NECESSARY USE OF Σ^1_1 INDUCTION IN A REVERSAL

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Abstract. Jullien's indecomposability theorem states that if a scattered countable linear order is indecomposable, then it is either indecomposable to the left, or indecomposable to the right. The theorem was shown by Montalbán to be a theorem of hyperarithmetic analysis, and then, in the base system RCA_0 plus Σ^1_1 induction, it was shown by Neeman to have strength strictly between weak Σ^1_1 choice and Δ^1_1 comprehension. We prove in this paper that Σ^1_1 induction is needed for the reversal of INDEC, that is for the proof that INDEC implies weak Σ^1_1 choice. This is in contrast with the typical situation in reverse mathematics, where reversals can usually be refined to use only Σ^0_1 induction.

§1. Introduction. This paper is concerned with a use of strong induction for a reversal, in reverse mathematics. We show that a use of Σ^1_1 induction made in a reversal proof in Neeman [6] is necessary. This appears to be the first time such a use is shown to be needed.

We refer the reader to Simpson [9] for a systematic treatment of reverse mathematics. Let us here recall that it deals with calibrating the strength of theorems of second order number theory (a.k.a. analysis). Strength is measured relative to a hierarchy of systems of axioms. It was realized early on that full induction is not needed in the base system. And so today the standard base system is RCA_0, consisting of the axioms of Peano arithmetic other than induction, Δ^0_1 comprehension, and induction limited to Σ^0_1 formulas. The subscript 0, in RCA_0 and in other stronger systems, indicates that the induction schema in the system is restricted to Σ^0_1 formulas.

This is not to say that stronger induction is not used in reverse mathematics. There are situations where theorems reverse to systems that include strong induction. For example, Simpson [8] shows that Σ^1_1 transfinite induction is equivalent to $\mathsf{ATR}_0 + \Sigma^1_1$ induction, but not equivalent to ATR_0. Nemoto [7] shows that determinacy for the Wadge pointclass immediately above Σ^0_2 in Baire space is also equivalent to $\mathsf{ATR}_0 + \Sigma^1_1$ induction. Medsalem–Tanaka [3] show that Δ^1_3-CA_0 plus Σ^1_1 induction proves Δ^0_3 determinacy, and that Σ^1_1 induction cannot be dropped.

But in all these situations, Σ^1_1 induction is used in proving the theorem, not in the reversal, that is not as part of the base system needed to extract strength from the theorem. It has typically been the case that reversals that (initially) use strong induction, can be refined to use only Σ^0_1 induction, and this indeed

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is the reason that by now the standard base system in reverse mathematics has only Σ^0_1 induction.

Here we present a situation where Σ^1_1 induction is used in a reversal, and the use is necessary.

Uses of strong induction in reversals, even uses that are not known to be necessary, are rare. But they do exist. For example Hirschfeldt–Shore [1] show that the combinatorial principles COH and CADS are equivalent in RCA_0 plus $B\Sigma^0_2$. $B\Sigma^0_2$ is in some sense an induction principle, being strictly between Σ^0_1 induction and Σ^0_2 induction. In RCA_0 alone COH implies CADS, and it is not known if the implication can be reversed.

Other uses, more closely related to the situation here, come from work of Montalbán [5, 4]. The work there is concerned with the strength of various results on countable orders, and many of the results are proved in the base system $\mathsf{RCA}_0+\Sigma^1_1$ induction, rather than RCA_0. To name just one, Montalbán proves, in RCA_0 plus Σ^1_1 induction, that Fraïssé's conjecture (FRA) is equivalent to a certain principle JUL of extendibility for linear orders. In RCA_0 alone, JUL implies FRA, and it is not known if the implication can be reversed.

Montalbán [4] also studies a particular theorem of Jullien [2], on indecomposability. Recall that a linear order is *scattered* if it does not embed the rationals. A *gap* in a linear order is a pair $\langle L, R\rangle$, which partitions the order, with L closed to the left, and R closed to the right. The gap is a *decomposition* if the full order does not embed into L and does not embed into R. The order is *indecomposable* if it has no decompositions, meaning that for every gap $\langle L, R\rangle$, the order embeds either into L or into R. The order is *indecomposable to the left* if for all gaps with $L \neq \emptyset$ the full order embeds into L. It is *indecomposable to the right* if the same holds with R instead of L. Jullien's indecomposability theorem, which Montalbán called INDEC, states that every scattered countable linear order which is indecomposable, is indecomposable to the left, or indecomposable to the right.

INDEC is of interest to us here because of another reversal that uses Σ^1_1 induction. Neeman [6] shows that the strength of INDEC lies strictly between weak Σ^1_1 choice and Δ^1_1 comprehension, and in particular INDEC implies weak Σ^1_1 choice. This strengthens work of Montalbán [4], who shows that INDEC has hyperarithmetic strength. For more on this we refer the reader to the two papers. Here let us only say that the proof in Neeman [6] that INDEC implies weak Σ^1_1 choice was carried out in the system RCA_0 plus Σ^1_1 induction.

It was generally expected that the uses of Σ^1_1 induction that we mentioned above could be removed, that technically more elaborate proofs would establish the reversals in RCA_0. This has been the typical case in reverse mathematics.

But it turns out that the use of Σ^1_1 induction in the reversal from INDEC to weak Σ^1_1 choice cannot be removed. We prove in this paper that:

THEOREM 1.1. *In the system $\mathsf{RCA}_0 + \Delta^1_1$ induction, INDEC does not imply weak Σ^1_1 choice.*

Note that neither INDEC nor weak Σ^1_1 choice implies Σ^1_1 induction. It is not that INDEC reverses to a system with Σ^1_1 induction, but that Σ^1_1 induction is needed as part of the base system for the reversal. To our knowledge this is the first time that strong induction is shown to be needed for a reversal in such a way.

The rest of the paper contains the proof of Theorem 1.1. We prove the theorem by constructing a model where INDEC and Δ^1_1 induction hold, yet weak Σ^1_1 choice fails. The model is non-standard of course. It is constructed using Steel forcing, and Section 2 includes some preliminaries on this. The model itself is defined in Section 3, where we also prove that it satisfies Δ^1_1 induction, and fails to satisfy weak Σ^1_1 choice. Finally in Section 4 we show that the models satisfies INDEC.

§2. Preliminaries. We briefly recall the forcing technique of Steel [10, 11]. This is a powerful technique for creating models of hyperarithmetic analysis. Our description of the actual poset follows Van Wesep [12].

Let $\prec$ be a recursive (illfounded) linear order on a recursive subset of ω, so that the wellfounded part of $\prec$ has order type ω_1^{ck}, and so that no hyperarithmetic sequence witnesses the illfoundedness of $\prec$.

Define a poset $\mathbb{P}$ as follows. Conditions are triples $p = \langle T_p, f_p, h_p\rangle$ where:

1. $T_p \subseteq \omega^{<\omega}$ is a finite tree.
2. f_p is a function from a finite subset of ω to T. Let $\mathrm{Dc}(f_p)$, the *downward closure* of $\mathrm{range}(f_p)$, be the set $\{f_p(i)\restriction j \mid i \in \mathrm{dom}(f_p), j \leq \mathrm{lh}(f_p(i))\}$ of initial segments of nodes in $\mathrm{range}(f_p)$.
3. h_p is a $\langle T_p, f_p\rangle$-tagging. I.e., h_p is a function from $T_p - \{\emptyset\} - \mathrm{Dc}(f_p)$ into $\mathrm{dom}(\prec)$, with $t \supsetneq s \rightarrow h_p(t) \prec h_p(s)$.

Conditions are ordered by reverse extension. $p \leq q$ iff $T_p \supseteq T_q$, $h_p \supseteq h_q$, and $(\forall i \in \mathrm{dom}(f_q))\ f_p(i)$ is defined and extends $f_q(i)$.

