the models we construct. We will thus need an iteration strategy to carry our construction through.

The existence of winning strategies for the good player in *general* iteration games is one of the central problems facing large cardinalists. Our own definition of iterability is restricted to the weak iteration games described above. These weak games are easier for "good" than the general games, and we have the following theorem of Martin–Steel [9]:

THEOREM 1.5 (Martin–Steel). *Let* V_η *be some sufficiently closed rank initial segment of* V. *Then countable elementary substructures of* V_η *are iterable (in the weak sense described above).*

We note that once one tries to prove determinacy of games somewhat longer than the continuously coded, for example games ending at ω_1 in L of the play, the weak iteration games described above no longer suffice for the constructions. The kind of iterability needed for games ending at ω_1 in L of the play was proved in Neeman [15]. For longer games, for example open games of length ω_1, it seems that nothing short of *general* iterability could suffice for the determinacy proofs. This is one of several examples of the great importance of general iterability.

§2. Auxiliary moves. Fix throughout this section some ZFC model M which has a Woodin cardinal δ. Assume that in V there are M–generics for $\mathrm{col}(\omega, \delta)$. Fix a name $\dot{A} \in M$ for a set of reals in $M^{\mathrm{col}(\omega,\delta)}$.

Work with some $x = \langle x_n \mid n < \omega \rangle \in \mathbb{R}$. We work to define an auxiliary game, $\mathcal{A}[x]$, of ω moves taken from M. In this game I tries to witness that $x \in \dot{A}[h]$ for some generic h. II tries to witness the opposite. We shall use this method of "witnessing" later on in our determinacy proofs. What we present here is a gentle guide to the definition of $\mathcal{A}[x]$. The actual definition can be found in [12, Chapter 1].

The format of the auxiliary game $\mathcal{A}[x]$ is presented in Diagram 8. All moves belong to M, and each rule should be read relativized to M.

$$\begin{array}{c|cccc} \mathrm{I} & \ldots \quad l_n, \mathcal{X}_n, p_n & & \ldots & \\ \hline \mathrm{II} & & \mathcal{F}_n, \mathcal{D}_n & & \ldots \end{array}$$

DIAGRAM 8. Round n of $\mathcal{A}[x]$.

In round n I plays:

- l_n, a number smaller than n, or $l_n =$ "new";
- $\mathcal{X}_n$, a set of names for reals of $M^{\mathrm{col}(\omega,\delta)}$; and
- p_n, a condition in $\mathrm{col}(\omega, \delta)$.

II plays:

- $\mathcal{F}_n$ a function from $\mathcal{X}_n$ into the ordinals; and
- $\mathcal{D}_n$, a function from $\mathcal{X}_n$ into $\{$dense sets in $\mathrm{col}(\omega, \delta)\}$.

Set $l = l_n$. If $l_n =$ "new" we make no requirements on I. Otherwise we require:

1. p_n extends p_l;
2. $\mathcal{X}_n \subset \mathcal{X}_l$.

We further require that for every name $\dot{x} \in \mathcal{X}_n$:

3. p_n forces "$\dot{x} \in \dot{A}$";
4. p_n forces "$\dot{x}(0) = \check{x_0}$," ......, "$\dot{x}(l) = \check{x_l}$"; and
5. p_n belongs to $\mathcal{D}_l(\dot{x})$.

We make the following requirement on II when $l_n \neq$ "new":

6. For every name $\dot{x} \in \mathcal{X}_n$, $\mathcal{F}_n(\dot{x}) < \mathcal{F}_l(\dot{x})$.

This completes the rules for round n.

If there is an h which is $\operatorname{col}(\omega, \delta)$–generic/$M$ and so that $x \in \dot{A}[h]$, then I can pick a name for x, play $\mathcal{X}_n$ containing this name, and play $p_n \in h$. Rule 6 ensures defeat for II. In other words, if there is an infinite run of $\mathcal{A}[x]$ where I played wisely enough, then there cannot be a name $\dot{x}$ and a generic h so that $x \in \dot{A}[h]$.

The game $\mathcal{A}[x]$ thus follows its stated intuitive goal—being a game in which II tries to witness that there is no generic h so that $x \in \dot{A}[h]$, while I tries to witness there is such h. This is consolidated below. In Section 2.1 we see that, if I plays wisely, then II's moves witness that $x \notin \dot{A}[h]$ for any generic h. Then in Section 2.2 we see that, if II plays wisely, then I's moves witness that x belongs to $j_b(\dot{A})[h]$, where $j_b(\dot{A})$ is some shifted image of $\dot{A}$, and h is generic for the collapse of the shifted δ.

Remark 2.1. Rather than play the sets $\mathcal{X}_n$ directly, I plays their *type*. I plays $\kappa_n < \delta$, and a set u_n of formulae with parameters in $M \| \kappa_n \cup \{\kappa_n, \delta, \dot{A}\}$.[2] We take $\mathcal{X}_n$ to be the set of names which satisfy all these formulae. The fact that this still allows I enough control over her choice of $\mathcal{X}_n$ has to do with our assumption that δ is a Woodin cardinal. We refer the reader to [12, Chapter 1] for precise details. $\mathcal{F}_n$ and $\mathcal{D}_n$ are played similarly.

Observe that all moves in $\mathcal{A}[x]$ are therefore elements of $M \| \delta$.

Note that the association $x \mapsto \mathcal{A}[x]$ is continuous: the rules governing the first $n+1$ rounds of $\mathcal{A}[x]$ depend only on $x \restriction n$. We in fact defined an association $s \mapsto \mathcal{A}[s]$; for $s \in \omega^{<\omega}$ we have $\mathcal{A}[s]$, a game of $\operatorname{lh}(s) + 1$ many rounds.

Definition 2.2. $\mathcal{A}$ denotes the map $(s \mapsto \mathcal{A}[s])$.

Our definition of $\mathcal{A}[s]$ from s takes place entirely in M. It follows that the map $\mathcal{A}$ belongs to M. This is important; it allows us to shift $\mathcal{A}$ using elementary embeddings which act on M. Given an elementary $j\colon M \to M^*$ we have the map $j(\mathcal{A})$ defined on $s \in \omega^{<\omega}$. For a real x (in V) we can then define $j(\mathcal{A})[x]$ in the natural way: $j(\mathcal{A})[x] = \bigcup_{n<\omega} j(\mathcal{A})[x \restriction n]$. We shall use such shiftings later on, see for example Section 2.2.

[2] By $M \| \kappa_n$ we mean $\mathrm{V}^M_{\kappa_n}$.

2.1. Generic runs. Fix some g which is $\mathrm{col}(\omega,\delta)$–generic/$M$. We alternate between thinking of g as a generic enumeration of δ, and as a generic enumeration of $M\|\,\delta$. (δ and $M\|\,\delta$ have the same cardinality in M.)

Working in $M[g]$ define $\sigma_{\mathrm{gen}}[x]$, a strategy for I in $\mathcal{A}[x]$, as follows: $\sigma_{\mathrm{gen}}[x]$ plays in each round the *first* (with respect to the enumeration g) legal move. (Remember that moves in $\mathcal{A}[x]$ are elements of $M\|\,\delta$; see Remark 2.1.)

The association $x \mapsto \sigma_{\mathrm{gen}}[x]$ is continuous; we are in fact defining a map $s \mapsto \sigma_{\mathrm{gen}}[s]$ for $s \in \omega^{<\omega}$. This map belongs to $M[g]$.

Definition 2.3. (Made with respect to a fixed g.) σ_{gen} denotes the map $(s \mapsto \sigma_{\mathrm{gen}}[s])$.

Lemma 2.4. *Suppose that there exists an infinite run of $\mathcal{A}[x]$, played according to $\sigma_{\mathrm{gen}}[x]$. Then $x \notin \dot{A}[g]$. (This is only useful if $x \in M[g]$.)*

Proof Sketch. Suppose for contradiction that $x \in \dot{A}[g]$. In particular $x \in M[g]$. We have some name $\dot{x}$ so that $\dot{x}[g] = x$ and $g \Vdash$"$\dot{x} \in \dot{A}$."

We have some infinite run of $\mathcal{A}[x]$, as displayed in Diagram 8. The run splits into **branches**: a branch is a sequence $\{n_k\}_{k<\omega}$ so that $l_{n_0} =$ "new" and $l_{n_k} = n_{k-1}$ for $k > 0$.

Note that $\dot{x}$ and conditions $p \in g$ satisfy rules 3–5 of $\mathcal{A}[x]$. The *genericity* of I's moves allows us to find a branch which realizes $\dot{x}$ and g. More precisely, a branch so that (a) $\dot{x} \in \mathcal{X}_{n_k}$ for all k; and (b) p_{n_k} belongs to g for all k. But then using rule 6 we get an infinite sequence of ordinals, a contradiction. $\dashv$

The key to the proof of Lemma 2.4 is the use of genericity in the last paragraph. We refer the reader to [12, Chapter 1] for a precise argument. The same proof can be used to show that in fact there is no generic h so that $x \in \dot{A}[h]$.

2.2. Pivots. We wish to phrase a lemma similar to Lemma 2.4, but now with a method of playing for II so that infinite runs put x *in* (something like) $\dot{A}[h]$. We cannot directly come up with moves for II in $\mathcal{A}[x]$. Instead we phrase another game which is similar to $\mathcal{A}[x]$ but easier for II, and come up with a method of playing for II in the easier game. This easier game is denoted $\mathcal{A}_{\mathrm{piv}}[x]$. Its format is presented in Diagrams 9 and 10.

I	$\dots$	$l_n, \mathcal{X}_n, p_n$		$\dots$	
II			$E_{2n}, E_{2n+1}, \mathcal{F}_n, \mathcal{D}_n$		$\dots$

Diagram 9. Round n of $\mathcal{A}_{\mathrm{piv}}[x]$.

At the start of round n we have a finite iteration tree $\mathcal{T}\restriction 2n{+}1$ on M ending with a model M_{2n}, an embedding $j_{0,2n}\colon M \to M_{2n}$, and a position P_n of n rounds in $j_{0,2n}(\mathcal{A})[x]$. During the round:

- I plays $l_n, \mathcal{X}_n, p_n$, a legal move in $j_{0,2n}(\mathcal{A})[x]$ following P_n.

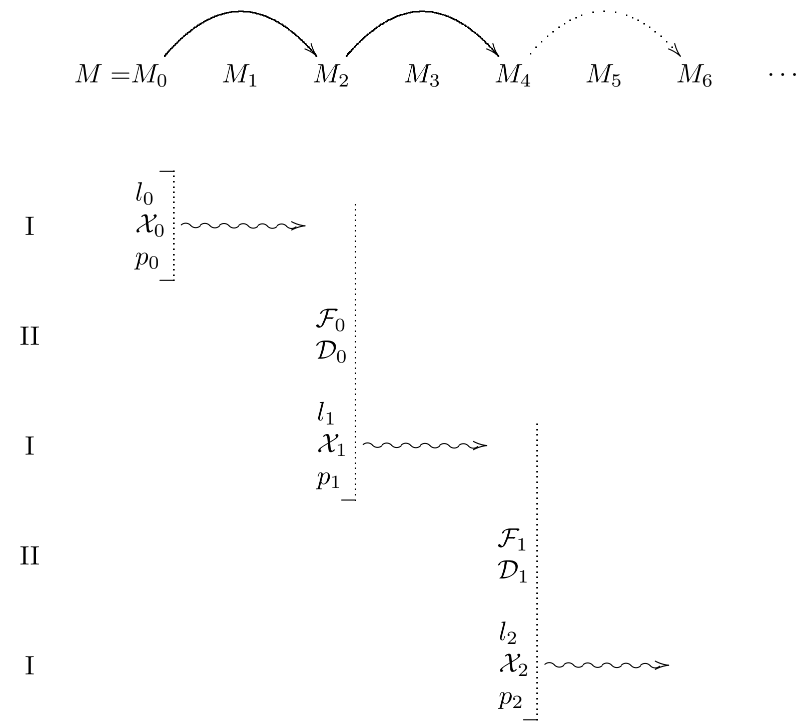

DIAGRAM 10. $\mathcal{A}_{\mathrm{piv}}[x]$, the dynamic view.

We extend the tree order $T\restriction 2n+1$ by setting $(2l_n+1)\ T\ (2n+1)$ if $l_n \neq$ "new" and $(2n)\ T\ (2n+1)$ otherwise. We set further $(2n)\ T\ (2n+2)$. We have now $T\restriction 2n+3$.

- II plays extenders E_{2n}, E_{2n+1}, which combined with our definition of $T\restriction 2n+3$ give rise to models M_{2n+1} and M_{2n+2}. (It is II's responsibility to make sure the domains of the extenders are within the level of agreement of the relevant models.)

We have an embedding $j_{2n,2n+2}\colon M_{2n} \to M_{2n+2}$. Let Q_n be the position $j_{2n,2n+2}(P_n—, l_n, \mathcal{X}_n, p_n)$.[3] This "shifting" of $P_n—, l_n, \mathcal{X}_n, p_n$ from M_{2n} to M_{2n+2} is indicated in squiggly arrows in Diagram 10.

- II plays $\mathcal{F}_n, \mathcal{D}_n$, a legal move in $j_{0,2n+2}(\mathcal{A})[x]$ following Q_n.

This completes the round. We let $\mathcal{T}\restriction 2n+3$ be the extended iteration tree (ending with M_{2n+2}), let $P_{n+1} = Q_n—, \mathcal{F}_n, \mathcal{D}_n$, and proceed to round $n+1$.

[3]We write $P_n—, l_n, \mathcal{X}_n, p_n$ to indicate that P_n is a *sequence* while l_n, $\mathcal{X}_n$, and p_n are singleton objects. Formally we should write $P_n{}^\frown\langle l_n, \mathcal{X}_n, p_n\rangle$, but we prefer to avoid the additional brackets.

Remark 2.5. We make one extra, technical demand on player II. We demand that all extenders used are taken from below δ, and have critical points larger than some pre-specified ordinal $\lambda < \delta$. For one example of how this is used (and which λ is specified) see Remark 4.6. For another example see Remark 4.9. Similar uses are made later, in Section 6.

We note as usual that the association $x \mapsto \mathcal{A}_{\text{piv}}[x]$ is continuous; for $s \in \omega^{<\omega}$ we get $\mathcal{A}_{\text{piv}}[s]$, a game of $\text{lh}(s) + 1$ many round.

Definition 2.6. We use $\mathcal{A}_{\text{piv}}$ to denote the map $(s \mapsto \mathcal{A}_{\text{piv}}[s])$.

As usual the map $\mathcal{A}_{\text{piv}} = (s \mapsto \mathcal{A}_{\text{piv}}[s])$ belongs to M.

Definition 2.7. A **pivot** for x is a pair $\mathcal{T}$, $\vec{a}$ so that:

1. $\mathcal{T}$ is a length ω iteration tree on M, with an even branch.
2. $\vec{a}$ is a run of $j_{\text{even}}(\mathcal{A})[x]$.
3. For every cofinal odd branch b of $\mathcal{T}$, there exists some h so that:
   (a) h is $\text{col}(\omega, j_b(\delta))$–generic/$M_b$; and
   (b) $x \in j_b(\dot{A})[h]$.

Any run of $\mathcal{A}_{\text{piv}}[x]$ produces $\mathcal{T}$, $\vec{a}$ which satisfy conditions 1 and 2. To be a pivot the run must further satisfy the crucial condition 3. Intuitively condition 3 states that x belongs to interpretations of "shifts" of the name $\dot{A}$. Our goal here is to phrase a lemma which complements Lemma 2.4 and the notions of the previous subsection. We can now say precisely what this means: we need a strategy which plays for II in $\mathcal{A}_{\text{piv}}[x]$ and always secures condition 3.

