HAPPY AND MAD FAMILIES IN $L(\mathbb{R})$

ITAY NEEMAN AND ZACH NORWOOD

Abstract. We prove that, in the choiceless Solovay model, every set of reals is H-Ramsey for every happy family H that also belongs to the Solovay model. This gives a new proof of Törnquist's recent theorem that there are no infinite mad families in the Solovay model. We also investigate happy families and mad families under determinacy, applying a generic absoluteness result to prove that there are no infinite mad families under AD^+.

1. Introduction

This paper joins a long line of research inspired by Mathias's celebrated 1977 paper *Happy Families* [14]. We focus on the existence of definable mad families, in which there has recently been a surge of interest. Mathias showed that there are no mad families in the Solovay-type model obtained by collapsing a Mahlo cardinal, and he asked whether there are mad families in the traditional Solovay model, obtained from only an inaccessible cardinal. That question remained open until Asger Törnquist answered it positively in 2015. Törnquist's proof is entirely combinatorial and makes no mention of the happy families Mathias used in his original argument. One purpose of this paper is to prove a strengthening of Törnquist's theorem (Theorem 16) that applies to Mathias's happy families. In contrast to Törnquist's proof, our proof is very similar to arguments in Mathias's original paper [14].

One of set theory's major projects is to show, often using strong assumptions, that pathological sets of reals cannot be easily defined. Theorems in this vein might assert that a specific type of pathological set cannot be Borel, or cannot be projective or ordinal-definable from reals in the presence of large cardinals, or cannot exist under AD. Some theorems of this sort have been established for infinite mad families; Törnquist asked whether AD implies that there is no infinite mad family. Mathias argues for a strong connection between the Ramsey property and the existence of mad families, so Törnquist's

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question seems closely related to the longstanding open question whether every set of reals must have the Ramsey property under AD. In Section 6 we give a partial answer (Theorem 28), confirming that there are no mad families under AD^+, a well-studied strengthening of AD. It is unknown whether AD and AD^+ are equivalent. In $L(\mathbb{R})$, AD implies AD^+.

2. Preliminaries

Definition 1. Two sets are **almost disjoint** if their intersection is finite. An **almost-disjoint family** is a family $A \subseteq [\omega]^\omega$ such that any two different members of A are almost disjoint. A **mad family** is an almost disjoint family that is maximal under inclusion. Equivalently, an almost-disjoint family A is maximal if and only if every $x \in [\omega]^\omega$ has infinite intersection with at least one member of A.

Remarks.

- According to our definition, any partition of ω into finitely many pieces is a mad family, but we will be interested in infinite mad families.
- Zorn's lemma implies that every almost-disjoint family extends to a mad family. By applying this fact to an infinite partition of ω, one obtains an infinite mad family. The use of AC in this argument is unavoidable, by Theorem 7 (Mathias).
- A straightforward diagonalization argument shows that no infinite mad family can be countable.

Crucial to the study of mad families has been Mathias's connection between mad families and the Ramsey property. In its classical form, Ramsey's theorem states that every coloring $\chi\colon [\omega]^2 \to 2$ has an infinite monochromatic set, a set $z \in [\omega]^\omega$ such that χ is constant on $[z]^2$. It's natural to seek mild conditions on a family $H \subseteq [\omega]^\omega$ sufficient to guarantee that the monochromatic set z could always be chosen to belong to H. The problem certainly isn't made easier by requiring H to be upward-closed under $\subseteq$, and a short exercise shows that the complement of H must be closed under finite unions. Reflecting on the proof of Ramsey's theorem reveals that one more condition on H would be enough for H to have at least one monochromatic set for every 2-dimensional coloring.

Definition 2. If $y_0 \supseteq y_1 \supseteq y_2 \supseteq \cdots$ is a decreasing sequence of subsets of ω, then we call a set $y_\infty \in [\omega]^\omega$ a **diagonalization** of the sequence $\langle y_n : n < \omega \rangle$ iff $f(n{+}1) \in y_{f(n)}$ for every $n < \omega$, where $f\colon \omega \to \omega$ is the increasing enumeration of y_∞.