We use $\mathbb{P}$ to force over the model $L_{\omega_1^{ck}}$. Let G be generic over this model. Let $T = T^G = \bigcup_{p\in G} T_p$ and define $h = h^G$ and $f(i) = f^G(i)$ similarly. We use $\dot{T}$, $\dot{f}$, and $\dot{h}$ for the canonical names for T, f, and h. T is a tree on ω, $B = B^G = \{f(i) \mid i \in \omega\}$ is a set of branches through T, and h "ranks" nodes of T which are not initial segments of branches in B, meaning that it embeds the order of reverse extension on these nodes to $\prec$. (If $\prec$ were wellfounded then h would witness that these nodes do not extend to branches of T.)

In talking about h, we identify each member of the wellfounded part of $\prec$ with its ordinal rank. Thus when we write $h(t) = \alpha$ we mean $h(t) = i$ for i whose order type in $\prec$ is α.

For each finite $F \subseteq B$ let $M_F = M_F^G$ be the model $L_{\omega_1^{ck}}(\{T\} \cup F)$. The subsets of ω which belong to M_F are precisely those which are hyperarithmetic in the join of $\{T\}\cup F$. The models of hyperarithmetic analysis that one typically produces using Steel forcing are unions of models of the form $M_F \cap (\omega \cup \mathcal{P}(\omega))$. (In our case we will use unions of non-standard models of a similar form.)

M_F has the tree T, but not the function h. For each $\alpha < \omega_1^{ck}$, the restriction of h to nodes t with $h(t) < \alpha$ does belong to M_F: genericity implies that $h(t)$, when wellfounded, is precisely equal to the rank of t in T, and for each $\alpha < \omega_1^{ck}$ the ranks up to α can be computed from T by recursion. But these recursions become increasingly complicated as α increases. If one is restricted to some bounded complexity below hyperarithmetic, then one cannot distinguish between sufficiently high values of h. A precise formulation of this symmetry is given in Lemma 2.3 below.

Let $A \subseteq \omega$ be finite. Let $F(A) = F^G(A)$ denote the set $\{f^G(i) \mid i \in A\}$. By induction on $\alpha < \omega_1^{ck}$ we define the A-*nice* names for elements of $L_\alpha(\{T\} \cup F(A))$, and the order of these names. The order of $\dot{x}$ is denoted $o(\dot{x})$, and we shall have $L_\alpha(\{T\} \cup F(A)) = \{\dot{x}[G] \mid \dot{x}$ is A-nice and $o(\dot{x}) < \alpha\}$. We start at level ω, with $L_\omega(\{T\} \cup F(A)) = L_\omega \cup \{T^G\} \cup F(A)$. $1_{\mathbb{P}}$ denotes $\langle \emptyset, \emptyset, \emptyset \rangle$, the weakest condition in $\mathbb{P}$.

- The A-nice names for elements of L_ω, for T^G, and for $f^G(i)$, $i \in A$, are simply the canonical $\mathbb{P}$-names for these objects. The order of these names is 0.
- Let $\alpha \geq \omega$. Let $\dot{z} = \{\langle \dot{x}, 1_{\mathbb{P}} \rangle \mid \dot{x}$ is A-nice and $o(\dot{x}) < \alpha\}$. (By induction, $\dot{z}$ names $L_\alpha(\{T\} \cup F(A))$.) If $\varphi(v_0, v_1, \ldots, v_k)$ is a formula, and $\dot{a}_1, \ldots, \dot{a}_k$ are A-nice names of order $< \alpha$, then $\{\langle \dot{u}, p \rangle \mid \dot{u}$ is A-nice, $o(\dot{u}) < \alpha$, and $p \Vdash$ "$\dot{z} \models \varphi[\dot{u}, \dot{a}_1, \ldots, \dot{a}_k]$"$\}$ is an A-nice name of order α.

It is clear that every element of $M_F = L_{\omega_1^{ck}}(\{T\} \cup F(A))$ has an A-nice name, and that $M_F = \{\dot{x}[G] \mid \dot{x}$ is A-nice$\}$.

A statement $\varphi(\dot{x}_1, \ldots, \dot{x}_k)$ in the forcing language is A-*nice* if $\dot{x}_i$ are A-nice, and all quantifiers of φ are bounded to range over A-nice names. When talking about $M_{F(A)}$ in the forcing language we shall only use A-nice statements.

An A-nice statement $\varphi(\dot{x}_1, \ldots, \dot{x}_k)$ is *ranked* if there is $\alpha < \omega_1^{ck}$ so that $o(\dot{x}_i) < \alpha$ and all quantifiers in φ are bounded to range over A-nice names of order $< \alpha$. The least α witnessing this is the *order* of $\varphi(\dot{x}_1, \ldots, \dot{x}_k)$. The *rank* of $\varphi(\dot{x}_1, \ldots, \dot{x}_k)$ is defined to be $\omega^2 \cdot o + \omega \cdot q + n$ where o is the order of $\varphi(\dot{x}_1, \ldots, \dot{x}_k)$, q is the number of quantifier in φ, and n the number of logical connectives. The definition is taken from Steel [11].

Claim 2.1. *For each $\alpha < \omega_1^{ck}$, the restriction of the forcing relation to A-nice statements of rank $< \alpha$ belongs to $L_{\omega_1^{ck}}$.*

Claim 2.1 is taken from Van Wesep [12] and relies on Van Wesep's definition of $\mathbb{P}$, which differs slightly from that of Steel [11].

Definition 2.2. Let $p, p^* \in \mathbb{P}$, $\eta < \omega_1^{ck}$. p^* is an η-*absolute A-reduct* of p if:

1. $T_p = T_{p^*}$ and $f_p(i) = f_{p^*}(i)$ for $i \in A$.
2. $h_{p^*}(s) = h_p(s)$ whenever either one of $h_{p^*}(s)$, $h_p(s)$ is defined and $< \eta$.

Lemma 2.3 (Steel [11]). *Let $\varphi(\dot{x}_1, \ldots, \dot{x}_k)$ be A-nice and ranked, with rank $\leq \eta < \omega_1^{ck}$. Suppose p^* is an $\omega\eta$-absolute A-reduct of p. Then $p \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_k)$ iff $p^* \Vdash \varphi(\dot{x}_1, \ldots, \dot{x}_k)$.*

Lemma 2.3 gives precise meaning to the statement that if one is restricted to complexity bounded below hyperarithmetic, in T and finitely many branches through it, then one cannot distinguish between values of h beyond a bounded level. It implies in particular that the only branches of T in M_F are the ones in F:

Claim 2.4 (Steel [11]). *Let $A \subseteq \omega$ be finite. Let $F = \{f^G(i) \mid i \in A\}$. Then the only branches of T which belong to M_F are those in F.*

PROOF. Suppose not. Let $\dot{b}$ be an A-nice name for a branch of T which is distinct from $f^G(i)$ for each $i \in A$. Let $p \in \mathbb{P}$ force this. Strengthening p, we may fix $n < \omega$ and a node t, and assume that p forces $\dot{b}\restriction\check{n} = \check{t}$, $t \in T_p$, and t is incompatible with $f_p(i)$ for each $i \in A$. This is a ranked statement. Let $\eta < \omega_1^{\mathrm{ck}}$ be its rank. (How large it is exactly depends on the order of $\dot{b}$.)

Let p^* be obtained from p by setting $T_{p^*} = T_p$, setting $f_{p^*} = f_p\restriction A$, and modifying h_p to produce h_{p^*} which agrees with h_p on nodes of rank $< \omega\eta$, but gives t a *wellfounded* rank (greater than or equal to η or course).