Fix some map $\varrho\colon \omega \to M\|\,\delta + 1$. Applying techniques of the kind used in Neeman [13]—which in turn builds on Martin–Steel [8]—it is possible to construct a strategy $\sigma_{\text{piv}}[\varrho, x]$ which plays for II in $\mathcal{A}_{\text{piv}}[x]$, and so that:

Lemma 2.8. *Suppose ϱ is* onto $M\|\,\delta{+}1$. *Then all runs according to $\sigma_{\text{piv}}[\varrho, x]$ are pivots.*

As usual the map $x \mapsto \sigma_{\text{piv}}[\varrho, x]$ is continuous in x. But we cannot expect this map to belong to M, since ϱ need not belong to M. This is why we include the extra variable ϱ. The map $\varrho, x \mapsto \sigma_{\text{piv}}[\varrho, x]$ is continuous, not only in x, but also in ϱ. For $s \in \omega^n$ and $\vartheta\colon n \to M\|\,\delta + 1$ we get a strategy $\sigma_{\text{piv}}[\vartheta, s]$ which plays for II in $\mathcal{A}_{\text{piv}}[s]$. We have

$$\sigma_{\text{piv}}[\varrho, x] = \bigcup_{n<\omega} \sigma_{\text{piv}}[\varrho \restriction n, x \restriction n].$$

Definition 2.9. σ_{piv} denotes the map $(\vartheta, s \mapsto \sigma_{\text{piv}}[\vartheta, s])$.

The construction of $\sigma_{\text{piv}}[\vartheta, s]$, indeed of the map $\vartheta, s \mapsto \sigma_{\text{piv}}[\vartheta, s]$, is phrased entirely in M. The map σ_{piv}, taken as a function in *two* variables, therefore belongs to M. This is important—it will allow us to shift this map using elementary embeddings which act on M. For an example of this see Section 4, particularly Remark 4.5.

For details on the construction of σ_{piv} and the proof of Lemma 2.8 we refer the reader to [12, Chapter 1]. Let us here only say that the construction draws heavily on the techniques of Martin–Steel [8], and that the assumption (earlier in this section) that δ is a Woodin cardinal is crucial.

§3. A first application, Σ^1_2 determinacy. As a first example we use the methods of Section 2 to prove Σ^1_2 determinacy. The result we obtain, Theorem 3.1, was previously proved by Woodin using different methods. It strengthens a result of Martin–Steel [8]. For more information on determinacy within the projective hierarchy we refer the reader to Neeman [14] and [13].

THEOREM 3.1. *Suppose there is an iterable class model M with a Woodin cardinal δ. Suppose further that $M \| \delta + 1$ is countable in* V. *Then Σ^1_2 determinacy holds.*

PROOF. Fix $A \subset \mathbb{R}$ a Σ^1_2 set, say the set of reals which satisfy a given Σ^1_2 statement ϕ. We wish to show that the standard game $G_\omega(A)$ is determined.

Fix M and δ which satisfy the hypothesis of Theorem 3.1. Let $\dot{A} \in M$ name the set of reals of $M^{\text{col}(\omega,\delta)}$ which satisfy ϕ in $M^{\text{col}(\omega,\delta)}$. We have the corresponding maps $\mathcal{A}$, σ_{gen}, $\mathcal{A}_{\text{piv}}$, and σ_{piv} of Section 2.

Working inside M we define a game G^*, played according to Diagram 11.

I	x_0	$l_0, \mathcal{X}_0, p_0$			$l_1, \mathcal{X}_1, p_1$		x_2	$\ldots$
II			$\mathcal{F}_0, \mathcal{D}_0$	x_1		$\mathcal{F}_1, \mathcal{D}_1$		

DIAGRAM 11. The game G^*.

I and II alternate playing natural numbers x_n, producing together $x = \langle x_n \mid n < \omega \rangle \in \mathbb{R}$. In addition they play auxiliary moves subject to the rules of $\mathcal{A}[x]$. If a player cannot follow these rules, she loses. Infinite runs of G^* are won by II.

REMARK 3.2. Our definition of G^* implicitly uses the continuity of the map $x \mapsto \mathcal{A}[x]$; in round n of G^* we only know $x \restriction n$, but this is enough to figure the rules for round n of $\mathcal{A}[x]$. Similarly, the fact that G^* exists *inside* M follows from the fact that $\mathcal{A} = (s \mapsto \mathcal{A}[s])$ belongs to M.

We will show that if I wins G^* in M then I wins $G(A)$ in V. Later on we will phrase a mirror image game H^*, and show that if II wins H^* in M then II wins $G(A)$ in V. Then we will use the determinacy of G^* and H^* in M—note G^* is an open game and H^* will be a closed game—to argue that one of these cases must hold.

CASE 1, if I wins G^* in M. Fix $\sigma^* \in M$ a winning strategy for I (the open player) in G^*. We wish to show that I wins $G_\omega(A)$ in V. Let us play $G_\omega(A)$ against an imaginary opponent. We describe how to play, and win.

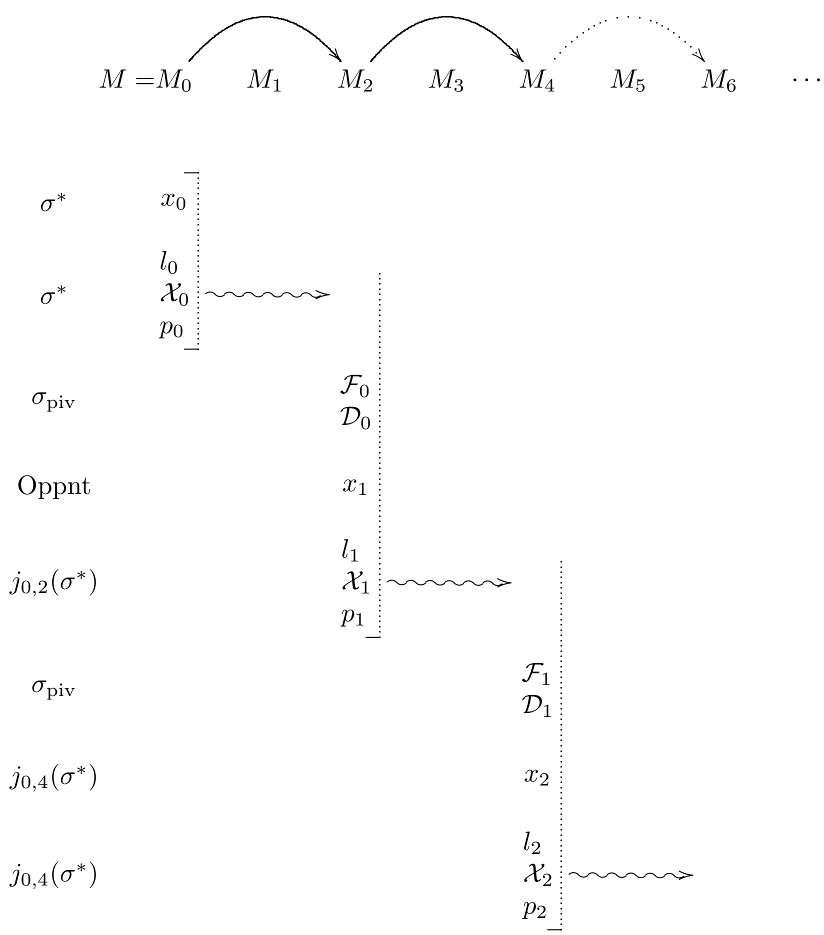

Diagram 12. The construction in case 1.

In V fix a surjection $\varrho\colon \omega \to M \| \delta + 1$. Our description takes the form of a construction in V. We construct a run $x \in \mathbb{R}$ of $G_\omega(A)$. At the same time we construct $\mathcal{T}$, $\vec{a}$, a run of $\mathcal{A}_{\text{piv}}[x]$. The participants in our construction are:

- The imaginary opponent: playing x_n for odd n.
- The strategy $\sigma_{\text{piv}}[\varrho, x]$: playing for II in $\mathcal{A}_{\text{piv}}[x]$.
- The strategy σ^* and its shifts along the even branch of $\mathcal{T}$: playing x_n for even n and playing for I in $\mathcal{A}_{\text{piv}}[x]$ (i.e., playing for I in shifts of $\mathcal{A}[x]$).

The time line of the construction is presented in Diagram 12. At the start of round n we have $x \restriction n$, $\mathcal{T} \restriction 2n+1$ ending with the model M_{2n}, and a position P_n of n rounds in $j_{0,2n}(\mathcal{A})[x \restriction n]$. If n is odd, our opponent opens the round playing x_n. If n is even $j_{0,2n}(\sigma^*)$ plays x_n. Then $j_{0,2n}(\sigma^*)$ plays an auxiliary move

$l_n, \mathcal{X}_n, p_n$, according to the rules of $j_{0,2n}(\mathcal{A})[x\restriction n]$ following the position P_n. At this point we apply $\sigma_{\text{piv}}[\varrho, x\restriction n]$ which creates the models M_{2n+1}, M_{2n+2}. Let $Q_n = j_{2n,2n+2}(P_n—, l_n, \mathcal{X}_n, p_n)$. $\sigma_{\text{piv}}[\varrho, x\restriction n]$ further plays $\mathcal{F}_n, \mathcal{D}_n$, a legal move for II in $j_{0,2n+2}(\mathcal{A})[x\restriction n]$ following Q_n. We let $P_{n+1} = Q_n—, \mathcal{F}_n, \mathcal{Q}_n$. This completes round n.

Once the construction is completed we let

$$a_n = j_{2n,\text{even}}(l_n, \mathcal{X}_n, p_n)—, j_{2n+2,\text{even}}(\mathcal{F}_n, \mathcal{D}_n).$$

We let $\vec{a} = \langle a_n \mid n < \omega\rangle$. Our construction is such that x and $\vec{a}$ form an infinite play of $j_{\text{even}}(G^*)$, which is played according to $j_{\text{even}}(\sigma^*)$. This play is created in V, since our opponent lives in V. If M_{even} were wellfounded the existence of such a play could be reflected into M_{even}. It could then be pulled back via j_{even} to yield the existence in M of an infinite play of G^* which is according to σ^*. But σ^* is a winning strategy for I, the open player in G^*; so there are no infinite plays according to σ^*. We conclude that M_{even} is illfounded.

Since M is iterable there must exist some cofinal wellfounded cofinal branch b through $\mathcal{T}$. b must be an odd branch. Our use of $\sigma_{\text{piv}}[\varrho, x]$ during the construction guarantees that $\mathcal{T}, \vec{a}$ is a pivot. Applying condition 3 of Definition 2.7 we conclude that there exists some h which is $\text{col}(\omega, j_b(\delta))$–generic/$M_b$ and so that:

$(*)$ $x \in j_b(\dot{A})[h]$.

This means that in $M_b[h]$ x satisfies the Σ^1_2 statement ϕ. By Shoenfield absoluteness x must also satisfy ϕ in V. (We are using here the wellfoundedness of M_b.) So $x \in A$ as required. This completes case 1. ⊣

Remark 3.3. Note the importance of continuity throughout our construction. In round n we are able to use $\sigma_{\text{piv}}[\varrho, x]$ despite only having knowledge of $x\restriction n+1$.

Remark 3.4. Note further the importance of having G^* and σ^* *inside* M. During the construction we shifted G^* and σ^* along the even branch of $\mathcal{T}$, using $j_{0,2n}(\sigma^*)$ in round n.

Let $\dot{B} \in M$ name the set of reals which do **not** satisfy ϕ in $M^{\text{col}(\omega,\delta)}$. Define $x \mapsto \mathcal{B}[x]$ and $x \mapsto \mathcal{B}_{\text{piv}}[x]$ as in Section 2, but changing $\dot{A}$ to $\dot{B}$ and **interchanging** I **and** II. We have strategies $\tau_{\text{gen}}[x]$ and $\tau_{\text{piv}}[\varrho, x]$ as before, but with the roles of I and II switched. (In particular, τ_{gen} is a strategy for II and τ_{piv} is a strategy for I.) These strategies satisfy Lemmas 2.4 and 2.8, but with $\dot{A}$ (in Lemma 2.4 and in condition 3b of Definition 2.7) changed to $\dot{B}$.

Working inside M we define a game H^*, played according to Diagram 13.

As before I and II alternate playing natural numbers x_n, producing together $x = \langle x_n \mid n < \omega\rangle \in \mathbb{R}$. This time they play auxiliary moves subject to the rules of $\mathcal{B}[x]$. If a player cannot follow these rules, she loses. This time infinite runs of the game are won by I.

$$
\begin{array}{c|cccccccc}
\mathrm{I} & x_0 & & \mathcal{F}_0, \mathcal{D}_0 & & & \mathcal{F}_1, \mathcal{D}_1 & x_2 & \dots \\
\hline
\mathrm{II} & & l_0, \mathcal{X}_0, p_0 & & x_1 & l_1, \mathcal{X}_1, p_1 & & &
\end{array}
$$

DIAGRAM 13. The game H^*.

CASE 2, if II wins H^* in M. Then an argument similar to that of case 1 shows that (in V) II has a strategy to get into $B = \mathbb{R} - A$. In other words, II wins $G_\omega(A)$ in V. $\dashv$

So far we showed:

- (In case 1.) If I wins G^* in M, then I wins $G_\omega(A)$ in V.
- (In case 2.) If II wins H^* in M, then II wins $G_\omega(A)$ in V.

It is now enough to check that one of these cases must occur. Suppose not, i.e., assume that in M II wins G^* and I wins H^*. Fix strategies σ^* and τ^* in M witnessing this. We intend to derive a contradiction.

We work in $M[g]$ to construct a real $x = \langle x_n \mid n < \omega \rangle$, an infinite play $\vec{a} = \langle a_{n-\mathrm{I}}, a_{n-\mathrm{II}} \mid n < \omega \rangle$ of $\mathcal{A}[x]$ ($a_{n-\mathrm{I}}$ denotes I's auxiliary move in round n; $a_{n-\mathrm{II}}$ denotes II's auxiliary move in round n), and an infinite play $\vec{b} = \langle b_{n-\mathrm{II}}, b_{n-\mathrm{I}} \mid n < \omega \rangle$ of $\mathcal{B}[x]$. We construct as follows:

- σ^* (playing for II in G^*) produces x_n for odd n, and $a_{n-\mathrm{II}}$ for all n.
- $\sigma_{\mathrm{gen}}[x]$ produces $a_{n-\mathrm{I}}$ for all n.
- τ^* (playing for I in H^*) produces x_n for even n and $b_{n-\mathrm{I}}$ for all n.
- $\tau_{\mathrm{gen}}[x]$ produces $b_{n-\mathrm{II}}$ for all n.

As usual continuity is important. Our use of $\sigma_{\mathrm{gen}}[x]$ and $\tau_{\mathrm{gen}}[x]$ in round n can be carried through since it only requires knowledge of $x \restriction n$. We note that the maps σ_{gen} and τ_{gen} exist in $M[g]$. Since σ^* and τ^* exist in M the entire construction can be carried inside $M[g]$.

Our use of σ_{gen} guarantees that $x \notin \dot{A}[g]$ (see Lemma 2.4). Since x belongs to $M[g]$ this means that x fails to satisfy ϕ in $M[g]$. Similarly our use of τ_{gen} guarantees that $x \notin \dot{B}[g]$, and this means that x fails to **not** satisfy ϕ in $M[g]$. But this is a contradiction. $\dashv$

§4. Games of length $\omega \cdot \omega$. Fix $C \subset \mathbb{R}^\omega$ a Σ^1_2 set, say the set of all sequences $\langle y_n \mid n < \omega \rangle \in \mathbb{R}^\omega$ which satisfy a given Σ^1_2 statement ϕ. Fix M and an increasing sequence $\langle \delta_1, \delta_2, \dots, \delta_\omega \rangle$ in M so that:

- M is a class model;
- M is iterable;
- Each δ_ξ, $1 \le \xi \le \omega$, is a Woodin cardinal in M; and
- $M \| \delta_\omega + 1$ is countable in V.

The existence of such an M is our large cardinal assumption. We work under this assumption to prove that $G_{\omega\cdot\omega}(C)$ is determined.

We work to define auxiliary games in M, analogous to the games G^* and H^* of Section 3. These games will be open and closed respectively, and hence determined. If in M I wins the analogue of G^*, we will show that in V I wins $G_{\omega\cdot\omega}(C)$. This is an analogue to case 1 in Section 3. If in M II wins the analogue of H^*, then by a parallel argument II wins $G_{\omega\cdot\omega}(C)$ in V. This is an analogue to case 2 in Section 3. Determinacy will follow once we verify, in Section 4.4, that one of these cases must occur. This is an analogue to the final argument in Section 3.