A set $H \subseteq [\omega]^\omega$ is called a **happy family** if it contains every cofinite set and satisfies the following three conditions:

(i) (upward-closure) If $x \in H$ and $y \supseteq x$, then $y \in H$ too.

(ii) (pigeonhole) If $x_0 \cup x_1 \cup \cdots \cup x_n \in H$, then $x_k \in H$ for some k.
(iii) (selectivity) Every decreasing sequence $y_0 \supseteq y_1 \supseteq \cdots$ of members of H has a diagonalization y_∞ in H.

Remarks.

- A brief meditation on this definition reveals that if y_∞ diagonalizes $\vec{y}$, then so does every infinite subset of y_∞.
- Clauses (i) & (ii) simply assert that the complement of H, together with all finite subsets of ω, forms an ideal on ω, so a family satisfying (i) & (ii) is called a **coideal**. Accordingly, happy families are often called **selective coideals**.

Noting that happy families have monochromatic sets for all two-dimensional colorings, Mathias began to investigate when they have monochromatic sets for infinite-dimensional colorings.

Definition 3. If H is a coideal (typically a happy family) and $X \subseteq [\omega]^\omega$ is a set of reals, then we say X is H-**Ramsey** if there is in H a monochromatic set for the coloring associated to X, that is, if there is $M \in H$ such that $[M]^\omega \subseteq X$ or $[M]^\omega \subseteq X^c$.

The following fact explains the connection and the terminology.

Proposition 4 (Mathias [14])**.** If $A \subseteq [\omega]^\omega$ is an infinite almost-disjoint family and $I(A)$ is the ideal generated by A, then $[\omega]^\omega \smallsetminus I(A)$ is a happy family.

If A is an infinite mad family, then a set belongs to $[\omega]^\omega \smallsetminus I(A)$ if and only if it has infinite intersection with infinitely many members of A.

Mathias provides two other prototypes of happy families.

- $[\omega]^\omega$ is a happy family.
- Every Ramsey ultrafilter is a happy family.

Notice that a set $X \subseteq [\omega]^\omega$ has the classical Ramsey property if and only if X is $[\omega]^\omega$-Ramsey, in our terminology.

This observation follows immediately from the definitions, but it deserves special emphasis.

Lemma 5. An infinite almost-disjoint family A is a mad family if and only if $I(A)^c$ is *not* $I(A)^c$-Ramsey.

Mathias used this observation to prove several results about the non-existence of definable mad families. His strategy, which we will emulate, is first to prove that definable (suitably understood) sets are H-Ramsey for every happy family H. Then, if a mad family A were definable, its corresponding happy family $I(A)$ would be too. These two facts are incompatible, by Lemma 5.

In his study of happy families in the Solovay model, Mathias established two theorems to support his conjecture that the Solovay model has no infinite mad families.

Theorem 6 (Mathias [14]). In the Solovay model, every set $x \in [\omega]^\omega$ has the Ramsey property, i.e., is $[\omega]^\omega$-Ramsey.

Theorem 7 (Mathias [14]). Suppose that G is generic over V for the Levy collapse of a Mahlo cardinal. In $V[G]$, every set $x \subseteq [\omega]^\omega$ that belongs to $L(\mathbb{R})$ is H-Ramsey for every happy family $H \in V[G]$. Consequently, there are no infinite mad families in $L(\mathbb{R})^{V[G]}$.

To prove that there are no infinite mad families in the Solovay model (obtained by collapsing an inaccessible cardinal), one might hope to imitate Mathias's proof of Theorem 7, but Eisworth showed that the conclusion of Theorem 7 is too strong for only an inaccessible.

Theorem 8 (Eisworth [6]). Suppose that CH holds and that every set $x \in [\omega]^\omega$ that belongs to $L(\mathbb{R})$ is H-Ramsey for every happy family H. Then $\aleph_1$ is Mahlo in L.