Then p^* is an $\omega\eta$-absolute A-reduct of p. By Lemma 2.3, p^* forces that $\dot{b}$ is a branch through $\dot{T}$, and $\dot{b}$ extends $\check{t}$. But now letting G^* be generic with $p^* \in G^*$, we get that $h^{G^*}(\dot{b}[G^*]\restriction j)$, $j > n$, is an infinite descending chain in $\prec$ below $h_{p^*}(t)$, contradicting the fact that $h_{p^*}(t)$ is wellfounded. $\dashv$

§3. The model, induction, and failure of weak choice. Let $M = L_{\omega_1^{\mathrm{ck}}}$, let G be generic over M for Steel's forcing $\mathbb{P}$, let $T = T^G$, and let $f = f^G$. T is a tree on ω, and each $f(i)$ is a branch through T. By genericity, for each $n < \omega$ there is a branch $f(i)$ whose first coordinate, $f(i)(0)$, is of the form (n, k) for some k. (Here and throughout, $(*, *)\colon \omega^2 \to \omega$ is some standard pairing function.) Let i_n be the least i witnessing this. Let $K = \{f(i_n) \mid n \in \omega\}$.

Let $\mathcal{F}$ be the collection of finite subsets of K. For each $F \in \mathcal{F}$ let $M_F = M(\{T\} \cup F)$. These are the usual settings for applications of Steel's forcing with the set K. Let $\mathcal{M}$ denote the function $F \mapsto M_F$.

Let $\mathfrak{H}$ be a standard (transitive) model of enough of ZFC, which is countable and contains all the objects above.

Let $\mathfrak{H}^*$ be a non-standard countable elementary extension of $\mathfrak{H}$, and let $j\colon \mathfrak{H} \to \mathfrak{H}^*$ be an elementary embedding. We use $\mathcal{M}^*$ to denote $j(\mathcal{M})$, and similarly with ω, M, T, K, and $\mathcal{F}$. $\omega^* = j(\omega)$ is an illfounded end extension of ω of course.

Each $F \in \mathcal{F}^*$ has a size, possibly non-standard, in ω^*. It is the unique $n \in \omega^*$ so that $\mathfrak{H}^* \models$"the size of F is n." We write $|F|$ to denote this n.

For each $F \in \mathcal{F}^*$ let $M^*_F = \mathcal{M}^*(F)$. This is the model $M^*(\{T^*\} \cup F)$, as computed in $\mathfrak{H}^*$. Let $\mathcal{U} = \{F \in \mathcal{F}^* \mid |F| \in \omega\}$. $\mathcal{U}$ thus consists of all elements of $\mathcal{F}^*$ which have standard size.

Let $N = \bigcup_{F \in \mathcal{U}} M^*_F$. N is the model we use to witness Theorem 1.1.

LEMMA 3.1. *N satisfies RCA_0 and Δ^1_1 induction.*

PROOF. The first order axioms hold in ω^* by elementarity of j. Δ^0_1 comprehension holds in all M^*_F by elementarity of j, and from this and the fact that $F_1, F_2 \in \mathcal{U} \to F_1 \cup F_2 \in \mathcal{U}$ it follows that the axiom holds in N. We prove that Δ^1_1 induction holds in N.

Let $\varphi(n)$ and $\psi(n)$ be a Σ^1_1 formulas, possibly with parameters. Suppose for simplicity that the parameters belong to $M^*_\emptyset$. In the general case one has to restrict below to l greater than the (standard) least size of an F so that the parameters belong to M^*_F.

Suppose that $N \models (\forall n)(\varphi(n) \leftrightarrow \neg\psi(n))$. Suppose that $N \models \varphi(0)$, and $N \models (\forall n)(\varphi(n) \to \varphi(n+1))$. We prove that $N \models (\forall n)\varphi(n)$.

For each $l \in \omega$ let $N_l = \bigcup_{F \subseteq K, |F|=l} M_F$. Let s be the function $l \mapsto N_l$. As $\mathfrak{H}$ satisfies a sufficient fragment of ZFC, we may assume s belongs to $\mathfrak{H}$. Let $s^* = j(s)$, and let N_l^* for $l \in \omega^*$ be $s^*(l)$.

Claim 3.2. *N is equal to $\bigcup_{l \in \omega} N_l^*$.*

Proof. Clear from the definitions. Note that the union is taken over standard l only. ⊣

Claim 3.3. *There exists a non-standard l so that $N_l^* \models (\forall n)(\varphi(n) \to \neg\psi(n))$.*

Proof. Suppose not. In other words suppose that for every non-standard l, $N_l^* \models (\exists n)(\varphi(n) \wedge \psi(n))$.

Recall that $\mathfrak{H}$ is a model of a sufficiently large fragment of ZFC. By elementarity of j, so is $\mathfrak{H}^*$. We may assume the fragment is large enough to define the set $A = \{l \in \omega^* \mid s^*(l) = N_l^* \models (\exists n)(\varphi(n) \wedge \psi(n))\}$. The set then belongs to $\mathfrak{H}^*$, and therefore has a smallest element, call it l_0. Since we assume for contradiction that all non-standard l belong to the set, l_0 must be standard. But then $N_{l_0}^* \subseteq N$. Since $(\exists n)(\varphi(n) \wedge \psi(n))$ is a Σ_1^1 formula it reflects from $N_{l_0}^*$ to N, so $N \models (\exists n)(\varphi(n) \wedge \psi(n))$, contradiction. ⊣

Work with l given by the last claim. Then for each $n \in \omega^*$:

$$\varphi^N(n) \to \varphi^{N_l^*}(n) \tag{1}$$
$$\to \neg\psi^{N_l^*}(n) \tag{2}$$
$$\to \neg\psi^N(n) \tag{3}$$
$$\to \varphi^N(n). \tag{4}$$

The implications (1) and (3) use Σ_1^1 reflection from N to N_l^*, which is a superset of N since l is non-standard. The implication (2) uses the last claim, and the implication (4) is the equivalence of φ and $\neg\psi$ in N.

Note that N_l^*, unlike N, belongs to $\mathfrak{H}^*$. Since $\mathfrak{H}^*$ thinks that ω^* is wellfounded, N_l^* is a model of full induction. Using the equivalence $\varphi^N(n) \leftrightarrow \varphi^{N_l^*}(n)$ given by the implications above, we may induct in N_l^* to conclude, from $N \models \varphi(0) \wedge (\forall n)(\varphi(n) \to \varphi(n+1))$, that $N \models (\forall n)\varphi(n)$. ⊣ (Lemma 3.1)

Let *finite weak Σ_1^1 choice* be the following schema: for each arithmetic formula $\varphi(k, x)$, if $(\forall k)(\exists! x)\varphi(k, x)$, then for all n there exists a sequence $\langle x_0, \dots, x_{n-1}\rangle$ so that $(\forall k < n)\varphi(k, x_k)$.

Finite weak Σ_1^1 choice is a consequence of weak Σ_1^1 choice. Indeed, under weak Σ_1^1 choice, if $(\forall k)(\exists! x)\varphi(k, x)$ then in fact there exists an infinite sequence $\langle x_k \mid k < \omega\rangle$ so that $(\forall k)\varphi(k, x_k)$.