4.1. Names. Let δ_∞ denote δ_ω. Let $\dot{A}_\infty \in M$ name the set of sequences $\langle y_n \mid n < \omega\rangle \in \mathbb{R}^\omega$ in $M^{\mathrm{col}(\omega,\delta_\infty)}$ which satisfy ϕ in $M^{\mathrm{col}(\omega,\delta_\infty)}$. For each $\langle y_n \mid n < \omega\rangle \in \mathbb{R}^\omega$ we have the associated auxiliary game $\mathcal{A}_\infty[y_n \mid n < \omega]$ of Section 2 corresponding to the name $\dot{A}_\infty$ and the Woodin cardinal δ_∞. (There is a slight abuse of notation here; formally we should think of $\langle y_n \mid n < \omega\rangle$ as coded by some real x.) We remind the reader that moves in $\mathcal{A}_\infty[y_n \mid n < \omega]$ are arranged so that I tries to witness $\langle y_n \mid n < \omega\rangle \in \dot{A}_\infty[h]$ for some h, while II tries to witness the opposite.

The association $\langle y_n \mid n < \omega\rangle \mapsto \mathcal{A}_\infty[y_n \mid n < \omega]$ is continuous, given by the map $\mathcal{A}_\infty$. This map belongs to M. We will talk about $\mathcal{A}_\infty[y_0, \dots, y_{k-1}]$, which we take to be a game of $k+1$ rounds. (Only a finite part of the reals $y_0, \dots, y_{k-1}$ is needed to determine the rules of this game.) We use $a^\infty_{0-\mathrm{I}}$, $a^\infty_{0-\mathrm{II}}$, $a^\infty_{1-\mathrm{I}}$, etc. to refer to moves in the games $\mathcal{A}_\infty[y_n \mid n < \omega]$, and use a^∞_n to denote $\langle a^\infty_{n-\mathrm{I}}, a^\infty_{n-\mathrm{II}}\rangle$. We use $\vec{a}_\infty = \langle a^\infty_n \mid n < \omega\rangle$ to refer to infinite runs of $\mathcal{A}_\infty[y_n \mid n < \omega]$.

DEFINITION 4.1. A ***k*-sequence** is a sequence

$$S = \langle y_0, \dots, y_{k-1}, a^\infty_0, \dots, a^\infty_{k-1}, \gamma\rangle$$

so that:

1. Each y_i is a real;
2. $a^\infty_0, \dots, a^\infty_{k-1}$ is a position in the auxiliary game $\mathcal{A}_\infty[y_0, \dots, y_{k-1}]$; and
3. γ is an ordinal.

DEFINITION 4.2. A **valid extension** for a k-sequence S as in Definition 4.1 is a triple $y_k, a^\infty_k, \gamma^*$ so that:

1. y_k is a real;
2. $a^\infty_k = \langle a^\infty_{k-\mathrm{I}}, a^\infty_{k-\mathrm{II}}\rangle$ where $a^\infty_{k-\mathrm{I}}$ and $a^\infty_{k-\mathrm{II}}$ are legal moves for I and II respectively in the game $\mathcal{A}_\infty[y_0, \dots, y_{k-1}]$ following $a^\infty_0, \dots, a^\infty_{k-1}$; and
3. γ^* is an ordinal smaller than γ.

We use $S\text{—}, y_k, a^\infty_k, \gamma^*$ to denote the $k+1$-sequence

$$\langle y_0, \dots, y_{k-1}, y_k, a^\infty_0, \dots, a^\infty_{k-1}, a^\infty_k, \gamma^*\rangle.$$

We remind the reader that $\mathcal{A}_\infty[y_0, \dots, y_{k-1}]$ is a game of $k+1$ rounds, so condition 2 of Definition 4.2 makes sense.

For expository simplicity fix for each $k < \omega$ some g_k which is $\mathrm{col}(\omega, \delta_k)$–generic/$M$. Below we define sets in $M[g_k]$ where strictly speaking we should

be defining names in $M^{\mathrm{col}(\omega,\delta_k)}$. For a tuple $a_0^\infty, \dots, a_{k-1}^\infty$ and an ordinal γ we work to describe $A_k[a_0^\infty, \dots, a_{k-1}^\infty, \gamma]$, a subset of $\mathbb{R}^k$ in $M[g_k]$. We shall then let $\dot{A}_k[a_0^\infty, \dots, a_{k-1}^\infty, \gamma]$ be the canonical name for this set.

We use the notation

$$\langle y_0, \dots, y_{k-1}, a_0^\infty, \dots, a_{k-1}^\infty, \gamma\rangle \in A_k$$

to mean that $\langle y_0, \dots, y_{k-1}\rangle$ belongs to $A_k[a_0^\infty, \dots, a_k^\infty, \gamma]$. We similarly think of $\dot{A}_k$ as a (class) name for the collection of tuples S so that $S \in A_k$. Thus we say

$$S = \langle y_0, \dots, y_{k-1}, a_0^\infty, \dots, a_{k-1}^\infty, \gamma\rangle \in \dot{A}_k[h]$$

to mean that $\langle y_0, \dots, y_{k-1}\rangle$ belongs to $\dot{A}_k[a_0^\infty, \dots, a_{k-1}^\infty, \gamma][h]$.

Let $\mathcal{A}_k[y_0, \dots, y_{k-1}, a_0^\infty, \dots, a_{k-1}^\infty, \gamma]$ be the auxiliary games corresponding to the name $\dot{A}_k[a_0^\infty, \dots, a_{k-1}^\infty, \gamma]$ and the Woodin cardinal δ_k. We use $\mathcal{A}_k[S]$ to refer to these games, and use $a_{0-\mathrm{I}}^k$, $a_{0-\mathrm{II}}^k$ etc. to denote moves in the games. These moves are arranged so that I tries to witness that S belongs to $\dot{A}_k[h]$ for some generic h, while II tries to witness the opposite.

Given $S = \langle y_0, \dots, y_{k-1}, a_0^\infty, \dots, a_{k-1}^\infty, \gamma\rangle$ a k-sequence, we define a game $G_k^*(S)$ played inside M according to Diagram 14.

I	$\gamma^*, a_{k-\mathrm{I}}^\infty$		$y_k(0)$	$a_{0-\mathrm{I}}^{k+1}$			$a_{1-\mathrm{I}}^{k+1}$		$y_k(2)\dots$
II		$a_{k-\mathrm{II}}^\infty$			$a_{0-\mathrm{II}}^{k+1}$	$y_k(1)$		$a_{1-\mathrm{II}}^{k+1}$	

DIAGRAM 14. The game $G_k^*(S)$.

I and II play

- γ^*,
- $a_k^\infty = \langle a_{k-\mathrm{I}}^\infty, a_{k-\mathrm{II}}^\infty\rangle$, and
- $y_k = \langle y_k(n) \mid n < \omega\rangle$

which form a valid extension of S. To be more precise we require:

1. γ^* is smaller than γ, in line with condition 3 of Definition 4.2.
2. $a_{k-\mathrm{I}}^\infty$ and $a_{k-\mathrm{II}}^\infty$ are legal moves for I and II respectively in $\mathcal{A}_\infty[y_0, \dots, y_{k-1}]$ following $a_0^\infty, \dots, a_{k-1}^\infty$. This is in line with condition 2 of Definition 4.2. Note that knowledge of y_k is not needed here.
3. $y_k(n)$ are natural numbers.

In addition I and II play auxiliary moves in the game $\mathcal{A}_{k+1}[S—, y_k, a_k^\infty, \gamma^*]$. If a player cannot follow these rules she loses. Infinite runs of the game are won by II.

DEFINITION 4.3. For $S \in M[g_k]$ set $S \in A_k$ iff S is a k-sequence and in $M[g_k]$ I has a winning strategy in $G_k^*(S)$.

Definition 4.3 at last specifies the sets $A_k[a_0^\infty, \dots, a_{k-1}^\infty, \gamma]$, and by extension the names $\dot{A}_k[a_0^\infty, \dots, a_{k-1}^\infty, \gamma]$.

REMARK 4.4. Our definition of the sets $A_k[a_0^\infty, \ldots, a_{k-1}^\infty, \gamma]$ is by induction on γ, not on k. To figure out whether $\langle y_0, \ldots, y_{k-1}\rangle \in M[g_k]$ belongs to the set $A_k[a_0^\infty, \ldots, a_{k-1}^\infty, \gamma]$ we need knowledge of the game $G_k^*(S)$ where $S = \langle y_0, \ldots, y_{k-1}, a_0^\infty, \ldots, a_{k-1}^\infty, \gamma\rangle$. For this we require knowledge of the auxiliary games $\mathcal{A}_{k+1}[S—, y_k, a_k^\infty, \gamma^*]$, but only for γ^* which are **smaller** than γ because of rule 1. Thus to determine $\dot{A}_k[a_0^\infty, \ldots, a_{k-1}^\infty, \gamma]$ we need knowledge of the names $\dot{A}_{k+1}[a_0^\infty, \ldots, a_{k-1}^\infty, a_k^\infty, \gamma^*]$, but only for $\gamma^* < \gamma$. We have this knowledge by induction.

Some words of motivation are due on Definition 4.3. Suppose that $S \in M[g_k]$ is a k-sequence and belongs to A_k. So I wins $G_k^*(S)$. Let us for a moment ignore the first round of $G_k^*(S)$. The remaining rounds essentially follow the rules of G^* of Section 3. (See Diagram 11 and the rules below it.) Our experience from Section 3 tells us that if I has a winning strategy for these rounds, then in V I has a strategy to enter some shift of $\dot{A}_{k+1}$. In other words, if S belongs to $A_k = \dot{A}_k[g_k]$ we expect to be able to produce y_k (working against an imaginary opponent who plays the odd half of y_k) so that $S—, y_k, a_k^\infty, \gamma^*$ belongs to $j_b(\dot{A}_{k+1})[h]$ for some iteration map j_b and some generic h.

This is a process of *perpetuation.* Membership in A_k allows us to aim for membership in a shift of $\dot{A}_{k+1}$.

And what about the first round of $G_k^*(S)$? This round too is related to the game G^* of Section 3, this time with the name $\dot{A}_\infty$. It is just one round out of this game, and our experience from Section 3 tells us that a winning strategy for I will allow us to aim into a shift of the name $\dot{A}_\infty$.

In short, membership in A_k allows us to (a) advance one round in witnessing that our sequence of reals belongs to a shift of $\dot{A}_\infty$; and (b) produce the next real, y_k, so that the resulting sequence belongs to a shift of $\dot{A}_{k+1}$. Once we entered a shift of $\dot{A}_{k+1}$ we can repeat the process, advancing an extra round towards $\dot{A}_\infty$ and entering a shift of $\dot{A}_{k+2}$, etc. At the end we make the full sequence of advances needed to witness membership in $\dot{A}_\infty$. This means that our sequence of reals (produced with the collaboration of some imaginary opponent playing for II) satisfies the Σ_2^1 statement ϕ. So we win the long game $G_{\omega\cdot\omega}(C)$, playing for I.

This argument is made more precise in Section 4.2. Then in Section 4.3 we phrase the mirror image argument and show under reversed circumstances that II wins the long game. The relationship between Sections 4.2 and 4.3 is analogous to the relationship between cases 1 and 2 in Section 3. Finally in Section 4.4 we show that either the circumstances of Section 4.2 (where I ends up winning) or the circumstances of Section 4.3 (where II ends up winning) must hold. This establishes the determinacy of $G_{\omega\cdot\omega}(C)$.

4.2. I wins. Suppose that there exists some γ so that in M I wins $G_0^*(\gamma)$. (Note that γ by itself is a 0-sequence.) We will show that I wins the original long game $G_{\omega\cdot\omega}(C)$ in V.

Fix $\sigma_0^* \in M$, a winning strategy for I (the open player) in $G_0^*(\gamma)$. Fix an imaginary opponent, playing for II in the long game $G_{\omega\cdot\omega}(C)$.

Recall that we have strategies $\sigma_{\mathrm{piv}-\infty}[y_n \mid n < \omega]$ corresponding to the name $\dot{A}_\infty$ (see Section 2.2).[4] Similarly we have strategies

$$\sigma_{\mathrm{piv}-k}[y_0, \dots, y_{k-1}, a_0^\infty, \dots, a_{k-1}^\infty, \gamma]$$

(which we denote $\sigma_{\mathrm{piv}-k}[S]$) corresponding to the names $\dot{A}_k[a_0^\infty, \dots, a_{k-1}^\infty, \gamma]$. These strategies are given by maps $\sigma_{\mathrm{piv}-\infty}$ and $\sigma_{\mathrm{piv}-k}$, continuous in the relevant reals and in the suppressed variable ϱ. The maps belong to M.

We will use σ_0^*, the strategies $\sigma_{\mathrm{piv}-1}, \sigma_{\mathrm{piv}-2}, \dots$, the strategy $\sigma_{\mathrm{piv}-\infty}$, and an iteration strategy for M, to play against the imaginary opponent and win.

Let us begin playing $G_{\omega\cdot\omega}(C)$. We divide the game into ω mega-rounds. In mega-round k we construct (among other things) the real y_k. At the start of mega-round k we will have:

(A) Reals $y_0, \dots, y_{k-1}$;
(B) An iterate M_k of M (the result of k iteration trees stacked one after the other) with iteration embedding $j_k \colon M \to M_k$;
(C) (If $k > 0$.) h_k which is $\mathrm{col}(\omega, j_k(\delta_k))$–generic/$M_k$;
(D) A position of k rounds in the game $j_k(\mathcal{A}_{\mathrm{piv}-\infty})[y_0, \dots, y_{k-1}]$, played according to $j_k(\sigma_{\mathrm{piv}-\infty})[y_0, \dots, y_{k-1}]$; and
(E) An ordinal γ_k.

Remark 4.5. With respect to (D) it is important to remember that the maps $\mathcal{A}_{\mathrm{piv}-\infty}$ and $\sigma_{\mathrm{piv}-\infty}$ belong to M. These maps can therefore be shifted via the embedding $j_k \colon M \to M_k$.

The position indicated in (D) includes an iteration tree $\mathcal{U}_k$ on M_k of length $2k+1$. We use $W_0^k, \dots, W_{2k}^k$ to denote the models of this tree, and $\pi_{*,*}^k$ to denote the embeddings. The position indicated in (D) further includes a position P_k^∞ of k rounds in the shift of $\mathcal{A}_\infty[y_0, \dots, y_{k-1}]$ to W_{2k}^k, namely in $(\pi_{0,2k}^k \circ j_k)(\mathcal{A}_\infty)[y_0, \dots, y_{k-1}]$.

Note that $\langle y_0, \dots, y_{k-1}, P_k^\infty, \gamma_k \rangle$ is a k-sequence in the sense of W_{2k}^k—this is simply a restatement of the fact that P_k^∞ is a position of k rounds in $(\pi_{0,2k}^k \circ j_k)(\mathcal{A}_\infty)[y_0, \dots, y_{k-1}]$. We use S_k to denote this k-sequence. We shall make sure that

(i) S_k belongs to $W_{2k}^k[h_k]$ (to $M = W_0^0$ if $k = 0$); and
(ii) In $W_{2k}^k[h_k]$ (in $M = W_0^0$ if $k = 0$) I wins $(\pi_{0,2k}^k \circ j_k)(G_k^*)(S_k)$.

In condition (i) we are saying that the reals $y_0, \dots, y_{k-1}$ belong to $W_{2k}^k[h_k]$. The rest of S_k is just a finite list of objects from W_{2k}^k.

Remark 4.6. The game $\mathcal{A}_{\mathrm{piv}-\infty}$ is played in the vicinity of the Woodin cardinal δ_∞, and all critical points used in the game are larger than some

[4] We are suppressing here and below the extra parameter ϱ. We shall comment on this in the discussion which follows Claim 4.10.

$$M_k = W_0^k \overset{\pi^k_{0,2k}}{\dashrightarrow} W_{2k}^k$$

DIAGRAM 15. At the start of mega-round k.

pre-specified ordinal $\lambda < \delta_\infty$. (See Remark 2.5.) The ordinal we specify is $\lambda = \sup\{\delta_k \mid k < \omega\}$. We know then that the models W_{2k}^k and $M_k = W_0^k$ agree beyond $j_k(\delta_k)$, so that our reference to $W_{2k}^k[h_k]$ in conditions (i) and (ii) makes sense. Moreover, the iteration tree $\mathcal{U}_k$ can then be regarded not just as a tree on M_k, but also as a tree on $M_k[h_k]$.