But really finite weak Σ_1^1 choice is an induction principle, and a fairly weak one at that. It is an innocent looking, weak consequence of Σ_1^1 induction:

Claim 3.4. *(In RCA_0.) Σ_1^1 induction implies finite weak Σ_1^1 choice.*

Proof. Let $\psi(n)$ be the statement that there exists a sequence $\langle x_0, \dots, x_{n-1}\rangle$ so that $(\forall k < n)\varphi(k, x_k)$. $\psi(n)$ is a Σ_1^1 statement. It is clearly true for $n = 0$. Further, assuming $(\forall k)(\exists! x)\varphi(k, x)$, $\psi(n+1)$ follows from $\psi(n)$ by taking any sequence witnessing $\psi(n)$, and concatenating to it the unique x witnessing

$\varphi(k,x)$. Thus, assuming $(\forall k)(\exists! x)\varphi(k,x)$, Σ^1_1 induction shows that $\psi(n)$ holds for all n. $\dashv$

REMARK 3.5. The only use of Σ^1_1 induction in the proof in Neeman [6] that INDEC implies weak Σ^1_1 choice, comes through an instance of Claim 3.4.

LEMMA 3.6. *Finite weak Σ^1_1 choice fails in N.*

PROOF. Let $\varphi(k,b)$ be the statement that b is a branch through T^* with first coordinate $b(0)$ of the form (k,i) for some i.

By definition of K and the elementarity of j, for every k in ω^* there exists a unique $b \in K^*$ so that $\varphi(k,b)$ holds. It follows that in N, $(\forall k)(\exists! b)\varphi(k,b)$.

But the statement $\psi(n)$, that there exists a sequence $\langle b_0,\dots,b_{n-1}\rangle$ so that $(\forall k < n)\varphi(k,b_k)$, fails in N for some n. Indeed it fails for all non-standard n. This is because $N = \bigcup_{F\in\mathcal{U}} M^*_F$, each $F \in \mathcal{U}$ has standard size, and every sequence of distinct branches through T^* in M^*_F has length at most the size of F by Claim 2.4. $\dashv$

COROLLARY 3.7. *Σ^1_1 induction and weak Σ^1_1 choice both fail in N.*

§4. INDEC. We begin the section with a lemma that holds in the standard model $\mathfrak{H}$. By the elementarity of j, the same lemma holds in $\mathfrak{H}^*$. Later we will use the lemma in $\mathfrak{H}^*$ to show that INDEC holds in N.

Let $F_0 \subseteq K$ be finite. Let $U \in M_{F_0}$ be a linear order on ω. For each $l < \omega$ let Left_l consist of all $x \in \mathrm{dom}(U)$ so that there is an embedding of U to the right of x in $M_{F_0\cup J}$ for some J of size $\le l$. We refer to any such J as a witness to the membership of x in Left_l. Let Right_l consist of $x \in \mathrm{dom}(U)$ so that there is an embedding of U to the left of x in $M_{F_0\cup J}$ for some J of size $\le l$. Again we refer to any such J as a witness to membership.

It is clear that Left_l is closed to the left, and Right_l is closed to the right.

LEMMA 4.1. *Suppose that $l < \omega$ is such that:*

1. *Each $x \in \mathrm{dom}(U)$ belongs to either Left_l or Right_l.*
2. *No $x \in \mathrm{dom}(U)$ belongs to both $\mathit{Left}_{2\cdot l}$ and $\mathit{Right}_{2\cdot l}$.*

Then $\langle \mathit{Left}_l, \mathit{Right}_l\rangle$ is a gap in U, and there is $E \subseteq K$ of size at most $2\cdot l$ so that Left_l and Right_l belong to $M_{F_0\cup E}$.

PROOF. If x belongs to both Left_l and Right_l, then letting J_{Left} and J_{Right} witness this we get that in $M_{F_0\cup J_{\mathit{Left}}\cup J_{\mathit{Right}}}$ there are embeddings of U both to the right of x and to the left of x, contradicting (2) as $J_{\mathit{Left}} \cup J_{\mathit{Right}}$ has size at most $2\cdot l$. So $\mathit{Left}_l \cap \mathit{Right}_l = \emptyset$. Using (1) we also have $\mathit{Left}_l \cup \mathit{Right}_l = \mathrm{dom}(U)$, so $\langle \mathit{Left}_l, \mathit{Right}_l\rangle$ partitions $\mathrm{dom}(U)$. From this and the closure of Left_l and Right_l to the left and right respectively, it follows that $\langle \mathit{Left}_l, \mathit{Right}_l\rangle$ is a gap in U. We continue to prove that it belongs to $M_{F_0\cup E}$ for some E of size at most $2\cdot l$.

Let E_{Left} be a set of the largest possible size so that for every $x \in \mathit{Left}_l$ there is J_{Left} witnessing the membership, with $E_{\mathit{Left}} \subseteq J_{\mathit{Left}}$. Define E_{Right} similarly. We intend to show that the gap $\langle \mathit{Left}_l, \mathit{Right}_l\rangle$ belongs to $M_{F_0\cup E_{\mathit{Left}}\cup E_{\mathit{Right}}}$. Since $E_{\mathit{Left}} \cup E_{\mathit{Right}}$ has size at most $2\cdot l$, this will complete the proof of the lemma.

CLAIM 4.2. *For every $x \in \mathit{Left}_l$, and any finite $A \subseteq K$, there is J witnessing the membership of x in Left_l with $J \supseteq E_{\mathit{Left}}$ and $J - E_{\mathit{Left}}$ disjoint from A. Similarly for Right_l and E_{Right}.*

PROOF. We prove the claim on Left_l. Suppose it fails for x. Fix $y_i \in \mathrm{dom}(U)$, $i < \omega$, all to the right of x, increasing and cofinal in Left_l. By cofinal we mean that every $z \in \mathit{Left}_l$ has some y_i to its right. Increasing of course means that $i < j$ implies $y_i <_U y_j$.

Since the claim fails for x, and y_i are all to the right of x, the claim also fails for each y_i. Thus, fixing J_i witnessing the membership of y_i in Left_l with $E_{\mathit{Left}} \subseteq J_i$, we know that $J_i - E_{\mathit{Left}}$ is not disjoint from A. Fix then for each $i < \omega$ some $b_i \in (J_i - E_{\mathit{Left}}) \cap A$. Since A is finite, there is a fixed $b \in A$ so that $b_i = b$ for infinitely many $i < \omega$. By thinning the sequence y_i, $i < \omega$, we may assume that in fact $b_i = b$ for all $i < \omega$.

Let $E^* = E_{\mathit{Left}} \cup \{b\}$. Then $E^* \subseteq J_i$ for each i. Using the fact that y_i, $i < \omega$, is cofinal in Left_l, it follows that for each $z \in \mathit{Left}_l$, there is a witness J for the membership with $E^* \subseteq J$. (One can take $J = J_i$ for any i so that y_i is to the right of z.) But since E^* is larger than E_{Left}, this contradicts the maximality in the definition of E_{Left}. ⊣

Our plan for the rest of the proof of Lemma 4.1 is this: Using standard Steel forcing techniques, for any $I \subseteq K$ we can (approximately) view $M_{F_0 \cup E_{\mathit{Left}} \cup E_{\mathit{Right}} \cup I}$ as a forcing extension of $M_{F_0 \cup E_{\mathit{Left}} \cup E_{\mathit{Right}}}$. $M_{F_0 \cup E_{\mathit{Left}} \cup E_{\mathit{Right}}}$ can identify the set of x which can be forced into Left_l with witness $E_{\mathit{Left}} \cup I$ so that I misses an arbitrary finite set in K, and similarly with Right_l. We intend to show that the sets of these x are exactly equal to Left_l and Right_l respectively, and hence Left_l and Right_l belong to $M_{F_0 \cup E_{\mathit{Left}} \cup E_{\mathit{Right}}}$. The main part of the argument goes into showing that the sets are disjoint. It is there that we use the fact that the set of branches added to E_{Left} to produce a witness can be made to avoid arbitrary finite subsets of K, and similarly with E_{Right}. We use particularly the fact that the sets can be made to avoid each other, as this allows combining the two extension, one forcing x into Left_l and the other forcing x into Right_l, and the combined extension leads to a contradiction.

Recall that we are working with a generic G for Steel's forcing, $T = T^G$, and $f = f^G$. For each $n < \omega$, there is exactly one branch b_n in K whose first coordinate $b_n(0)$ has the form (n, j) for some j. b_n is equal to $f^G(i_n)$ for i_n is least so that $f^G(i_n)(0) = (n, j)$ for some j. Let S be the function $n \mapsto i_n$. Let C_{Left} be the set of n so that $b_n = f^G(i_n) \in E_{\mathit{Left}}$, and define C_{Right} similarly. (So $E_{\mathit{Left}} = \{f^G(i_n) \mid n \in C_{\mathit{Left}}\} = F(S''C_{\mathit{Left}})$ and similarly with E_{Right}.) Let C_0 be such that $F_0 = \{f^G(i_n) \mid n \in C_0\} = F(S''C_0)$. Recall that $M_{F(I)}$ for $I \subseteq \omega$ denotes the model M_F where $F = \{f^G(i) \mid i \in I\}$.

We know by the previous claim and the conditions of the lemma that:

I. For every $x \in \mathrm{dom}(U)$ and every finite $A \subseteq \omega$, there exists a set $D \subseteq \omega$ of size $\leq l$, so that either: $D \supseteq C_{\mathit{Left}}$, $D - C_{\mathit{Left}}$ avoids A, and in $M_{F(S''(C_0 \cup D))}$ there is an embedding of U to the right of x; or $D \supseteq C_{\mathit{Right}}$, $D - C_{\mathit{Right}}$ avoids A, and in $M_{F(S''(C_0 \cup D))}$ there is an embedding of U to the left of x.

II. There is no x and no D of size $\leq 2 \cdot l$ so that in $M_{F(S''(C_0 \cup D))}$ there are embeddings of U both to the left and to the right of x.

Fix a condition $p_0 \in G$ forcing these statements to hold. In the forcing languages, these are statements about $\dot{U}$ naming $U \in M_{F_0}$, $\dot{f}$ naming the function f^G, $\dot{S}$ naming the function $S = n \mapsto i_n$, $\check{C}_0$, $\check{C}_{Left}$, $\check{C}_{Right}$, and $\check{l}$.

Claim 4.3. *$q \in \mathbb{P}$ forces $\dot{S}(\check{n}) = \check{i}$ iff $\mathrm{dom}(f_q) \supseteq i+1$, $\mathrm{lh}(f_q(k)) \geq 1$ for each $k \leq i$, and i is least so that $f_q(i)(0)$ has the form (n, j) for some j.*

Proof. Both directions are clear. For the right-to-left direction, note that any q which fails to satisfy the condition on the right can be extended to q' forcing $\dot{S}(\check{n}) < \check{i}$. $\dashv$

Let $I_0 = S''(C_0)$. We may assume, by strengthening p_0 if needed, that p_0 forces a value for $\dot{S}''(n)$ for each $n \in C_0$. The domain of U is ω, and we may assume that p_0 forces this. $\dot{U}$ names an element of $M_{F(I_0)}$, and we may therefore assume that it is I_0-nice. We work below with $I \supseteq I_0$ (even when this is not mentioned explicitly). For such I, there is a natural I-nice name which is forced equal to $\dot{U}$. Abusing notation we do not distinguish between this name and $\dot{U}$ itself.

Claim 4.4. *For every condition $p \leq p_0$ in $\mathbb{P}$, every finite $A \subseteq \omega$, and every $x \in \omega$, there exists a condition $q \leq p$, a set $D \subseteq \omega$ of size $\leq l$, a finite set $I \subseteq \omega$, and an I-nice name $\dot{\sigma}$ so that:*

1. *q forces $\dot{S}''(\check{C}_0 \cup \check{D}) = \check{I}$.*
2. *Either $D \supseteq C_{Left}$, $D - C_{Left}$ avoids A, and q forces $\dot{\sigma}$ to be an embedding of $\dot{U}$ to the right of $\check{x}$, or $D \supseteq C_{Right}$, $D - C_{Right}$ avoids A, and q forces $\dot{\sigma}$ to be an embedding of $\dot{U}$ to the left of $\check{x}$.*

Proof. This is immediate from the fact that p_0 forces (I) above. Let us only recall that the elements of $M_{F(I)}$ are precisely those that have I-nice names. $\dashv$

The claim holds in the ground model $M = M_{\omega_1^{ck}}$. Its clauses are Δ_1 over the model. (The first condition involves only hereditarily finite sets by Claim 4.3. The statement being forced in the second clause involves only quantifiers over ω, and using the local definability of the forcing relation, the clause can be checked in L_α for any sufficiently closed α with $\dot{U}, \dot{\sigma} \in L_\alpha$.) Since M is admissible, it follows that there is $\theta < \omega_1^{ck}$, so that:

III. For every $p \leq p_0$ in $\mathbb{P}$, every finite $A \subseteq \omega$, and every $x \in \omega$, there are witnesses q, D, I, and $\dot{\sigma}$ for Claim 4.4, with $\dot{\sigma} \in L_\theta$.

We work with such an ordinal θ fixed for the rest of the proof. Increasing θ if needed, we may assume that $\dot{U} \in L_\theta$ and that θ is closed under ordinal addition and multiplication. It follows that, for an I-nice name $\dot{\sigma}$ in L_θ, the forcing formulas "$\dot{\sigma}$ is an embedding of $\dot{U}$ to the right of $\check{x}$" and "$\dot{\sigma}$ is an embedding of $\dot{U}$ to the left of $\check{x}$" are ranked, and have rank less than θ. Thus:

Claim 4.5. *Let $\dot{\sigma}$ be an I-nice name in L_θ. Let $q, q' \in \mathbb{P}$ with q' a θ-absolute I-reduct of q. Then "$\dot{\sigma}$ is an embedding of $\dot{U}$ to the right of $\check{x}$" is forced by q iff it is forced by q', and similarly with embeddings to the left.*

Claim 4.6. *Let $C \subseteq \omega$ be finite, and let $S_1, S_2 \colon C \to \omega$ both be one-to-one. Let $q \in \mathbb{P}$, and suppose that there is $i_0 \in \mathrm{dom}(f_q)$ so that $f_q(i_0)$ has first coordinate* *of the form (n_0, j) with $n_0 \notin C$. Suppose that q forces $\dot{S}(\check{n}) = \check{S}_1(\check{n})$ for each $n \in C$. Suppose that $q \Vdash \varphi(\tau_1, \ldots, \tau_k)$, where each τ_j is ${S_1}''D_j$-nice for some $D_j \subseteq C$. Then there is a condition $q^* \in \mathbb{P}$, and names $\tau_1^*, \ldots, \tau_k^*$, so that:*

- *$T_{q^*} = T_q$ and $h_{q^*} = h_q$. If S_1 and S_2 agree on n then f_{q^*} and f_q agree on $i = S_1(n) = S_2(n)$. If S_1 and S_2 agree on D_j then $\tau_j^* = \tau_j$.*
- *q^* forces $\dot{S}(\check{n}) = \check{S}_2(\check{n})$ for each $n \in C$.*
- *$q^* \Vdash \varphi(\tau_1^*, \ldots, \tau_k^*)$.*
- *If τ_j belongs to L_θ then so does τ_j^*.*
- *τ_j^* is ${S_2}''D_j$-nice.*

Proof. Let $\pi\colon \operatorname{dom}(f_q) \to \omega$ be an injection chosen so that $\pi(S_1(n)) = S_2(n)$ for each $n \in C$, and $\pi(i) > \max({S_2}''C)$ for all other $i \in \operatorname{dom}(f_q)$. Define q' setting $T_{q'} = T_q$, $h_{q'} = h_q$, and $f_{q'}(\pi(i)) = f_q(i)$ for each $i \in \operatorname{dom}(f_q)$. Note that by choice of π, the only elements in $\operatorname{dom}(f_{q'})$ below $\max({S_2}''C)$ are those in ${S_2}''C$. Define τ_j^* to be the name resulting from τ_j by replacing references to $\dot{f}_i$ with references to $\dot{f}(\pi(i))$.