We begin with $M_0 = M$, and $\gamma_0 = \gamma$. Condition (ii) holds because of our case assumption, that I wins $G_0^*(\gamma)$ in M. Let us handle mega-round k. Our models at the start of mega-round k are presented in Diagram 15. Our situation at the end of the mega-round is presented in Diagram 16.

Using condition (ii) we have $\sigma_k^* \in W_{2k}^k[h_k]$, a winning strategy for I (the open player) in $(\pi^k_{0,2k} \circ j_k)(G_k^*)[S_k]$.

(a) To open mega-round k this strategy plays γ^*_{k+1}.

The game $(\pi^k_{0,2k} \circ j_k)(G_k^*)[S_k]$ now proceeds with one round—round k—from the shifted $\mathcal{A}_\infty$, followed by all ω rounds from the shifted $\mathcal{A}_{k+1}$ together with natural number moves to produce y_k. Folding in constructions of the kind done in Section 3 we create:

(b) The models W^k_{2k+1} and W^k_{2k+2} extending $\mathcal{U}_k$, and the embedding $\pi^k_{2k,2k+2}$;
(c) A position of $k+1$ rounds in the shift of $\mathcal{A}_\infty$ to W^k_{2k+2}, extending $\pi^k_{2k,2k+2}(P_k^\infty)$;
(d) The real y_k;
(e) A length ω iteration tree $\mathcal{T}_k$ on W^k_{2k+2}, a cofinal odd branch b_k through it, the direct limit model W_{b_k}, and the direct limit embedding j_{b_k}; and
(f) h_{k+1} which is generic over W_{b_k} for the collapse of $(j_{b_k} \circ \pi^k_{0,2k+2} \circ j_k)(\delta_{k+1})$.

We let P^∞_{k+1} denote the position of (c), shifted to W_{b_k} via j_{b_k}. We let γ_{k+1} denote γ^*_{k+1} of (a) shifted to W_{b_k} via $j_{b_k} \circ \pi^k_{2k,2k+2}$. As in Section 3 we get

$(*)$ $\langle y_0, \dots, y_k\rangle \in \dot{A}^s_{k+1}[P^\infty_{k+1}, \gamma_{k+1}][h_{k+1}]$,

where $\dot{A}^s_{k+1}$ denotes $\dot{A}_{k+1}$ shifted to W_{b_k}.

REMARK 4.7. Our construction in (b) and (c) simply extends the position of (D) to $k+1$ rounds. This is nothing more than an adaptation of the construction in round k of case 1 in Section 3, using the shifts of δ_∞ and $\mathcal{A}_\infty$.

REMARK 4.8. Our construction in (d)–(f) is an adaptation of the entire argument of case 1 in Section 3, using the shifts of δ_{k+1} and $\mathcal{A}_{k+1}$. We make the following notes:

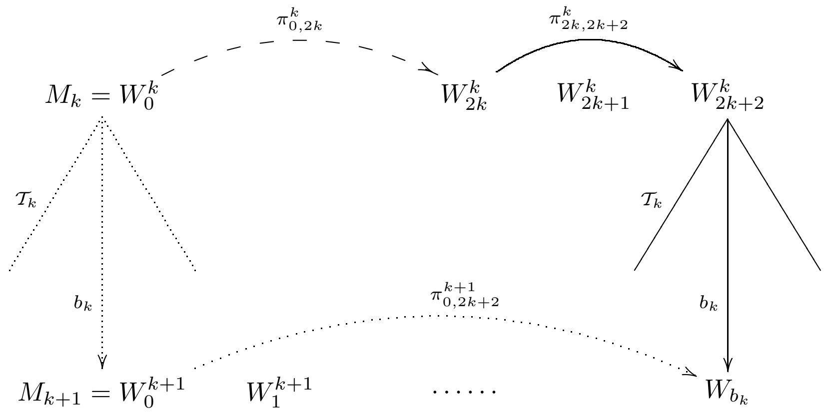

DIAGRAM 16. Mega-round k.

The real y_k is constructed as a collaborative process involving our imaginary opponent and shifts of the strategy $\pi^k_{2k,2k+2}(\sigma^*_k)$ along the even branch of $\mathcal{T}_k$. As a reminder of this we refer the reader to Diagram 12.

Remember that we have not a single name $\dot{A}_{k+1}$, but a whole class of them. Our construction in (d)–(f) uses the strategy $\sigma_{\text{piv}-k+1}$ which corresponds to $\dot{A}_{k+1}[X]$ (shifted to W^k_{2k+2}) where "X" consists of the position created in (b) and the shift to W^k_{2k+2} of the ordinal of (a). It is this use of $\sigma_{\text{piv}-k+1}$ which gives h_{k+1} for (f) and secures $(*)$, see Section 2.2.

The branch b_k is the work of an iteration strategy for M which we pick at the outset. (Recall that our initial assumptions on M included iterability.) In particular the wellfoundedness of W_{b_k} is guaranteed.

REMARK 4.9. The starting model for (d)–(f) is $W^k_{2k+2}[h_k]$. The generic extension here is *important.* Remember that as part of the construction we must shift the strategy $\pi^k_{2k,2k+2}(\sigma^*_k)$ along the even branch of the iteration tree $\mathcal{T}_k$. (See Diagram 12 and Remark 3.4.) W^k_{2k+2} does not contain this strategy; $\mathcal{T}_k$ must therefore act on $W^k_{2k+2}[h_k]$.

This is where we use the fact that δ_{k+1} is greater than δ_k, so that h_k is a "small generic" compared to the shift of δ_{k+1}. With Remark 2.5 this allows us to make sure that all extenders used in $\mathcal{T}_k$—a tree created in the vicinity of δ_{k+1}—have critical points above the shift of δ_k. $\mathcal{T}_k$ then extends to act on $W^k_{2k+2}[h_k]$.

We have so far the embeddings indicated in solid lines in Diagram 16. The top horizontal line represents the tree $\mathcal{U}_k$ and its extension by two extra models. This tree has critical points above the shift of $\sup\{\delta_1, \delta_2, \ldots\}$ (see Remark 4.6), and hence certainly above the shift of δ_{k+1}. The vertical tree on the right is our $\mathcal{T}_k$. It has critical points below its Woodin cardinal, the shift of δ_{k+1}.

Using these relations between the critical points, standard commutativity allows us to switch the order of $\mathcal{T}_k$ and the extended $\mathcal{U}_k$. We can first apply $\mathcal{T}_k$—which we may regard as a tree on M_k—and then apply the image of the extended $\mathcal{U}_k$. This new order is represented in dotted lines in Diagram 16.

We let $\mathcal{U}_{k+1}$ be the image of the extended $\mathcal{U}_k$ (this image is presented in dots on the lower line of Diagram 16). The final model of this tree, W^{k+1}_{2k+2}, is precisely equal to W_{b_k}. We are now in a position to start mega-round $k+1$. Conditions (i) and (ii) hold because of $(*)$ above.

Two points about our construction in mega-round k should be recorded for future reference. We have:

(†) $\mathcal{U}_{k+1}$ extends $j_{b_k}(\mathcal{U}_k)$; and
(‡) $\gamma_{k+1} < (j_{b_k} \circ \pi^k_{2k,2k+2})(\gamma_k)$.

(‡) follows from our use of I's strategy σ^*_k, because of rule 1 in the game G^*_k (see also condition 3 of Definition 4.2). This is now our second use of this rule. The first one was in Remark 4.4.

Once the construction is over we are left with a sequence of reals $\langle y_n \mid n < \omega\rangle$, and a sequence of iteration trees $\mathcal{T}_k$ presented in Diagram 17 giving rise to a direct limit M_∞. Our use of an iteration strategy to pick the branches b_k during the construction guarantees the wellfoundedness of M_∞.

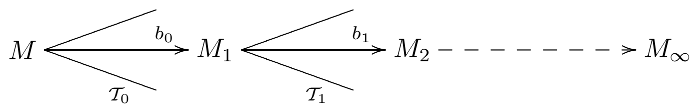

DIAGRAM 17. At the end.

We have further for each k the finite tree $\mathcal{U}_k$ on M_k. Let $\mathcal{U}_\infty$ on M_∞ be the natural limit of these trees, specifically the union of the trees $j_{k,\infty}(\mathcal{U}_k)$. This makes sense because of (†). $\mathcal{U}_\infty$ has an even branch, consisting of the models W^∞_{2k}. Let W^∞_{even} denote the direct limit along this branch. W^∞_{even} is in fact equal to the direct limit of the models W^k_{2k} under the embeddings $j_{b_k} \circ \pi^k_{2k,2k+2} \colon W^k_{2k} \to W^{k+1}_{2(k+1)}$. (We use here the same kind of commutativity that allowed us to "switch order" from the solid and broken lines to the dotted lines in Diagram 16.) Condition (‡) tells us that this last direct limit is illfounded.

So we have $\mathcal{U}_\infty$, a length ω iteration tree on M_∞, with an illfounded even branch. The iteration strategy for M, faced with $\mathcal{U}_\infty$, is forced to produces a cofinal odd branch c. Let W^∞_c be the direct limit, and let $\pi_c \colon M_\infty \to W^\infty_c$ be the direct limit embedding. Note W^∞_c, played by an iteration strategy, is wellfounded.

Now $\mathcal{U}_\infty$ is part of a play according to $j_{0,\infty}(\sigma_{\text{piv}-\infty})[y_n \mid n < \omega]$—this was part of our construction, see (D) above. Our use of $j_{0,\infty}(\sigma_{\text{piv}-\infty})[y_n \mid n < \omega]$ guarantees that there exists some h_∞ so that:

1. h_∞ is $\text{col}(\omega, (\pi_c \circ j_{0,\infty})(\delta_\infty))$–generic/$W^\infty_c$; and

2. $\langle y_n \mid n < \omega\rangle \in (\pi_c \circ j_{0,\infty})(\dot{A}_\infty)[h_\infty]$.

From condition 2 we see that $\langle y_n \mid n < \omega\rangle$ satisfies the Σ^1_2 statement ϕ, inside $W_c^\infty[h_\infty]$. By absoluteness ϕ is satisfied in V. This means that $\langle y_n \mid n < \omega\rangle$ belongs to the payoff set C, and is won by I, as required. This completes the argument. We proved:

Claim 4.10. *Suppose that there exists γ so that* I *wins $G^*_0(\gamma)$ in M. Then* I *wins $G_{\omega\cdot\omega}(C)$ in* V. $\dashv$

Before closing let us comment on our suppression throughout of the parameter ϱ. We worked during the construction with the map $\sigma_{\text{piv}-\infty}$, which exists in M and could thus be shifted via embeddings acting on M. This map takes two parameters: x, which was interpreted by the sequence $\langle y_n \mid n < \omega\rangle$ in our construction; and ϱ, which was suppressed.

To be precise we should add the following to the list (A)–(D) of objects constructed:

(F) A function $\vartheta_k \colon k \to M_k \| j_k(\delta_\infty) + 1$.

We should also insert the parameter ϑ_k in (D), replacing the occurrence of "$j_k(\sigma_{\text{piv}-\infty})[y_0, \dots, y_{k-1}]$" with "$j_k(\sigma_{\text{piv}-\infty})[\vartheta_k, y_0, \dots, y_{k-1}]$."

The functions ϑ_k should be constructed so that ϑ_{k+1} extends $j_k(\vartheta_k)$. This allows us at the end to set

$$\varrho = \bigcup_{k<\omega} j_{k,\infty}(\vartheta_k).$$

ϱ is then a function from ω into $M_\infty \| j_{0,\infty}(\delta_\infty) + 1$, and $\mathcal{U}_\infty$ is part of a play according to $j_{0,\infty}(\sigma_{\text{piv}-\infty})[\varrho, y_n \mid n < \omega]$.

Most importantly, we should (using standard book-keeping) construct the functions ϑ_k so that ϱ ends up being *onto*. This is necessary for our application of Lemma 2.8. It was Lemma 2.8 that gave us conditions 1 and 2 above.

4.3. II **wins.** Here we mirror the development of Section 4.2, just as case 2 of Section 3 mirrored case 1. Let $\dot{B} \in M$ name the set of sequences $\langle y_n \mid n < \omega\rangle \in \mathbb{R}^\omega$ in $M^{\text{col}(\omega,\delta_\infty)}$ which do **not** satisfy ϕ. Let $\mathcal{B}_\infty[y_n \mid n < \omega]$ be the associated auxiliary games, but with the roles of I **and** II **interchanged**.

We use $\dot{B}_\infty$ and $\mathcal{B}_\infty$ as our starting points here, instead of $\dot{A}_\infty$ and $\mathcal{A}_\infty$. Define names $\dot{B}_k$ and games $H^*_k(T)$ to parallel the names $\dot{A}_k$ and games $G^*_k(S)$ of Section 4.1, only switching the roles of I and II, and using $\mathcal{B}_\infty$ (which corresponds to the negation of ϕ) instead of $\mathcal{A}_\infty$. We very briefly outline these definitions.

The game $H^*_k(T)$ is played according to Diagram 18.

y_k, $b^\infty_k = \langle b^\infty_{k-\text{II}}, b^\infty_{k-\text{I}}\rangle$, and γ^* must form a valid extension of T. (See Definition 4.2, with $\mathcal{A}_\infty$ changed to $\mathcal{B}_\infty$.) $b^{k+1}_{n-\text{II}}$ and $b^{k+1}_{n-\text{I}}$ are auxiliary moves in $\mathcal{B}_{k+1}[T\frown, y_k, b^\infty_k, \gamma^*]$. Infinite runs of the game are won by I.

For $T \in M[g_k]$ set $T \in B_k$ iff T is a k-sequence (with $\mathcal{A}_\infty$ changed to $\mathcal{B}_\infty$) and II wins $H^*_k(T)$ in $M[g_k]$. This definition mirrors Definition 4.3.

I		$b^{\infty}_{k-\mathrm{I}}$	$y_k(0)$		$b^{k+1}_{0-\mathrm{I}}$			$b^{k+1}_{1-\mathrm{I}}$ $y_k(2)\dots$
II	$\gamma^*, b^{\infty}_{k-\mathrm{II}}$			$b^{k+1}_{0-\mathrm{II}}$		$y_k(1)$	$b^{k+1}_{1-\mathrm{II}}$	

DIAGRAM 18. The game $H^*_k(T)$, which mirrors $G^*_k(S)$ of Diagram 14.

It determines the names $\dot{B}_k$ and by extension the auxiliary games $\mathcal{B}_k$. The definition is by induction on γ, see Remark 4.4.

An argument which mirrors that of Section 4.2 gives:

CLAIM 4.11. *Suppose that there exists γ so that* II *wins $H^*_0(\gamma)$ in M. Then* II *wins $G_{\omega\cdot\omega}(C)$ in* V. $\dashv$

4.4. Otherwise. To prove that $G_{\omega\cdot\omega}$ is determined it is now enough to verify that the hypotheses of Claims 4.10 and 4.11 cannot both fail.

Suppose for contradiction that they do, i.e., assume that for every γ II wins $G^*_0(\gamma)$ in M and I wins $H^*_0(\gamma)$ in M. We intend to derive a contradiction. Our argument here is similar to the final argument in Section 3, where we constructed a real x which neither satisfied, nor failed to satisfy, the statement ϕ. Here we shall construct a sequence $\langle y_n \mid n<\omega\rangle \in \mathbb{R}^\omega$ which neither satisfies nor fails to satisfy ϕ. The reader may wish to compare our construction here with the final construction in Section 3.

Fix $g_\infty \in$ V which is $\mathrm{col}(\omega,\delta_\infty)$–generic/$M$. Replacing the generics g_k if needed, we may assume that each g_k belongs to $M[g_{k+1}]$, and that the sequence $\langle g_k \mid k<\omega\rangle$ belongs to $M[g_\infty]$.

Pick ordinals $\gamma_{\min} < \gamma_{\max}$, substantially larger than δ_∞, so that

$$M\|(\gamma_{\min}+\omega) \models \varphi[\vec{c},\gamma_{\min}] \iff$$
$$M\|(\gamma_{\max}+\omega) \models \varphi[\vec{c},\gamma_{\max}]$$

for any formula φ and any parameter $\vec{c} \in (M\|\delta_\infty+\omega)^{<\omega}$. These ordinals will serve as *indiscernibles*.