It is easy to check that the last three conditions in the claim hold for $q^* = q'$; this is just a symmetry argument. It is clear also that the first condition holds. We now extend q' to a condition q^* in such a way that the second condition holds.

Define $T_{q^*} = T_{q'}$, $h_{q^*} = h_{q'}$, and $f_{q^*}(i) = f_{q'}(i)$ for $i \in \operatorname{dom}(f_{q'})$. By assumption of the claim, there is $i_0 \in \operatorname{dom}(f_q)$ so that $f_q(i_0)$ has first coordinate of the form (n_0, j) with $n_0 \notin C$. For all $i < \max({S_2}''C)$ which are not in ${S_2}''C$, and hence not in $\operatorname{dom}(f_{q'})$, define $f_{q^*}(i) = f_q(i_0)$. It is now easy to check that q^* forces $\dot{S}(\check{n}) = \check{S}_2(\check{n})$ for each $n \in C$. That $f_{q^*}(S_2(n))$ has first coordinate of the form (n, j) follows from the definitions using the fact that $S_2(n) = \pi(S_1(n))$ and that the first coordinate of $f_q(S_1(n))$ has this form. That $S_2(n)$ is least with this property—indeed unique with this property up to $\max({S_2}''C)$—follows from the definitions and the fact that $n_0 \neq n$. ⊣

Let $\bar{C} = C_0 \cup C_{Left} \cup C_{Right}$, and let $\bar{I} = S''(\bar{C})$. We work inside $M_{F(\bar{I})}$, and aim to show that $Left_l$ and $Right_l$ belong to the model. The model has the tree $T = T^G$, and the branches $f^G(i_n)$ for $n \in \bar{C} = C_0 \cup C_{Left} \cup C_{Right}$. It does not have any other branches, nor does it have the rank function h^G. But it does have the restriction of this function to nodes of ranks $< \theta$, since this restriction is hyperarithmetic in T.

Define $\bar{G}$ to be the set of conditions $p \in \mathbb{P}$ extending p_0 and so that:

- For each $n \in \bar{C}$, p forces $\dot{S}(\check{n}) = S(n)\check{}$.
- $T_p \subseteq T = T^G$.
- $f_p(i) \subseteq f^G(i)$ for each $i \in \bar{I}$.
- If $h^G(t) < \theta$, then $h_p(t) = h^G(t)$. If $h^G(t) \geq \theta$ then $h_p(t) \geq \theta$.

In the last condition, as usual, we adopt the convention that $h_p(t) = \infty > \theta$ for $t \in \operatorname{Dc}(f_p)$, and similarly with G.

Note that $\bar{G}$ belongs to $M_{F(\bar{I})}$. It serves as an approximation in the model to the actual generic G.

Claim 4.7. *Let q_1 and q_2 belong to $\bar{G}$, with $T_{q_1} = T_{q_2}$. Let $I \subseteq \operatorname{dom}(f_{q_2})$. Then there is $q^* \in \bar{G}$ so that:*

- $T_{q^*} = T_{q_1} = T_{q_2}$.
- $f_{q^*}(i) = f_{q_1}(i)$ *for* $i \in \operatorname{dom}(f_{q_1}) - I$.
- $f_{q^*}(i) = f_{q_2}(i)$ *for* $i \in I$.
- h_{q^*} *agrees with* h_{q_1} *on nodes where either one is smaller than* θ. *Similarly with* h_{q_2}.

Proof. Define T_{q^*} and f_{q^*} subject to the first three conditions. Since q_1 and q_2 both belong to $\bar{G}$, h_{q_1} and h_{q_2} both agree with h^G on nodes where any of them is smaller than θ. Let h_{q^*} take their common value on these nodes. On other nodes define h_{q^*} in any arbitrary way that makes $q^* = \langle T_{q^*}, h_{q^*}, f_{q^*}\rangle$ a condition. $\dashv$

Let $*_{\mathit{Left}}(x, A, q)$ denote the statement that there is a set $D \subseteq \omega$ of size $\leq l$, a finite set $I \subseteq \omega$, and an I-nice name $\dot{\sigma}$ in L_θ, so that q forces $\dot{S}''(\check{C}_0 \cup \check{D}) = \check{I}$, $D \supseteq C_{\mathit{Left}}$, $D - C_{\mathit{Left}}$ avoids A, and q forces $\dot{\sigma}$ to be an embedding of $\dot{U}$ to the right of $\check{x}$. Let $*_{\mathit{Right}}(x, A, q)$ denote the corresponding statement with embedding to the left.

Define *LeftAp*, intended to be an approximation to Left_l inside $M_{F(\bar{C})}$, to be the set of x so that for every finite $A \subseteq \omega$, there is a condition $q < p_0$ in $\bar{G}$ so that $*_{\mathit{Left}}(x, A, q)$ holds. Define *RightAp* similarly using $*_{\mathit{Right}}$.

Both *LeftAp* and *RightAp* belong to $M_{F(\bar{C})}$, since they can be obtained from $\bar{G}$ and the restriction of the forcing relation to θ ranked statements. To complete the proof of the lemma, it is enough to show that they are equal to Left_l and Right_l respectively.

Claim 4.8. *$\mathit{Left}_l \subseteq$ LeftAp. Similarly, $\mathit{Right}_l \subseteq$ RightAp.*

Proof. We prove only the first claim. Fix $x \in \mathit{Left}_l$. Fix a finite $A \subseteq \omega$. We have to find a condition $q < p_0 \in \bar{G}$ so that $*_{\mathit{Left}}(x, A, q)$ holds. Since every condition $< p_0$ that belongs to G, and forces sufficient information about $\dot{S}$, belongs also to $\bar{G}$, it is enough to find $q \in G$.

Let R be the set of conditions $q < p_0$ for which there is a set $D \subseteq \omega$ of size $\leq l$, a finite set $I \subseteq \omega$, and an I-nice name $\dot{\sigma}$ in L_θ, so that q forces $\dot{S}''(\check{C}_0 \cup \check{D}) = \check{I}$ and either:

(i) $D \supseteq C_{\mathit{Left}}$, $D - C_{\mathit{Left}}$ avoids A, and q forces $\dot{\sigma}$ to be an embedding of $\dot{U}$ to the right of $\check{x}$; or
(ii) $D \supseteq C_{\mathit{Right}}$, $D - C_{\mathit{Right}}$ avoids A, and q forces $\dot{\sigma}$ to be an embedding of $\dot{U}$ to the left of $\check{x}$.

By condition (III) the set R is dense in $\mathbb{P}$ below p_0. Since G is generic it follows that there is $q \in G \cap R$. Fix such q, and let D, I, and $\dot{\sigma}$ witness that $q \in R$. We are done if we can show that the membership of q in R holds through (i) above, rather than (ii), because D, I, and $\dot{\sigma}$ then witness $*_{\mathit{Left}}(x, A, q)$.

Suppose for contradiction that (ii) holds. Then since $q \in G$, it follows that in $M_{F(S''(C_0 \cup D))}$ there is an embedding of U to the left of x and therefore $x \in \mathit{Right}_l$. This is a contradiction, since x was assumed to be in Left_l, and we saw at the start of the proof of Lemma 4.1 that Left_l and Right_l are disjoint. $\dashv$

Claim 4.9. *LeftAp and RightAp are disjoint.*

Proof. Suppose not. Fix x which belongs to both *LeftAp* and *RightAp*. Let $q_{Left} < p_0$ in $\bar{G}$ be such that $*_{Left}(x, A, q_{Left})$ holds with $A = \bar{C} = C_0 \cup C_{Left} \cup C_{Right}$. Let D_{Left}, I_{Left}, and $\dot{\sigma}_{Left}$ witness this, and let $H_{Left} = D_{Left} - C_{Left}$. Let $q_{Right} < p_0$ in $\bar{G}$ be such that $*_{Right}(x, A, q_{Right})$ holds with $A = \bar{C} \cup H_{Left}$. Let D_{Right}, I_{Right}, and $\dot{\sigma}_{Right}$ witness this, and let $H_{Right} = D_{Right} - C_{Right}$. We then have:

1. q_{Left} forces "$\dot{\sigma}_{Left}$ is an embedding of $\dot{U}$ to the right of $\check{x}$."
2. q_{Right} forces "$\dot{\sigma}_{Right}$ is an embedding of $\dot{U}$ to the left of $\check{x}$."
3. $\dot{\sigma}_{Left}$ is I_{Left}-nice, and q_{Left} forces $\check{I}_{Left} = \dot{S}(\check{C}_0 \cup \check{C}_{Left} \cup \check{H}_{Left})$.
4. $\dot{\sigma}_{Right}$ is I_{Right}-nice, and q_{Right} forces $\check{I}_{Right} = \dot{S}(\check{C}_0 \cup \check{C}_{Right} \cup \check{H}_{Right})$.
5. H_{Left} and H_{Right} are disjoint, and both are disjoint from $\bar{C} = C_0 \cup C_{Left} \cup C_{Right}$.

Using condition (5) we work to combine q_{Left} and q_{Right} into a single condition $q^* < p_0$, that forces there to be embeddings of $\dot{U}$ both to the right of $\check{x}$ and to its left, in $M_{F(\dot{S}''(\check{C}_0 \cup \check{C}_{Left} \cup \check{C}_{Right} \cup \check{H}_{Left} \cup \check{H}_{Right}))}$. Since $C_{Left} \cup C_{Right} \cup H_{Left} \cup H_{Right}$ has size at most $2 \cdot l$, this will contradict the fact that p_0 forces condition (II) above.

We begin with some cosmetic modifications to q_{Left} and q_{Right}. Since both q_{Left} and q_{Right} belong to $\bar{G}$, we may by extending the conditions assume that $T_{q_{Left}} = T_{q_{Right}}$, and that $f_{q_{Left}}(i) = f_{q_{Right}}(i)$ for all $i \in \bar{I}$. Again by extending the conditions we may assume that there is $i \in \text{dom}(f_{q_{Left}})$ so that $f_{q_{Left}}(i)$ has first coordinate (n, j) for $n \notin \bar{C} \cup H_{Left} \cup H_{Right}$, and similarly with $f_{q_{Right}}$. (This is needed for the application of Claim 4.6, which we use next.)

Let S_{Left} be the partial function defined by $S_{Left}(n) = i$ iff q_{Left} forces $\dot{S}(\check{n}) = \check{i}$, and define S_{Right} similarly. Note that S_{Left} and S_{Right} agree on $\bar{C}$, since both q_{Left} and q_{Right} belong to $\bar{G}$. Modifying q_{Left} and σ_{Left} through applications of Claim 4.6, we may assume that:

(i) $\min(S_{Left}{''}H_{Left}) > \max(S_{Left}{''}\bar{C}) = \max(S_{Right}{''}\bar{C})$.

Modifying q_{Right} and σ_{Right} we may assume further that:

(ii) $\min(S_{Right}{''}(H_{Right} \cup H_{Left})) > \max(S_{Left}{''}H_{Left})$.

We are of course using the fact that H_{Left} and H_{Right} are disjoint from $\bar{C}$ in making the modifications. The applications of Claim 4.6 preserve membership of the conditions in $\bar{G}$, because no changes are made to S_{Left} and S_{Right} on $\bar{C}$. They also preserve conditions (1)–(4) for the modified objects, with (modified) $I_{Left} = S_{Left}{''}(C_0 \cup C_{Left} \cup H_{Left})$ and $I_{Right} = S_{Right}{''}(C_0 \cup C_{Right} \cup H_{Right})$.

Let $q^* \in \bar{G}$ be obtained through an application of Claim 4.7 to q_{Left} and q_{Right}, with $f_{q^*}(i) = f_{q_{Left}}(i)$ for $i \in S_{Left}{''}H_{Left}$, and $f_{q^*}(i) = f_{q_{Right}}(i)$ for all other i.

Then q^* is a θ-absolute $S_{Right}{''}(\bar{C} \cup H_{Right})$ reduct of q_{Right}, and a θ-absolute $S_{Left}{''}(\bar{C} \cup H_{Left})$ reduct of q_{Left}. It follows by Claim 4.5 that:

(iii) q^* forces both $\dot{\sigma}_{Left}$ is an embedding of $\dot{U}$ to the right of $\check{x}$, and $\dot{\sigma}_{Right}$ is an embedding of $\dot{U}$ to the left of $\check{x}$.

Let $S^*(n)$ be the partial function defined by $S^*(n) = i$ iff q^* forces $\dot{S}(\check{n}) = \check{i}$. By (i), all i on which f_{q^*} differ from $f_{q_{Right}}$ are greater than $\max(S_{Left}{''}\bar{C}) = \max(S_{Right}{''}\bar{C})$. It follows that:

(iv) $S^*(n) = S_{Right}(n) = S_{Left}(n)$ for all $n \in \bar{C}$.

Let $n \in H_{Left}$. By (ii), the least i such that the first coordinate of $f_{q_{Right}}(i)$ has the form (n, j) is greater than $S_{Left}(n)$. f_{q^*} is equal to $f_{q_{Right}}$ except on $i \in S_{Left}{}''H_{Left}$, where it is equal to $f_{q_{Left}}$. And of course the first coordinate of $f_{q_{Left}}(S_{Left}(n))$ has the form (n, j), as q_{Left} forces $\dot{S}(\check{n}) = \check{S}_{Left}(\check{n})$. It follows that:

(v) For $n \in H_{Left}$, $S^*(n) = S_{Left}(n)$.

By definition of q^*, the only i on which f_{q^*} differs from $f_{q_{Right}}$ are those in $S_{Left}{}''H_{Left}$. For these i, the first coordinate of $f_{q^*}(i)$ is of the form (n, j) with $n \in H_{Left}$. Since H_{Left} and H_{Right} are disjoint, it is therefore not of the form (n, j) with $n \in H_{Right}$. It follows from this that:

(vi) $S^*(n) = S_{Right}(n)$ for $n \in H_{Right}$.

It is important to note here that we made crucial use of the fact that H_{Left} and H_{Right} are *disjoint*. Ultimately it is through Claim 4.2 that we are able to get to a situation where H_{Left} and H_{Right} are indeed disjoint.

By conditions (iv) and (v), q^* forces $\dot{S}''(\check{C}_0 \cup \check{C}_{Left} \cup \check{H}_{Left}) = \check{S}''_{Left}(\check{C}_0 \cup \check{C}_{Left} \cup \check{H}_{Left}) = \check{I}_{Left}$. From this, condition (3), and condition (iii), it follows that q^* forces that there exists an embedding of $\dot{U}$ to the right of $\check{x}$ in $M_{F(\dot{S}''(\check{C}_0 \cup \check{C}_{Left} \cup \check{H}_{Left}))}$.