We work in $M[g_\infty]$ to construct $\langle y_n \mid n<\omega\rangle \in \mathbb{R}^\omega$; an infinite play $\vec{a}_\infty = \langle a^\infty_{n-\mathrm{I}}, a^\infty_{n-\mathrm{II}} \mid n<\omega\rangle$ of $\mathcal{A}_\infty[y_n \mid n<\omega]$; and an infinite play $\vec{b}_\infty = \langle b^\infty_{n-\mathrm{II}}, b^\infty_{n-\mathrm{I}} \mid n<\omega\rangle$ of $\mathcal{B}[y_n \mid n<\omega]$. We use the following notation:

$$S_k = \langle y_0,\dots,y_{k-1}, a^\infty_0,\dots,a^\infty_{k-1},\gamma_{\min}\rangle$$
$$S'_k = \langle y_0,\dots,y_{k-1}, a^\infty_0,\dots,a^\infty_{k-1},\gamma_{\max}\rangle.$$

(Note the switch from $\gamma_{\min}$ in S_k to $\gamma_{\max}$ in S'_k.) We use T_k and T'_k similarly, with $\vec{b}_\infty$ instead of $\vec{a}_\infty$.

We intend to maintain the following conditions:

1. (For $k\ge 1$.) $y_0,\dots,y_{k-1}$ belong to $M[g_k]$;
2. In $M[g_k]$ (in M if $k=0$) II wins $G^*_k(S_k)$; and
3. In $M[g_k]$ (in M if $k=0$) I wins $H^*_k(T_k)$.

We construct in mega-rounds. At the start of mega-round k we will have conditions 1–3 for k. Note that for $k = 0$ conditions 2 and 3 hold because of our initial case assumption in this subsection.

Let us begin mega-round k. Using the indisernibility of $\gamma_{\min}$ and $\gamma_{\max}$ conditions 2 and 3 tell us that II wins $G^*_k(S'_k)$ (note the switch to S'_k) and I wins $H^*_k(T'_k)$. Fix strategies σ^*_k and τ^*_k in $M[g_k]$ (in M if $k = 0$) witnessing this. We play the games $G^*_k(S'_k)$ and $H^*_k(T'_k)$. Both games start with an ordinal move, γ^*. In both games we play $\gamma^* = \gamma_{\min}$. Note that this is a legal move since $\gamma_{\min} < \gamma_{\max}$. We continue the games as follows:

- $\sigma_{\text{gen}-\infty}[y_0, \dots, y_{k-1}]$ plays $a^\infty_{k-\mathrm{I}}$ in $G^*_k(S'_k)$, and σ^*_k plays $a^\infty_{k-\mathrm{II}}$.
- Similarly, $\tau_{\text{gen}-\infty}[y_0, \dots, y_{k-1}]$ plays $b^\infty_{k-\mathrm{II}}$ in $H^*_k(T'_k)$, and τ^*_k plays $b^\infty_{k-\mathrm{I}}$.

This completes the first round. We pass to the remaining ω rounds which involve auxiliary moves from $\mathcal{A}_{k+1}$ and $\mathcal{B}_{k+1}$.

- σ^*_k, playing for II in $G^*_k(S'_k)$, produces $y_k(n)$ for odd n, and $a^{k+1}_{n-\mathrm{II}}$ for all n.
- $\sigma_{\text{gen}-k+1}[S'_k\frown, y_k, a^\infty_k, \gamma_{\min}]$ produces $a^{k+1}_{n-\mathrm{I}}$ for all n.
- τ^*_k, playing for I in $H^*_k(T'_k)$, produces $y_k(n)$ for even n and $b^{k+1}_{n-\mathrm{I}}$ for all n.
- $\tau_{\text{gen}-k+1}[T'_k\frown, y_k, b^\infty_k, \gamma_{\min}]$ produces $b^{k+1}_{n-\mathrm{II}}$ for all n.

$\sigma_{\text{gen}-\infty}$, $\sigma_{\text{gen}-k+1}$, $\tau_{\text{gen}-\infty}$, and $\tau_{\text{gen}-k+1}$ are the generic strategies defined in Section 2.1. As usual continuity is important; for example in the last item we are using $\tau_{\text{gen}-k+1}[T'_k\frown, y_k, b^\infty_k, \gamma_{\min}]$ at a stage where we only know $y_k \restriction n+1$.

The reader should consult Diagrams 14 and 18 to verify that the above strategies between them cover all moves in the games $G^*_k(S'_k)$ and $H^*_k(T'_k)$. (Well, except for the first move $\gamma^* = \gamma_{\min}$ which we decided on ourselves.) The conditions above therefore complete the construction in mega-round k. The reader may consult the final stages of Section 3 for a simpler example of a similar argument.

In mega-round k we used σ^*_k and τ^*_k, which exist in $M[g_k]$ (in M if $k = 0$); and the maps $\sigma_{\text{gen}-k+1}$ and $\tau_{\text{gen}-k+1}$, which exist in $M[g_{k+1}]$. The real y_k produced in mega-round k therefore belongs to $M[g_{k+1}]$.

Our use of the generic strategy $\sigma_{\text{gen}-k+1}[S'_k\frown, y_k, a^\infty_k, \gamma_{\min}]$ guarantees that $S_{k+1} = S'_k\frown, y_k, a^\infty_k, \gamma_{\min}$ does not belong to $\dot{A}_{k+1}[g_{k+1}]$. Since y_k and hence S_{k+1} belong to $M[g_{k+1}]$ we conclude (see Definition 4.3) that I does not win $G^*_{k+1}(S_{k+1})$. Now $G^*_{k+1}(S_{k+1})$ is an open game, hence determined. Thus II must win $G^*_{k+1}(S_{k+1})$. This secures condition 2 for $k + 1$.

Similarly our use of $\tau_{\text{gen}-k+1}[T'_k\frown, y_k, b^\infty_k, \gamma_{\min}]$ guarantees that $T_{k+1} = T'_k\frown, y_k, b^\infty_k, \gamma_{\min}$ does not belong to $\dot{B}_{k+1}[g_{k+1}]$, and this secures condition 3 for $k + 1$. We are now in a position to start mega-round $k + 1$.

Once completed the construction leaves us with $\langle y_n \mid n < \omega\rangle \in \mathbb{R}^\omega$ and infinite plays $\vec{a}_\infty$ of $\mathcal{A}_\infty[y_n \mid n < \omega]$ and $\vec{b}_\infty$ of $\mathcal{B}_\infty[y_n \mid n < \omega]$. Note that everything we did took place in $M[g_\infty]$. (Here we are using the fact that $\langle g_k \mid k < \omega\rangle \in M[g_\infty]$.) These sequences therefore belong to $M[g_\infty]$.

Our use of $\sigma_{\text{gen}-\infty}[y_n \mid n < \omega]$ during the construction ensures that $\langle y_n \mid n < \omega\rangle$ does not belong to $\dot{A}_\infty[g_\infty]$. Since $\langle y_n \mid n < \omega\rangle$ belongs to $M[g_\infty]$ we conclude that $\langle y_n \mid n < \omega\rangle$ fails to satisfy ϕ, our original Σ^1_2 statement, inside $M[g_\infty]$. Similarly our use of $\tau_{\text{gen}-\infty}[y_n \mid n < \omega]$ ensures that $\langle y_n \mid n < \omega\rangle$ fails to **not** satisfy ϕ. This is a contradiction.

4.5. Summary. Claim 4.10, Claim 4.11, and the construction of Section 4.4 together give the following theorem:

THEOREM 4.12. *Suppose that there exist M and an increasing sequence $\langle \delta_1, \delta_2, \dots, \delta_\omega\rangle$ in M so that:*
- *M is a class model;*
- *M is iterable;*
- *Each δ_ξ, $1 \leq \xi \leq \omega$, is a Woodin cardinal of M; and*
- *$M\| \delta_\omega + 1$ is countable in V.*

Then all games $G_{\omega\cdot\omega}(C)$ where C is Σ^1_2 are determined. ⊣

§5. Pivots revisited.

In this section we return to our definition of auxiliary moves, and make some adjustments. These adjustments will be needed later on, in Section 6. We begin in Section 5.1 with a minor modification to the games $\mathcal{A}[x]$. We describe the modification and its effect on the notions of generic runs and pivots. Then in Section 5.2 we handle the more serious adjustment. We describe a game $\mathcal{A}_{\text{mix}}$, a variant of $\mathcal{A}_{\text{piv}}$, and use this game to define the notion of a *mixed* pivot. Mixed pivots will be used in the proof of determinacy of continuously coded games.

5.1. Modified auxiliary moves. Work as in Section 2 with a model M which has a Woodin cardinal δ. Fix $\dot{A} \in M$, a name for a subset of $(M\| \delta)^\omega \times \omega^\omega$ in $M^{\text{col}(\omega,\delta)}$. Note already here the change from Section 2, where we had a name for a set of reals, i.e., a subset of ω^ω.

Work with $x \in \mathbb{R}$. We define an auxiliary game $\mathcal{A}[x]$ displayed in Diagram 19. We use a_n to denote $\langle l_n, u_n, p_n, w_n\rangle$, the sequence of moves in round n, and let $\vec{a} = \langle a_n \mid n < \omega\rangle$. Moves in $\mathcal{A}[x]$ are elements of $M\| \delta$, so that $\vec{a}$ belongs to $(M\| \delta)^\omega$. A run $\vec{a}$ of $\mathcal{A}[x]$ is arranged so that I tries to witness that $\langle \vec{a}, x\rangle \in \dot{A}[h]$ for some generic h, while II tries to witness the opposite. Note the change from Section 2, where we dealt with "x" rather than $\langle \vec{a}, x\rangle$.

I	l_0, u_0, p_0		l_1, u_1, p_1		$\dots$	
II		w_0		w_1		$\dots$

DIAGRAM 19. Outline of $\mathcal{A}[x]$.

Moves in $\mathcal{A}[x]$ are elements of M, and each rule should be read relativized to M. In round n I plays:
- $l = l_n$, a number smaller than n, or l_n = "new";

- a *type* u_n which codes $\mathcal{X}_n$, a set of **pairs** of $M^{\mathrm{col}(\omega,\delta)}$–names; and
- p_n, a condition in $\mathrm{col}(\omega,\delta)$.

II plays a *type* w_n which codes $\mathcal{F}_n, \mathcal{D}_n$ where:

- $\mathcal{F}_n$ is a function from $\mathcal{X}_n$ into the ordinals; and
- $\mathcal{D}_n$ is a function from $\mathcal{X}_n$ into $\{$dense sets in $\mathrm{col}(\omega,\delta)\}$.

We remind the reader of Remark 2.1. Already in Section 2 the moves $\mathcal{X}_n$ and $\mathcal{F}_n, \mathcal{D}_n$ were coded by types. This part is not new. We didn't say much about the coding in Section 2, referring the reader to [12, Chapter 1] instead. We adopt the same attitude here. Let us only note that the types u_n and w_n are essentially elements of $M\|\delta$. This is important. It means that $a_n = \langle l_n, u_n, p_n, w_n\rangle$ is an element of $M\|\delta$, so that $\vec{a}$ is an element of $(M\|\delta)^\omega$.

If $l_n =$ "new" we make no requirements on I. Otherwise we demand that p_n extends p_l, that $\mathcal{X}_n \subset \mathcal{X}_l$, and that for every pair $\langle \dot{a}, \dot{x}\rangle \in \mathcal{X}_n$:

1. p_n forces "$\langle \dot{a}, \dot{x}\rangle \in \dot{A}$";
2. p_n forces "$\dot{a}(0) = \check{a_0}$,"....,"$\dot{a}(l) = \check{a_l}$";
3. p_n forces "$\dot{x}(0) = \check{x_0}$,"....,"$\dot{x}(l) = \check{x_l}$"; and
4. p_n belongs to $\mathcal{D}_l(\dot{a}, \dot{x})$.

We make the following demand on II when $l_n \neq$ "new":

5. $\mathcal{F}_n(\dot{a}, \dot{x}) < \mathcal{F}_l(\dot{a}, \dot{x})$ for every pair $\langle \dot{a}, \dot{x}\rangle \in \mathcal{X}_n$.

Remark 5.1. Note the addition of condition 2, stating that $\dot{a}$ must name the actual run of $\mathcal{A}[x]$, $\vec{a}$. This is the condition which distinguishes our game here from the game in Section 2. Other than this the rules are essentially the same.

Condition 2 makes sense; $\vec{a}$ is an element of $(M\|\delta)^\omega$ and may potentially be named by $\dot{a}$. Observe that condition 2 in round n only involves $a_0, \dots, a_l$, which are already known. It poses no greater hardship to the players than condition 3. The arguments (not) presented in Section 2 thus go through essentially unmodified. The curious reader can find these arguments in [12, Chapter 1]. Let us briefly go over the results of these arguments.

Fix some g which is $\mathrm{col}(\omega,\delta)$–generic/$M$. As in Section 2.1 we let $\sigma_{\mathrm{gen}}[x]$ be the strategy which plays in each round the first, with respect to g, legal move. The map $x \mapsto \sigma_{\mathrm{gen}}[x]$ is continuous, given by some $\sigma_{\mathrm{gen}} = (s \mapsto \sigma_{\mathrm{gen}}[s])$ which belongs to $M[g]$. We have:

Lemma 5.2. *Suppose that $\vec{a}$ is an infinite run of $\mathcal{A}[x]$ played according to $\sigma_{\mathrm{gen}}[x]$. Then $\langle \vec{a}, x\rangle \notin \dot{A}[g]$. (This is only useful if both $\vec{a}$ and x belong to $M[g]$.)*

This should be compared with Lemma 2.4. Where now we have $\langle \vec{a}, x\rangle \notin \dot{A}[g]$, Lemma 2.4 had $x \notin \dot{A}[g]$.

Definition 2.7 can be adapted to our new game by changing condition 3 to:

3. For every cofinal odd branch b of $\mathcal{T}$ there exists some h so that:
   (a) h is $\mathrm{col}(\omega, j_b(\delta))$–generic/$M_b$; and

(b) $\langle \vec{a}, x\rangle \in j_b(\dot{A})[h]$.

(Note the change from x to $\langle \vec{a}, x\rangle$ in condition 3b.)

As in Section 2 there are strategies $\sigma_{\text{piv}}[\varrho, x]$ which are guaranteed to produce pivots. But when proving determinacy of continuously coded games this is not enough. We shall need stronger strategies than those given by σ_{piv}, capable of handling what we call *mixing*.

5.2. Mixed pivots. Instead of working with a single name $\dot{A}$ as before, we work here with a collection of names. Fix some ordinal ν. Fix a map $\dot{A} = (\gamma \mapsto \dot{A}[\gamma])$ assigning to each ordinal $\gamma < \nu$ a name $\dot{A}[\gamma]$ for a subset of $(M\|\,\delta)^\omega \times \omega^\omega$ in $M^{\text{col}(\omega,\delta)}$. We assume that the map $\dot{A}$ belongs to M.

We shall henceforth suppress mention of ν. When we say "for each γ" we mean for each $\gamma < \nu$. We generally think of ν as some very large ordinal. Indeed, if it weren't for our desire to work with sets rather than classes we would take $\nu = \text{ON}$.

For each γ we have the map $x \mapsto \mathcal{A}[\gamma, x]$ of Section 5.1, associated to the name $\dot{A}[\gamma]$. We regard it now as a map $\gamma, x \mapsto \mathcal{A}[\gamma, x]$. This map, which belongs to M, is continuous in x.

Working with reference to the map $\dot{A}$, we define for each $x \in \mathbb{R}$ the game $\mathcal{A}_{\text{mix}}[x]$ played according to Diagram 20. As usual the association is continuous, given by a map $\mathcal{A}_{\text{mix}} = (s \mapsto \mathcal{A}_{\text{mix}}[s])$ which belongs to M.

I		$f(n), \mathcal{T}\restriction f(n)+1, \gamma_n$	l_n, p_n, u_n		$\cdots$
II	$\cdots\cdots$			$E_{f(n)}, E_{f(n)+1}, w_n$	$\cdots$

Diagram 20. Round n of the game $\mathcal{A}_{\text{mix}}[x]$.

At the start of round n we have a number $e(n)$, an iteration tree $\mathcal{T}\restriction e(n)+1$ ending with the model $M_{e(n)}$, and a position $P_n = \langle a_0, \dots, a_{n-1}\rangle$ in $M_{e(n)}$. For $n = 0$ we set $e(0) = 0$, $M_0 = M$, $P_0 = \emptyset$.

The time line of round n is presented in Diagrams 20 and 21. At the start of round n player I:

- Plays some $f(n) \geq e(n)$;
- Extends $\mathcal{T}\restriction e(n)+1$ to $\mathcal{T}\restriction f(n)+1$;
- Plays an ordinal γ_n so that P_n is a legal position in $j_{0,f(n)}(\mathcal{A})[\gamma_n, x]$.

The rest of the round follows the usual rules of $\mathcal{A}_{\text{piv}}$ (see Section 2.2): I plays a move in $j_{0,f(n)}(\mathcal{A})[\gamma_n, x]$ following the position P_n; II shifts this move to the model $M_{f(n)+2}$—this is illustrated by the squiggly arrow in Diagram 21—and replies there. We let $a_n = j_{f(n),f(n)+2}(l_n, u_n, p_n)\frown w_n$. Note the shifting of l_n, u_n, p_n from $M_{f(n)}$ to $M_{f(n)+2}$. We let $P_{n+1} = P_n\frown a_n = \langle a_0, \dots, a_{n-1}, a_n\rangle$, let $e(n+1) = f(n)+2$, and proceed to the next round.

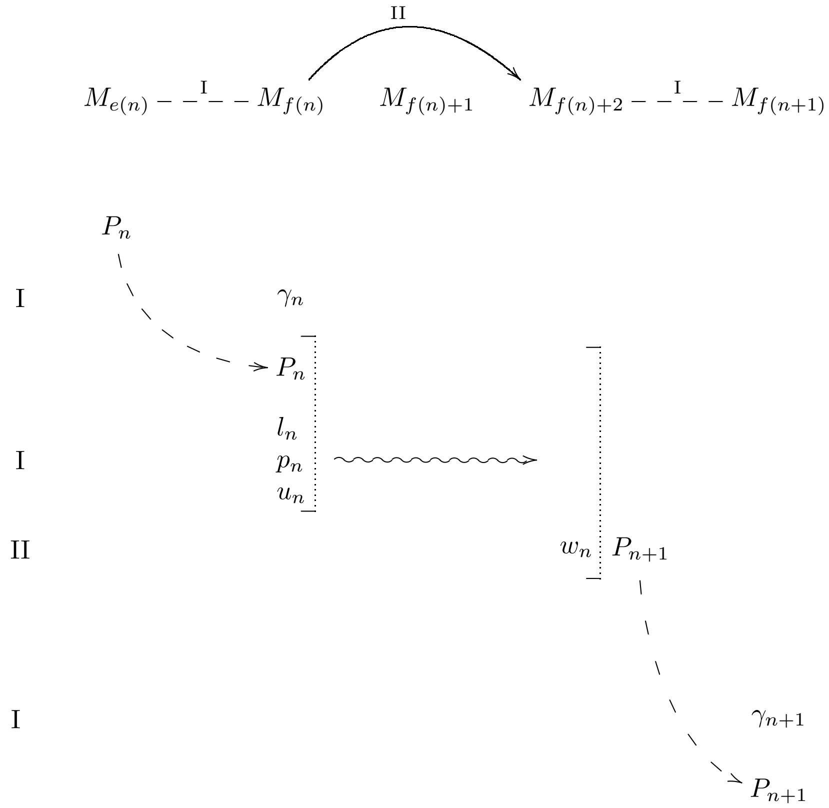

DIAGRAM 21. Round n of $\mathcal{A}_{\text{mix}}[x]$ and the beginning of round $n+1$.

REMARK 5.3. Suppose I fixes some $\gamma_0 \in M$ and always plays $f(n) = e(n)$ (so that no extension of $\mathcal{T} \restriction e(n)+1$ is needed) and $\gamma_n = j_{0,f(n)}(\gamma_0)$. Then the game degenerates into $\mathcal{A}_{\text{piv}}$ associated to the real x and the name $\dot{A}[\gamma_0]$.

$\mathcal{A}_{\text{mix}}$ is thus a variant of $\mathcal{A}_{\text{piv}}$ which gives some extra control to player I: I *may* play $f(n) > e(n)$, inserting her own interval of models into the tree $\mathcal{T}$, and I *may* pick a new ordinal γ_n to work with.

In line with Remark 5.3 we make the following definition:

DEFINITION 5.4. Round n is said to **contain mixing** if $f(n) > e(n)$; or (when $n > 0$) $f(n) = e(n)$ but $\gamma_n \neq j_{f(n-1),e(n)}(\gamma_{n-1})$.

There are some technical restrictions on the moves by the two players, not explained above. (For example the critical point of $j_{f(n),f(n)+2}$ must be large enough that $a_0, \ldots, a_{n-1}$ are not moved. This is why we take $P_{n+1} = P_n$—$, a_n$, and not $P_{n+1} = j_{f(n),f(n)+2}(P_n)$—$, a_n$.) The reader may find the exact rules

in [12, Chapter 1]. Here we only comment on how these rules affect the branch structure of $\mathcal{T}$.

Suppose $\vec{a} = \langle a_n \mid n < \omega \rangle$ and $\mathcal{T}$ are given by a run of $\mathcal{A}_{\text{mix}}[x]$. A cofinal branch of $\mathcal{T}$ is **even** if it contains arbitrarily high nodes from $\{f(n) \mid n < \omega\}$.[5] Otherwise the branch is **odd**. Note that a mixed $\mathcal{T}$ may have many cofinal even branches; this has to do with *not* requiring $e(n)\ T\ f(n)$ in the rules of the game. How about the odd branches? The predecessors of nodes $\{f(n)+1, f(n)+2 \mid n < \omega\}$ are determined by the moves $\{l_n \mid n < \omega\}$ in the manner of Section 2.2 (see the rules following Diagram 9). There are extra rules now on player I limiting the way she may choose predecessors for nodes in $\bigcup_{n<\omega}(e(n), f(n)]$. The main point of these rules is to make sure that the following condition holds:

(o) Suppose b is a cofinal odd branch of $\mathcal{T}$. Then there is a sequence $\langle n_k \mid k < \omega \rangle$ so that:
  - $l_{n_0} =$ "new";
  - $l_{n_k} = n_{k-1}$ for $k > 0$; and
  - $\langle f(n_k) + 1 \mid k < \omega \rangle$ is a tail-end of b.

Thus the tree structure on the odd models is essentially the same structure we had in Section 2.2.

We use $n(b)$ to denote the n_0 given by condition (o). We use $f(b)$ to denote $f(n_0)$. Note that $f(b)$ is the largest node in b which belongs to $\{f(n) \mid n < \omega\}$. We think of $f(b)$ as the **even root** of the odd branch b (though it needn't be an even number, see footnote 5).

DEFINITION 5.5. A **mixed pivot** for x is a run of $\mathcal{A}_{\text{mix}}[x]$ (given by $\vec{a}$, $\vec{f}$, $\vec{\gamma}$, and $\mathcal{T}$ say) with the property that for every cofinal odd branch b of $\mathcal{T}$ there exists some h so that:

1. h is $\text{col}(\omega, j_b(\delta))$–generic/$M_b$; and
2. $\langle \vec{a}, x \rangle \in j_b(\dot{A})[\gamma_b][h]$, where $\gamma_b = j_{f(b),b}(\gamma_{n(b)})$.

The reader should compare the conditions of Definition 5.5 to condition 3 listed immediately following Lemma 5.2. The difference is that here we work not with a single name but with a collection of names. So we have to say which γ to use in condition 2 of Definition 5.5. The γ we take is the one which corresponds to I's move at the even root of b.

Recall that the main point in Section 2.2 was the existence of strategies $\sigma_{\text{piv}}[\varrho, x]$ which produced pivots. Similar strategies exist in our current situation. For each real x and each map $\varrho\colon \omega \to M \| \delta + 1$ there is a strategy $\sigma_{\text{mix}}[\varrho, x]$, playing for II in $\mathcal{A}_{\text{mix}}[x]$. The association is continuous, given by a map $\sigma_{\text{mix}} = (\vartheta, s \mapsto \sigma_{\text{mix}}[\vartheta, s])$. This map belongs to M. Most importantly we have:

[5] If there is no mixing in $\mathcal{T}$ this gives precisely the branch $0, 2, 4, \ldots$, and hence the terminology. The terminology may be slightly confusing since in general the numbers $f(n)$ needn't actually be even. One can avoid the confusion by adding the requirement "$f(n)$ is even" on I, and canceling the extra hardship by letting player I "pad" in her iteration trees.

LEMMA 5.6. *Suppose* $\varrho\colon \omega \to M\|(\delta+1)$ *is onto. Then all runs according to* $\sigma_{\text{mix}}[\varrho, x]$ *are mixed pivots for* x.

In the future we shall use Lemma 5.6 as before we had used Lemma 2.8. Note that Lemma 2.8 is really a special case of our current Lemma 5.6. This follows from Remark 5.3. We refer the reader to [12, Chapter 1] for more details on the construction of σ_{mix}. The construction involves only minor modifications to the construction of σ_{piv}.

§6. Games of continuously coded length. Fix a continuous function $\nu\colon \mathbb{R} \to \mathbb{N}$. Fix $C \subset \mathbb{R}^{<\omega_1}$ which is Σ^1_2 in the codes (see Section 1.1). We work to prove, or at least sketch a proof of, the determinacy of $G_{\text{cont}-\nu}(C)$. Our proof will build on the constructions presented in Sections 3 and 4, and will use the notions of Section 5.

Let us say that an extender E **overlaps** δ if $\text{dom}(E)$ is smaller than δ, and the ultrapower embedding by E sends $\text{dom}(E)$ above δ.

Fix M, $\delta < \delta_\infty$ in M, and an extender $E \in M$ which overlaps δ, so that:

1. M is a class model;
2. M is iterable;
3. δ and δ_∞ are Woodin cardinals of M;
4. $M\|\,\delta_\infty + 1$ is countable in V; and
5. E is strong enough that $M\|\,\delta + 1 \subset \text{Ult}(M, E)$.

The existence of such a model is our large cardinal assumption.

Let N denote $\text{Ult}(M, E)$, and let $\pi\colon M \to N$ denote the ultrapower embedding. Let δ' denote $\pi(\delta)$. For expository simplicity fix g which is $\text{col}(\omega, \delta)$–generic/$M$, and g_∞ which is $\text{col}(\omega, \delta_\infty)$–generic/$M$.

CLAIM 6.1. g *is also* $\text{col}(\omega, \delta)$*–generic over* $N = \text{Ult}(M, E)$*. If* x *is a* real *which belongs to* $M[g]$*, then* x *belongs also to* $N[g]$.

PROOF. The proof is immediate. We only note that condition 5 is crucial for the second part. ⊣

Remember that in Section 4 we needed an increasing sequence of Woodin cardinals. The reason was explained in Remark 4.9. Roughly speaking we wanted $y_0, \ldots, y_{k-1}$ to belong to a small generic extension relative to the Woodin cardinal used in mega-round k. Here we have the single Woodin cardinal δ, but Claim 6.1 tells us that we can use E to manufacture a "next" Woodin cardinal δ' *and* have the current real x belong to a small generic extension relative to δ'.

6.1. Names. Recall that C, the payoff set, is assumed to be Σ^1_2 in the codes. Fix a Σ^1_2 set $A \subset \mathbb{R} \times \mathbb{R}$ so that $\langle y_\xi \mid \xi \leq \alpha \rangle \in C$ iff $\langle \ulcorner y_\xi \mid \xi < \alpha \urcorner, y_\alpha \rangle \in A$. Fix a Σ^1_2 statement ϕ so that $\langle x, y \rangle \in A$ iff $\langle x, y \rangle$ satisfies ϕ. Recall that $\nu\colon \mathbb{R} \to \mathbb{N}$ is assumed to be continuous. Fix a function $\bar\nu\colon \omega^{<\omega} \to \mathbb{N}$ so that $\nu(y) = n$ iff $\exists i\, \bar\nu(y \restriction i) = n$. Without loss of generality $\bar\nu$, which is essentially a

real number, belongs to M. (We can always absorb $\bar{\nu}$ into a generic extension of an iterate of M of size much less than δ.)

Let $\dot{A}_\infty$ be the canonical name for the set of pairs $\langle x, y\rangle \in \mathbb{R}^2$ in $M^{\mathrm{col}(\omega,\delta_\infty)}$ which satisfy the Σ^1_2 statement ϕ. We have the associated auxiliary games $\mathcal{A}_\infty[x, y]$, of the kind presented in Section 2, where I tries to witness $\langle x, y\rangle \in \dot{A}[h_\infty]$ for some generic h_∞ and II tries to witness the opposite.

For each ordinal γ we define a name $\dot{A}[\gamma]$ for a subset of $(M\|\,\delta)^\omega \times \omega^\omega$ in $M^{\mathrm{col}(\omega,\delta)}$. Following notation similar to that of Section 4.1 we write $\langle \vec{a}, x, \gamma\rangle \in \dot{A}[h]$ to mean that $\langle \vec{a}, x\rangle \in \dot{A}[\gamma][h]$. We use $\mathcal{A}[\gamma, x]$ to denote the auxiliary game of Section 5, associated to the name $\dot{A}[\gamma]$ and the real x. A run $\vec{a}$ of this game is an attempt by I to witness that $\langle \vec{a}, x\rangle \in \dot{A}[\gamma][h]$—in other words that $\langle \vec{a}, x, \gamma\rangle \in \dot{A}[h]$—for some h, and an attempt by II to witness the opposite.

Definition 6.2. A **code** is any real x which has the form $\ulcorner y_\xi \mid \xi < \alpha\urcorner$ for some α and some position $\langle y_\xi \mid \xi < \alpha\rangle$ in $G_{\mathrm{cont}-\nu}$.

Following the ideas of Section 4.1 we work in $M[g]$ to define open games, denoted here $G^*(\vec{a}, x, \gamma)$. We then set:

Definition 6.3. For $\langle \vec{a}, x\rangle \in (M\|\,\delta)^\omega \times \omega^\omega$ in $M[g]$ put $\langle \vec{a}, x\rangle \in A[\gamma]$ iff x is a code and I wins $G^*(\vec{a}, x, \gamma)$ in $M[g]$. Let $\dot{A}[\gamma]$ be the canonical name for $A[\gamma]$.

As in Section 4 the definition is by induction on γ. The game $G^*(\vec{a}, x, \gamma)$, which we define shortly, will make reference to the names $\dot{A}[\gamma^*]$—indeed to the map $\gamma^* \mapsto \dot{A}[\gamma^*]$—but only for $\gamma^* < \gamma$.

Fix an ordinal γ, a code $x = \ulcorner y_\xi \mid \xi < \alpha\urcorner$, and a sequence $\vec{a} \in (M\|\,\delta)^\omega$. Suppose that $\vec{a}$ and x belong to $M[g]$. The game $G(\vec{a}, x, \gamma)$ is played in two parts, parts (F) and (M) described below. (F) stands for "finishing" and (M) stands for "main." In part (M) we use $\mathcal{A}$ to denote the map $\gamma^*, x^* \mapsto \mathcal{A}[\gamma^*, x^*]$, which we assume known for $\gamma^* < \gamma$. The map is continuous in x^* and belongs to M. Recall that $N = \mathrm{Ult}(M, E)$ and $\pi\colon M \to N$ is the ultrapower embedding. We use $\mathcal{A}'$ to denote $\pi(\mathcal{A})$. Similarly we use δ' to denote $\pi(\delta)$ and γ' to denote $\pi(\gamma)$.

(F) I and II collaborate as usual playing a real $y_\alpha = \langle y_\alpha(i) \mid i < \omega\rangle$. In addition they play auxiliary moves subject to the rules of $\mathcal{A}_\infty[x, y_\alpha]$.

The players stay in part (F) until, if ever, $i < \omega$ is reached so that $\bar{\nu}(y_\alpha\restriction i)$ is defined. If and when this happens we set $n_\alpha = \bar{\nu}(y_\alpha\restriction i)$. If there exists $\xi < \alpha$ so that $n_\alpha = \nu(y_\xi)$, the players simply continue with part (F). Otherwise they set $a' = \pi(\vec{a}\restriction n_\alpha)$ and pass to part (M):

(M) 1. I plays γ^* so that $\gamma^* < \gamma'$ and a' is a legal position in $\mathcal{A}'[\gamma^*, x]$.
2. The players collaborate to form the real y_α, continuing from the point they left in part (F).