Similarly q^* forces $\dot{S}''(\check{C}_0 \cup \check{C}_{Right} \cup \check{H}_{Right}) = \check{S}''_{Right}(\check{C}_0 \cup \check{C}_{Right} \cup \check{H}_{Right}) = \check{I}_{Right}$ by conditions (iv) and (vi), and using conditions (4) and (iii) it follows that q^* forces that there exists an embedding of $\dot{U}$ to the left of $\check{x}$ in $M_{F(\dot{S}''(\check{C}_0 \cup \check{C}_{Right} \cup \check{H}_{Right}))}$.

Thus, q^* forces that in the model $M_{F(\dot{S}''(\check{C}_0 \cup \check{C}_{Left} \cup \check{H}_{Left} \cup \check{C}_{Right} \cup \check{H}_{Right}))}$ there are embeddings of $\dot{U}$ both to the right of $\check{x}$ and to its left. But since $C_{Left} \cup H_{Left} \cup C_{Right} \cup H_{Right}$ has size at most $2 \cdot l$, and since q^* extends p_0, this is a contradiction to the fact that p_0 forces (II). $\dashv$

Since $\langle Left_l, Right_l \rangle$ partition the domain of U, the last two claims establish that $LeftAp = Left_l$ and $RightAp = Right_l$. $LeftAp$ and $RightAp$ belong to the model $M_{F(S''\bar{C})} = M_{F(S''(C_0 \cup C_{Left} \cup C_{Right}))} = M_{F_0 \cup E_{Left} \cup E_{Right}}$, and therefore so do $Left_l$ and $Right_l$. Since $E_{Left} \cup E_{Right}$ has size at most $2 \cdot l$, this completes the proof of the lemma. $\dashv$ (Lemma 4.1)

With Lemma 4.1 at hand, we can proceed to prove that INDEC holds in N. In the proof we will use the shift of the lemma to $\mathfrak{H}^*$.

Lemma 4.10. *INDEC holds in N.*

Proof. Suppose not. Then there is $F_0 \in \mathcal{F}^*$ of standard size, and a linear order $U \in M^*_{F_0}$ on ω^*, so that U is scattered in N, indecomposable in N, yet not indecomposable to the left and not indecomposable to the right in N.

For each $l \in \omega^*$ let $Left^*_l$ consist of all $x \in \mathrm{dom}(U)$ so that there is an embedding of U to the right of x in $M^*_{F_0 \cup J}$ for some $J \in \mathcal{F}^*$ with $|J| \leq l$. Define $Right^*_l$ similarly with embeddings to the left. Note that the functions $Left^* = (l \mapsto Left^*_l)$ and $Right^*$ are defined from F_0 in $\mathfrak{H}^*$ in exactly the way that $Left$ and $Right$ were define in $\mathfrak{H}$ at the start of the section.

Claim 4.11. *There is a standard* l *so that each* x *in* $\mathrm{dom}(U)$ *belongs to either* Left^*_l *or* Right^*_l*.*

Proof. Working in $\mathfrak{H}^*$ define L to be the set of all $l \in \omega^*$ so that every x in $\mathrm{dom}(U)$ belongs to either Left^*_l or Right^*_l. From the fact that U is indecomposable in N, and the definition that $N = \bigcup_{J\in\mathcal{F}^*,|J|\in\omega} M^*_{F_0\cup J}$, it follows that every non-standard l belongs to L. Since L belongs to $\mathfrak{H}^*$, it has a minimal element in $\mathfrak{H}^*$. Since L contains all non-standard l, this minimal element must be standard. ⊣

We continue to work with l given by the last claim.

Claim 4.12. *No* x *in* $\mathrm{dom}(U)$ *belongs to both* $\mathit{Left}^*_{2\cdot l}$ *and* $\mathit{Right}^*_{2\cdot l}$*.*

Proof. Suppose for contradiction that x belongs to both $\mathit{Left}^*_{2\cdot l}$ and $\mathit{Right}^*_{2\cdot l}$. Then in $\bigcup_{J\in\mathcal{F}^*,|J|\leq 2\cdot l} M^*_{F_0\cup J}$ there are embeddings of U both to the right of x and to the left. Since l is standard, $2\cdot l$ is standard, and therefore $\bigcup_{J\in\mathcal{F}^*,|J|\leq 2\cdot l} M^*_{F_0\cup J}$ is contained in N. Hence in N there are embeddings of U both to the right of x and to the left. But from this, it follows by standard arguments (see [4, Lemma 1.17] or the last paragraph in the proof of [6, Claim 2.9]) that U is not scattered in N, contradiction. ⊣

Lemma 4.1, which we proved above, holds in the standard structure $\mathfrak{H}$. By elementarity of $j\colon \mathfrak{H} \to \mathfrak{H}^*$, a parallel lemma holds in $\mathfrak{H}^*$. The last two claims establish the assumptions of the parallel lemma, applied to U and l. It follows by the lemma that $\langle \mathit{Left}^*_l, \mathit{Right}^*_l\rangle$ is a gap in U, and belongs to $\bigcup_{J\in\mathcal{F}^*,|J|\leq 2\cdot l} M^*_{F_0\cup J}$. In particular the gap belongs to N. Now standard arguments lead to a contradiction: Since U is indecomposable in N, it must embed into either Left^*_l or Right^*_l, in N. Suppose for definitiveness that σ is an embedding of U into Left^*_l in N. Let x belong to Right^*_l, so that the range of σ is to the left of x. Then σ^2 embeds U to the left of $\sigma(x)$, and since $\sigma(x)$ belongs to Left^*_l, there is also an embedding of U to the right of $\sigma(x)$ in N. From embeddings of U both to the left and right of $\sigma(x)$ it follows that U is not scattered. ⊣ (Lemma 4.10)

REFERENCES

[1] Denis R. Hirschfeldt and Richard A. Shore, *Combinatorial principles weaker than Ramsey's theorem for pairs*, ***J. Symbolic Logic***, vol. 72 (2007), no. 1, pp. 171–206.

[2] Pierre Jullien, *Contribution à l'étude des types d'ordre dispersés*, ***Ph.D. thesis***, Marseille, 1969.

[3] MedYahya Ould MedSalem and Kazuyuki Tanaka, Δ^0_3*-determinacy, comprehension and induction*, ***J. Symbolic Logic***, vol. 72 (2007), no. 2, pp. 452–462.

[4] Antonio Montalbán, *Indecomposable linear orderings and hyperarithmetic analysis*, ***J. Math. Log.***, vol. 6 (2006), no. 1, pp. 89–120.

[5] ———, *On the equimorphism types of linear orderings*, ***Bull. Symbolic Logic***, vol. 13 (2007), no. 1, pp. 71–99.

[6] Itay Neeman, *The strength of jullien's indecomposability theorem*, ***J. Math. Log.***, vol. 8 (2009), no. 1, pp. 93–119.

[7] Takako Nemoto, *Determinacy of Wadge classes and subsystems of second order arithmetic*, ***MLQ Math. Log. Q.***, vol. 55 (2009), no. 2, pp. 154–176.

[8] Stephen G. Simpson, Σ^1_1 *and* Π^1_1 *transfinite induction*, ***Logic Colloquium '80 (Prague, 1980)***, Stud. Logic Foundations Math., vol. 108, North-Holland, Amsterdam, 1982, pp. 239–253.

[9] ———, ***Subsystems of second order arithmetic***, second ed., Perspectives in Logic, Cambridge University Press, Cambridge, 2009.

[10] John R. Steel, *Determinateness and subsystems of analysis*, ***Ph.D. thesis***, University of California, Berkeley, 1977.

[11] ———, *Forcing with tagged trees*, ***Ann. Math. Logic***, vol. 15 (1978), no. 1, pp. 55–74.

[12] Robert Van Wesep, *Subsystems of second-order arithmetic, and descriptive set theory under the axiom of determinateness*, ***Ph.D. thesis***, University of California, Berkeley, 1977.

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