We set $x^* = \ulcorner\langle y_\xi \mid \xi < \alpha\rangle\frown y_\alpha\urcorner$. x^* is obtained continuously as y_α is played out. Regardless of the end value of y_α we know by Property

1.3 that x and x^* agree to n_α. Using the continuity of $\mathcal{A}'$ and rule (M1) we see that a' is a legal position in $\mathcal{A}'[\gamma^*, x^*]$, again regardless of the end value of y_α.

3. While forming y_α the players play auxiliary moves subject to the rules of $\mathcal{A}'[\gamma^*, x^*]$, starting from the position a'.

If a player cannot follow these rules she loses. Infinite runs are won by II.

Remark 6.4. As a whole, the game $G^*(\vec{a}, x, \gamma)$ is defined inside $M[g]$. But part (M) can be defined in the smaller model $N[g]$. (The parameters needed to phrase part (M) are x, which is used in defining the continuous $y_\alpha \mapsto x^*$, $\mathcal{A}'$, γ', and a'. The last three parameters belong to N, and x belongs to $N[g]$ by Claim 6.1.)

This completes the inductive definition of $G^*(\vec{a}, x, \gamma)$, and with it the inductive definition of the names $\dot{A}[\gamma]$. We make the following notes on motivation:

$G^*(\vec{a}, x, \gamma)$ consists of two separate parts. So long as it seems that α is the last round of the long game $G_{\mathrm{cont}-\nu}(C)$—so long as $\nu(y_\alpha)$ is not defined or defined and equal to a previous n_ξ—the players follow the "finishing" part. What they do in this part is aim for the Σ^1_2 payoff set. I tries to witness that $\langle x, y_\alpha \rangle$ satisfies the Σ^1_2 statement ϕ, while II tries to witness the opposite.

Once (if ever) it becomes clear that α is not the last round in the game, the players pass to the "main" part. What they do is pass to the ultrapower $N = \mathrm{Ult}(M, E)$ where they have the next Woodin cardinal $\pi(\delta)$. They play auxiliary moves in the vicinity of $\pi(\delta)$. We use $\vec{a}^*$ to denote these auxiliary moves. Rule (M3) is such that $\vec{a}^*$ must extend $a' = \pi(\vec{a}\restriction n_\alpha)$.

Note that I's goal in $\vec{a}^*$ is to witness that $\langle \vec{a}^*, x^*, \gamma^* \rangle \in \pi(\dot{A})[h^*]$ for some h^* which is generic over N for the collapse of $\pi(\delta)$. II's goal is to witness the opposite. We draw the reader's attention to the similarity with Section 4. Here too we have a process of *perpetuation*. Membership in $\dot{A}[\gamma][h]$ allows I to aim for membership in a shift of $\pi(\dot{A})[\gamma^*]$. But here we have an additional ingredient. The witness $\vec{a}^*$ agrees with the shift of the witness $\vec{a}$ up to n_α. Using Claim 1.2 this will allow us to argue that the witnesses converge at limit stages.

6.2. I **wins.** Suppose that there exists some γ so that in M I wins the open game $G(\emptyset, \ulcorner\emptyset\urcorner, \gamma)$. We claim that in this case I wins the long game $G_{\mathrm{cont}-\nu}(C)$ in V.

Fix an imaginary opponent playing for II in $G_{\mathrm{cont}-\nu}(C)$. Working against the imaginary opponent we construct:

(A) Reals $y_\xi \in \mathbb{R}$. We set $x_\alpha = \ulcorner y_\xi \mid \xi < \alpha \urcorner$;
(B) Iterates M_α of M, with embeddings $\tau_{0,\alpha} \colon M \to M_\alpha$;
(C) Mixed pivots $\vec{a}_\alpha, \vec{f}_\alpha, \vec{\gamma}_\alpha, \mathcal{T}_\alpha$ for x_α over the model M_α, played according to $\tau_{0,\alpha}(\sigma_{\mathrm{mix}})[x_\alpha]$; and
(D) Sequences $\vec{\eta}_\alpha = \{\eta^\alpha_i\}_{i<\omega}$ witnessing that $\mathcal{T}_\alpha$ is continuously illfounded on its "even nodes," namely on nodes in $\{f_\alpha(n) \mid n < \omega\}$.

We use $\mathfrak{P}_\alpha$ to denote the mixed pivot of (C). In (D) we mean that $j^\alpha_{k,l}(\eta^\alpha_k) > \eta^\alpha_l$ whenever k, l both belong to $\{f_\alpha(n) \mid n < \omega\}$ and $k\ T_\alpha\ l$. ($j^\alpha_{*,*}$ here are the iteration embeddings forming part of the tree $\mathcal{T}_\alpha$.) The existence of a sequence $\vec{\eta}_\alpha$ of this kind implies that all the cofinal even branches of $\mathcal{T}_\alpha$ lead to illfounded direct limits, forcing the iteration strategy to pick an odd branch.

The construction of the objects (A)–(D) is similar to the previous constructions in Section 4.2 and in case 1 of Section 3. We shall not present it in great detail. Instead we concentrate on the two points which are new. We explain how to carry the construction through limits, and how and why *mixed* pivots appear in the construction.

Let us first consider the matter of limit stages. Fix a limit ordinal λ, and suppose that all objects up to λ were constructed. This includes the models M_ξ and reals y_ξ for $\xi < \lambda$. Let M_λ be the direct limit of the models M_ξ, $\xi < \lambda$. Let $x_\lambda = \ulcorner y_\xi \mid \xi < \lambda \urcorner$. Our construction below λ will satisfy the following *agreement* condition, which traces to the inclusion of a' in rule (M3) above.

(i) $\mathfrak{P}_{\alpha+1}$ agrees with the shifted image of $\mathfrak{P}_\alpha$ up to n_α, and similarly for the sequence $\vec{\eta}_{\alpha+1}$. To be more precise:

$$\begin{aligned}
\vec{a}_{\alpha+1}\restriction n_\alpha &= \tau_{\alpha,\alpha+1}(\vec{a}_\alpha\restriction n_\alpha);\\
\vec{f}_{\alpha+1}\restriction n_\alpha &= \vec{f}_\alpha\restriction n_\alpha;\\
\vec{\gamma}_{\alpha+1}\restriction n_\alpha &= \tau_{\alpha,\alpha+1}(\vec{\gamma}_\alpha\restriction n_\alpha);\\
\mathcal{T}_{\alpha+1}\restriction e_\alpha(n_\alpha)+1 &= \tau_{\alpha,\alpha+1}(\mathcal{T}_\alpha\restriction e_\alpha(n_\alpha)+1);\ \text{ and}\\
\vec{\eta}_{\alpha+1}\restriction e_\alpha(n_\alpha)+1 &= \tau_{\alpha,\alpha+1}(\vec{\eta}_\alpha\restriction e_\alpha(n_\alpha)+1).
\end{aligned}$$

($e_\alpha(n_\alpha)$ is 0 if $n_\alpha = 0$, and $f(n_\alpha - 1) + 2$ otherwise. See Section 5.2.)

It is this agreement condition that carries us through the limit. By Claim 1.2 $n_\alpha \to \infty$ as $\alpha \to \lambda$. This, (i), and our pending definition at limit stages imply that the mixed pivots $\tau_{\alpha,\lambda}(\mathfrak{P}_\alpha)$ *converge* as $\alpha \to \lambda$. We let $\mathfrak{P}_\lambda$ be their limit. By Remark 1.4 the reals x_α converge to x_λ as $\alpha \to \lambda$. Each $\tau_{\alpha,\lambda}(\mathfrak{P}_\alpha)$ is a play according to $\tau_{0,\lambda}(\sigma_{\text{mix}})[x_\alpha]$, because of (C). The plays $\tau_{\alpha,\lambda}(\mathfrak{P}_\alpha)$ must thus converge to a play according to $\tau_{0,\lambda}(\sigma_{\text{mix}})[x_\lambda]$. In other words $\mathfrak{P}_\lambda$ is a play according to $\tau_{0,\lambda}(\sigma_{\text{mix}})[x_\lambda]$. In particular $\mathfrak{P}_\lambda$ is a mixed pivot for x_λ over M_λ.

A similar limit construction allows us to define $\vec{\eta}_\lambda$, and argue that (D) is satisfied. This completes the construction at the limit stage λ.

In sum, several factors combine to carry us through limit stages. One is the convergence given by Remark 1.4. Another is the continuity of all the different maps we defined in Sections 2 and 5. A third is the agreement between $\mathfrak{P}_{\alpha+1}$ and $\tau_{\alpha,\alpha+1}(\mathfrak{P}_\alpha)$.

Remark 6.5. The reader should compare the formation of $\mathfrak{P}_\lambda$ to the formation of $\mathcal{U}_\infty$ in Section 4.2. $\mathcal{U}_\infty$ was formed in parts spread over previous stages of the construction. Each stage contributed an extra round to the formation. $\mathfrak{P}_\lambda$ too is formed in parts spread over previous stages of the construction. But

now the exact contribution of each stage is not set in advance. It depends on the behavior of the n_α-s, which in turn depends on the players. This extra flexibility in setting the break lines in the formation of limit pivots is the key to handling games of *variable* length.

Let us now consider the successor stage. We have the model M_α; the mixed pivot $\mathfrak{P}_\alpha$ of (C); and the ordinal sequence $\vec{\eta}_\alpha$ of (D). Our goal is to construct $M_{\alpha+1}$; the real y_α, which gives rise to the code $x_{\alpha+1}$; the mixed pivot $\mathfrak{P}_{\alpha+1}$; and the sequence $\vec{\eta}_{\alpha+1}$.

To start we use the iteration strategy to pick a cofinal branch b_α through $\mathcal{T}_\alpha$. We let Q_α denote the direct limit along b_α. The sequence $\vec{\eta}_\alpha$ of (D) forces the iteration strategy to pick an **odd** branch. We have the models presented in Diagram 22.

$$\begin{tikzcd} M \arrow[rr, dashed, "\tau_{0,\alpha}"] & & M_\alpha \arrow[r, "b_\alpha", "\mathcal{T}_\alpha"'] & Q_\alpha \end{tikzcd}$$

Diagram 22. At the start of round α.

For simplicity assume that $\mathfrak{P}_\alpha$ does not contain any mixing (see Remark 5.3 and Definition 5.4). So $f_\alpha(n) = 2n$ and there is some single γ^α so that $\gamma_n^\alpha = j_{0,2n}^\alpha(\gamma^\alpha)$ for all n. Since b_α is an odd branch we know that there is some h_α so that:

1. h_α is $\mathrm{col}(\omega, (j_{b_\alpha} \circ \tau_{0,\alpha})(\delta))$–generic/$Q_\alpha$; and
2. $\langle \vec{a}_\alpha, x_\alpha \rangle \in (j_{b_\alpha} \circ \tau_{0,\alpha})(\dot{A})[j_{b_\alpha}(\gamma^\alpha)][h_\alpha]$.

Using condition 2 and Definition 6.3 we get

3. In $Q_\alpha[h_\alpha]$, I wins the game $(j_{b_\alpha} \circ \tau_{0,\alpha})(G^*)(\vec{a}_\alpha, x_\alpha, j_{b_\alpha}(\gamma^\alpha))$.

Fix $\sigma_\alpha^* \in Q_\alpha[h_\alpha]$ witnessing condition 3. Let us use G_α^* to denote the game $(j_{b_\alpha} \circ \tau_{0,\alpha})(G^*)(\vec{a}_\alpha, x_\alpha, j_{b_\alpha}(\gamma^\alpha))$ of condition 3.

We divide now into two cases. Suppose first that in playing G_α^* we stay within part (F)—the "finishing" part. In this case we are essentially playing the game G^* of Section 3. Our construction in this case is similar to the construction in case 1 of Section 3. We use σ_α^* together with the appropriate image of $\sigma_{\mathrm{piv}-\infty}$ to play against the imaginary opponent. The construction produces the real y_α, and makes sure that $\langle x_\alpha, y_\alpha \rangle$ satisfies the Σ_2^1 statement ϕ. The fact that we stayed within part (F) tells us that α is the last round in our run of $G_{\mathrm{cont}-\nu}(C)$. The fact that $\langle x_\alpha, y_\alpha \rangle$ satisfies ϕ tells us that $\langle y_\xi \mid \xi \leq \alpha \rangle \in C$. $\langle y_\xi \mid \xi \leq \alpha \rangle$ is thus won by I, and our task for this subsection has been achieved.

So suppose that while playing G_α^* we enter part (M)—the "main" part. Let P denote our position when entering part (M). P determines $y_\alpha \restriction i$ for some i, and $y_\alpha \restriction i$ suffices to determine n_α. Let E_α denote $(j_{b_\alpha} \circ \tau_{0,\alpha})(E)$, where E is our

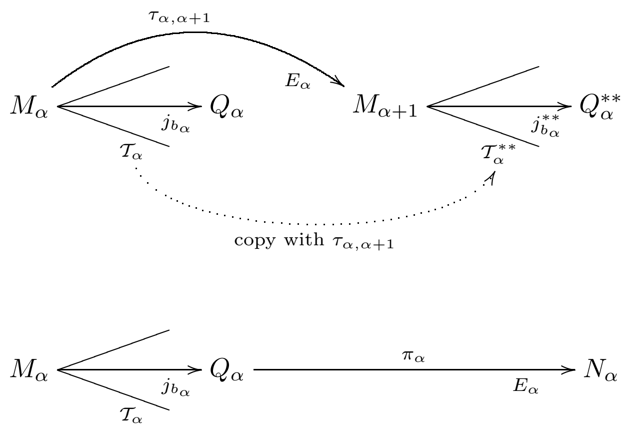

DIAGRAM 23. E_α applied to Q_α (lower line); and E_α applied to M_α (upper line) followed by copying.

original extender fixed at the beginning of Section 6. Let $N_\alpha = \mathrm{Ult}(Q_\alpha, E_\alpha)$ and let π_α be the ultrapower embedding. This is presented in the lower line of Diagram 23. Let $a'_\alpha = \pi_\alpha(\vec{a}_\alpha \restriction n_\alpha)$. Note how we follow the definitions listed just before the rules of part (M). Let G^{**}_α be the game obtained from G^*_α by starting from the position P. G^{**}_α is played according to the rules of part (M).

REMARK 6.6. G^{**}_α exists in $N_\alpha[h_\alpha]$. This follows from Remark 6.4, which in turn traces back to the strength of E assumed in condition 5 at the start of Section 6.

Note that G^{**}_α is an open game. Note further that G^{**}_α, being a tail-end of G^*_α played from a position according to σ^*_α, is won by I. Using Remark 6.6 we may fix a winning strategy σ^{**}_α for player I so that:

($\sharp$) σ^{**}_α belongs to $N_\alpha[h_\alpha]$.

We wish to use σ^{**}_α in much the same way we had used similar strategies in the past, combining it with moves given by some σ_{piv} or in our case σ_{mix}. Our problem is this: The starting position in G^{**}_α already includes auxiliary moves, the moves in a'_α. But we do not know that these auxiliary moves correspond to any starting position in the formation of a pivot. One particular aspect of our problem is the following: Auxiliary moves which correspond to pivots have some odd models around them. a'_α belongs to N_α and we have no odd models around N_α. N_α was not created as part of an iteration tree, it is simply the ultrapower of Q_α by E_α.

To solve this problem we try to look at N_α from a different perspective. Let $M_{\alpha+1}$ be the ultrapower by E_α of the model M_α rather than Q_α. Let $\tau_{\alpha,\alpha+1} \colon M_\alpha \to M_{\alpha+1} = \mathrm{Ult}(M_\alpha, E_\alpha)$ be the ultrapower map. This is presented in the upper left part of Diagram 23.

Remark 6.7. To form this ultrapower we need some agreement between M_α and Q_α. Now $\mathcal{T}_\alpha$ is part of a pivot corresponding to $\delta_\alpha = \tau_{0,\alpha}(\delta)$. We can arrange that the critical points in $\mathcal{T}_\alpha$ are larger than any pre-specified λ below δ_α (see Remark 2.5). We take $\lambda = \tau_{0,\alpha}(\mathrm{dom}(E))$. This ensures that all critical points in $\mathcal{T}_\alpha$ are above $\mathrm{dom}(E_\alpha)$, and so M_α and Q_α are in sufficient agreement that $E_\alpha \in Q_\alpha$ can be applied to M_α.

Use $\tau_{\alpha,\alpha+1}$ to copy $\mathcal{T}_\alpha$, a tree on M_α, to a tree on $M_{\alpha+1}$. Let $\mathcal{T}^{**}_\alpha$ denote the copied tree. Let $\vec{a}^{**}_\alpha$ denote the result of copying $\vec{a}_\alpha$, which is formed in models of $\mathcal{T}_\alpha$, to the models of $\mathcal{T}^{**}_\alpha$. While we are at it, let $\mathfrak{P}^{**}_\alpha$ be the result of copying the entire pivot $\mathfrak{P}_\alpha$ via $\tau_{\alpha,\alpha+1}$. Let Q^{**}_α be the direct limit of the models of $\mathcal{T}^{**}_\alpha$ along b_α, and let $j^{**}_{b_\alpha}$ be the direct limit embedding. These copies of Q_α and j_{b_α} are presented in the upper right part of Diagram 23.

Fact 6.8. Q^{**}_α equals N_α. Moreover $\vec{a}^{**}_\alpha$ equals $\pi_\alpha(\vec{a}_\alpha)$ and $\pi_\alpha \circ j_{b_\alpha} = j^{**}_{b_\alpha} \circ \tau_{\alpha,\alpha+1}$.

Remark 6.9. Fact 6.8 assumes some closure conditions on the extenders used in $\mathcal{T}_\alpha$. One can build these closure conditions into the construction of σ_{mix}. Alternatively one can use a weaker version of Fact 6.8 which holds in general. We refer the reader to [12, Chapter 4] for details.

Fact 6.8 is the answer to our problem. It tells us that $\pi_\alpha(\vec{a}_\alpha)$ does correspond to a pivot, the pivot $\mathfrak{P}^{**}_\alpha$. It follows that $a'_\alpha = \pi_\alpha(\vec{a}_\alpha \restriction n_\alpha)$ corresponds to $\mathfrak{P}^{**}_\alpha \restriction n_\alpha$.

Let $\mathfrak{P}^{***}_\alpha$ denote $\mathfrak{P}^{**}_\alpha \restriction n_\alpha$. This includes $\mathcal{T}^{***}_\alpha = \mathcal{T}^{**}_\alpha \restriction 2n_\alpha + 1$ and $a'_\alpha = \vec{a}^{**}_\alpha \restriction n_\alpha$. $\mathfrak{P}^{***}_\alpha$ represents a position of n_α rounds in $\tau_{0,\alpha+1}(\mathcal{A}_{\mathrm{mix}})[x_\alpha]$, played according to $\tau_{0,\alpha+1}(\sigma_{\mathrm{mix}})[x_\alpha]$. Since x_α and $x_{\alpha+1}$ agree to n_α (regardless of the end value of y_α) $\mathfrak{P}^{***}_\alpha$ is also a position in $\tau_{0,\alpha+1}(\mathcal{A}_{\mathrm{mix}})[x_{\alpha+1}]$, played according to $\tau_{0,\alpha+1}(\sigma_{\mathrm{mix}})[x_{\alpha+1}]$.

$\mathfrak{P}^{***}_\alpha$ will be our starting position when using $\tau_{0,\alpha+1}(\sigma_{\mathrm{mix}})[x_{\alpha+1}]$ for the construction of $\mathfrak{P}_{\alpha+1}$.

Remark 6.10. Note that starting the construction of $\mathfrak{P}_{\alpha+1}$ from $\mathfrak{P}^{***}_\alpha$—a restriction of $\tau_{\alpha,\alpha+1}(\mathfrak{P}_\alpha)$—has the pleasant side effect of securing the agreement condition, condition (i) above, which was used at limit stages.

Fix some $k < \omega$ which belongs to the odd branch b_α, is larger than $2n_\alpha$, and is large enough that $\vec{a}^{**}_\alpha \restriction n_\alpha$ has a pre-image in Q^{**}_k. (We use $Q^{**}_{\dots}$ to denote the models of $\mathcal{T}^{**}_\alpha$.) Let $\vec{a}^{***}_\alpha$ be this pre-image. Pick k large enough that G^{**}_α and σ^{**}_α, which belong to $N_\alpha[h_\alpha] = Q^{**}_\alpha[h_\alpha]$, have pre-images in $Q^{**}_k[h_\alpha]$. Let G^{***}_α and σ^{***}_α be these pre-images. Note our use here of Remark 6.6 and the condition ($\sharp$) following it.

G^{***}_α is an open game played according to rules (M1)–(M3), from the starting position $\vec{a}^{***}_\alpha$. σ^{***}_α is a winning strategy for I in this game.

From this point onward we continue along the lines of past constructions. We combine σ^{***}_α, $\tau_{0,\alpha+1}(\sigma_{\mathrm{mix}})[x_{\alpha+1}]$, and the imaginary opponent to create

y_α and $\mathfrak{P}_{\alpha+1}$. There is one difference though. We don't start the construction from zero. We start it from $\mathfrak{P}^{***}_\alpha$ which already contains n_α rounds according to $\tau_{0,\alpha+1}(\sigma_{\text{mix}})[x_{\alpha+1}]$.

The construction starts at round n_α. Let us go over this round. σ^{***}_α, in accordance with rule (M1), plays an ordinal γ^*. We have by that rule:

(†) $\gamma^* < (j^{**}_{0,k} \circ \tau_{\alpha,\alpha+1})(\gamma^\alpha)$; and

(‡) $\vec{a}^{***}_\alpha$ is a position in the auxiliary game $(j^{**}_{0,k} \circ \tau_{0,\alpha+1})(\mathcal{A})[\gamma^*, x_{\alpha+1}]$.

We now play round n_α of the mixed game $\tau_{0,\alpha+1}(\mathcal{A}_{\text{mix}})[x_{\alpha+1}]$ (see Diagram 21), continuing from the position given by $\mathfrak{P}^{***}_\alpha$. We play for I, and we intend to *mix*.

To begin, we play $f_{\alpha+1}(n_\alpha) = k$ and $\mathcal{T}_{\alpha+1}\restriction k{+}1 = \mathcal{T}^{**}_\alpha \restriction k{+}1$. Note that here already we have mixing, since k is larger than $2n_\alpha$. Next we play $\gamma^{\alpha+1}_{n_\alpha} = \gamma^*$. This is a legal move because of (‡).

The rest of the construction follows precisely the lines of case 1 in Section 3, except that the starting point is the model Q^{**}_k. σ^{***}_α and its shifts provide moves for I, $\tau_{0,\alpha+1}(\sigma_{\text{mix}})$ provides auxiliary moves for II, and the imaginary opponent provides natural number moves for II. These characters combined produce $\mathfrak{P}_{\alpha+1}$. We omit further details, and only point out that in shifting σ^{***}_α we use the fact that it belongs to $Q^{**}_k[h_\alpha]$, and the fact that $Q^{**}_k[h_\alpha]$ is a small extension relative to $\tau_{0,\alpha+1}(\text{dom}(E))$—hence relative to the critical points used in $\mathcal{T}_{\alpha+1}$, see Remark 6.7. The first fact traces back to condition (♯) above, which in turn traces back to condition 5 at the start of this section. The second fact traces back to our initial assumption that E overlaps δ.

Remark 6.11. At the start of the construction we made the simplifying assumption that $\mathfrak{P}_\alpha$ does not contain any mixing. Still, we ended with $\mathfrak{P}_{\alpha+1}$ which does contain mixing. Mixed pivots are therefore an essential part of the construction.

We point out that $\mathfrak{P}_{\alpha+1}$ contains mixing in round n_α, but does not contain mixing in any round above n_α. (Rounds below n_α depend on $\mathfrak{P}_\alpha \restriction n_\alpha$, which in general may contain mixing.) This is a general pattern at successor stages.

The case of a limit ordinal λ is different. $\mathfrak{P}_\lambda$ is the limit of the mixed pivots $\tau_{\alpha,\lambda}(\mathfrak{P}_\alpha)$, and can contain mixing in cofinally many rounds.

Finally, note that every time a mixing is initiated, some "smaller ordinal" is produced by (†) above. (Without the simplifying assumption that $\mathfrak{P}_\alpha$ does not contain mixing, the statement of (†) becomes more involved. $(j^{**}_{0,k} \circ \tau_{\alpha,\alpha+1})(\gamma^\alpha)$ is replaced by the pre-image to Q^{**}_k of the ordinal corresponding to the even root of b_α, see condition 2 of Definition 5.5.) These ordinals are used to create the sequences $\vec{\eta}_\alpha$ of (D), ensuring the agreement in the last item of (i) so that (D) holds at limits.

Remark 6.12. The base case of $\alpha = 0$ is similar to the successor case; our initial assumption, that I wins $G^*(\emptyset, \ulcorner\emptyset\urcorner, \gamma)$, is similar to condition 3 above and the construction starts from there. We leave this case to the reader.

6.3. Closing arguments. So far we defined the games $G^*(\vec{a}, x, \gamma)$, open games played in $M[g]$. We showed that if there exists γ so that I wins $G^*(\emptyset, \ulcorner\emptyset\urcorner, \gamma)$ then I wins $G_{\text{cont}-\nu}(C)$ in V. This work is analogous to the developments of Sections 4.1 and 4.2. To complete the proof of determinacy we must:

1. Define the mirror image games $H^*(\vec{b}, x, \gamma)$;
2. Show that if there exists γ so that II wins $H^*(\emptyset, \ulcorner\emptyset\urcorner, \gamma)$ then II wins $G_{\text{cont}-\nu}(C)$ in V; and
3. Derive a contradiction from the assumption that for all ordinals γ, II wins $G^*(\emptyset, \ulcorner\emptyset\urcorner, \gamma)$ and I wins $H^*(\emptyset, \ulcorner\emptyset\urcorner, \gamma)$.

The first two tasks are routine. Task 3 is an analogue of our work in Section 4.4. Working with σ_{gen} and τ_{gen} we construct a run of $G_{\text{cont}-\nu}$ in $M[g_\infty]$ which fails to satisfy ϕ, and fails to satisfy $\neg\phi$. The argument is an adaptation of the one in Section 4.4, but the adaptation is not entirely straightforward; some additional work is necessary. The precise details can be found in [12, Chapter 4]. Once task 3 is completed we get:

THEOREM 6.13. *Suppose that there exist M, $\delta < \delta_\infty \in M$, and $E \in M$ overlapping δ, which satisfy conditions 1–5 listed at the beginning of Section 6. Then all games $G_{\text{cont}-\nu}(C)$ where ν is continuous and C is Σ^1_2 in the codes are determined.* $\dashv$

6.4. Summary. We end with several observations about the proof of Theorem 6.13. Two of these observations show how the theorem can be improved somewhat.

In some sense our construction is a method for converting an iteration strategy into a winning strategy for I in $G_{\text{cont}-\nu}(C)$. (The mirror image construction of task 2 converts an iteration strategy into a winning strategy for II.) Note that the iteration trees in Section 6.2 are of the kind presented in Diagram 7, the "second" kind. The iteration strategy we use during the construction must therefore apply to the second kind iteration game. In contrast, Section 4 only used games of the first kind.

Next we note that the large cardinal assumption in Theorem 6.13 can be weakened without forcing great change to the proof. Suppose there exist M and $\delta < \delta_\infty$ which satisfy conditions 1–4 listed at the beginning Section 6 and satisfy the following weakened version of condition 5:

w5. For every $X \in M\| \delta + 1$ there exists an extender E in M overlapping δ and strong enough that $X \in \text{Ult}(M, E)$.

(In the original condition 5 one extender E worked for all $X \in M\| \delta + 1$.)

The proof of Theorem 6.13 can be repeated, almost verbatim, under this weaker assumption. Our main use of condition 5 was in Remark 6.6 and the condition ($\sharp$) which followed it. This use traced back to condition 5 through Claim 6.1; we needed to know that the real $x_\alpha \in Q_\alpha[h_\alpha]$ belonged also to $\text{Ult}(Q_\alpha, E_\alpha)[h_\alpha]$. Given a real $x_\alpha \in Q_\alpha[h_\alpha]$, the weak condition 5 can also be

used to find an extender E_α so that $x_\alpha \in \mathrm{Ult}(Q_\alpha, E_\alpha)[h_\alpha]$. So we can adjust the construction to only use the weak condition 5. (Note that with the weak condition we cannot expect a single extender to handle all reals. Thus we cannot at the outset fix $E \in M$ and always let $E_\alpha = \tau_{0,\alpha}(E)$. Instead we must let the extenders vary.)

Theorem 6.13 applies to games $G_{\mathrm{cont}-\nu}(C)$ where ν is continuous, i.e., $\mathbf{\Sigma}^0_1$ measurable. Our final note is that the theorem can be strengthened to apply to ν which are $\mathbf{\Sigma}^0_2$ measurable.

Fix $\nu\colon \mathbb{R} \to \mathbb{N}$ which is $\mathbf{\Sigma}^0_2$ measurable. For each $n \in \mathbb{N}$ the pre-image $\nu^{-1}\{n\}$ is $\mathbf{\Sigma}^0_2$. Let $\mathcal{C}$ include all the closed sets which participate in the unions defining the sets $\nu^{-1}\{n\}$, $n < \omega$. Without loss of generality the real parameter which defines ν belongs to M. Working in $M[g]$ and using the unraveling techniques of Martin [7], find a covering (R, π, φ) of $\omega^{<\omega}$ which unravels each of the sets in $\mathcal{C}$. Moves in the game on R are subsets of ω^ω in $M[g]$. For each $n \in \mathbb{N}$, the pre-image $(\pi^{-1} \circ \nu^{-1})\{n\}$ is *open*. Revise the rule of part (F) in Section 6.1 so that instead of forming $y_\alpha = \langle y_\alpha(i) \mid i < \omega\rangle$ by directly playing on $\omega^{<\omega}$, the players play on R. Part (F) continues until, if ever, the players enter one of the sets $(\pi^{-1} \circ \nu^{-1})\{n\}$, $n < \omega$. Note that the revision makes sense because these sets are open. If the players enter $(\pi^{-1} \circ \nu^{-1})\{n\}$ we set $n_\alpha = n$ and, if n_α is new, pass to part (M). The rules of part (M) are as before, except for rule (M2). What does it mean now to form y_α "continuing from the point" left in part (F)? The moves in part (F) give us some position in R to continue from. A position in R includes some initial segment $y_\alpha \restriction i$ of y_α, and some **commitment** T; T is a subtree of $\omega^{<\omega}$ and both players are committed to staying inside T. Revise rule (M2) to say that I and II play on $\omega^{<\omega}$ continuing from $y_\alpha \restriction i$ *and* must stay inside the tree T.

These revisions to parts (F) and (M) redefine the games $G^*(\vec{a}, x, \gamma)$. Using the techniques of Martin [7] one can adapt the construction of Section 6.2 to the new games, and complete the determinacy proof.

Remark 6.14. In adapting the construction of Section 6.2 we must take care to preserve Remark 6.6 and the subsequent condition ($\sharp$). Tracing back we must preserve Remark 6.4. Let us check that Remark 6.4 applies to the revised part (M). The revised part (M) is defined from the parameters listed in Remark 6.4, plus the additional parameter T. T, a commitment in the covering R, is a subset of $\omega^{<\omega}$ in $M[g]$. The strength given by condition 5 is enough to make sure that it belongs to $N[g]$, as required.

Note that we have here a limitation on the size of moves permitted in R. This in turn limits the complexity of functions ν which we can handle. To handle functions in pointclasses above $\mathbf{\Sigma}^0_2$ we would need stronger agreement between $M[g]$ and $N[g]$ than the one given by condition 5.

Combining the observations above we get the following strengthening of Theorem 6.13:

THEOREM 6.15. *Suppose that there exist M and $\delta < \delta_\infty \in M$ which satisfy the following conditions:*

1. *M is a class model;*
2. *M is iterable;*
3. *δ and δ_∞ are Woodin cardinals of M;*
4. *$M \| \delta_\infty + 1$ is countable in V; and*

w5. *For every $X \in M \| \delta + 1$ there exists an extender E in M overlapping δ and strong enough that $X \in \mathrm{Ult}(M, E)$.*

Then all games $G_{\text{cont}-\nu}(C)$ where ν is $\boldsymbol{\Sigma}^0_2$ measurable and C is Σ^1_2 in the codes are determined. $\dashv$

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