Eisworth's theorem doesn't seem to leave any room for proving a version of Theorem 7 that uses only an inaccessible, and Törnquist's purely combinatorial proof seems to confirm this suspicion. But a closer examination of Eisworth's proof suggests an alternative approach. Assuming that $\aleph_1$ is not Mahlo in L, Eisworth builds Ramsey ultrafilters U such that not every set in $L(\mathbb{R})$ can be U-Ramsey. These ultrafilters certainly aren't definable, in the sense that they won't belong to $L(\mathbb{R})$. Our chief innovation on this question is to consider only definable happy families.

In Section 4 we will prove:

Theorem. If G is generic over V for the Levy collapse of an inaccessible cardinal, then in $V[G]$ every set $X \subseteq [\omega]^\omega$ that belongs to $L(\mathbb{R})$ is H-Ramsey for every happy family $H \in L(\mathbb{R})$.

Using Lemma 5, we obtain as a corollary Törnquist's theorem [25] that there are no infinite mad families in $L(\mathbb{R})^{V[G]}$, the Solovay model. Törnquist was the first to show that the nonexistence of infinite mad families is consistent relative to an inaccessible cardinal. Recently, Horowitz and Shelah [10] have shown that the theory $\mathsf{ZF}+$"there is no infinite mad family" is in fact equiconsistent with ZFC.

The main difficulty in adapting Mathias's arguments to our needs is that, while a coideal H may be a happy family in $V[G]$, its intersection with an intermediate extension $V[G \upharpoonright \alpha] \subseteq V[G]$ is unlikely to be closed under diagonalizations. For this reason, there may not be (in $V[G \upharpoonright \alpha]$) any Ramsey ultrafilters $U \subseteq H$ to guide Mathias forcing over M. Mathias forcing guided by an ultrafilter $U \subseteq H$ is the primary tool Mathias uses to prove that a set is H-Ramsey, but it has the Mathias and Prikry properties if and only if U is Ramsey.[1] Consequently, we will need a different poset to serve the same purpose in the absence of Ramsey ultrafilters.

3. Diagonalization forcing

We use standard notation for sequences, so, for instance, $\mathrm{lh}(s)$ denotes the length of s (which, for us, will always be the same as its domain). It will be convenient to refer to the last term of s by $\mathrm{last}(s)$, i.e., $\mathrm{last}(s) = s(\mathrm{lh}(s) - 1)$.

A set y_∞ **almost diagonalizes** a $\subseteq$-decreasing sequence $\vec{y} = \langle y_0, y_1, \dots \rangle$ if there is some $n < \omega$ such that $y_\infty \setminus n$ diagonalizes $\vec{y}$. Just as Mathias forcing guided by an ultrafilter generically adds a set almost-included in every measure-one set, we need a poset that generically adds a set almost-diagonalizing every decreasing sequence of measure-one sets.

Fix a nonprincipal ultrafilter U on ω. (Crucially, we do not assume that U is a Ramsey ultrafilter.)

Definition 9. Conditions in $\mathbb{P}_U$ are pairs $\langle s, \vec{y} \rangle$ such that

(i) $s \in \omega^{<\omega}$ is a finite, strictly increasing sequence,
(ii) $\vec{y} = (y_0, y_1, \dots)$ is an $\subseteq$-decreasing sequence of sets, each in U.

Say $\langle t, \vec{z} \rangle \leq \langle s, \vec{y} \rangle$ iff

(iii) s is an initial segment of t,
(iv) $z_n \subseteq y_n$ for every $n \geq \mathrm{last}(t)$, and
(v) $t(n+1) \in y_{t(n)}$ for every $n \in [\mathrm{lh}(s), \mathrm{lh}(t) - 1)$.

If $p = \langle s, \vec{y} \rangle$, then we will often write $\mathrm{stem}(p) = s$. Following convention, we write $q \leq_0 p$ to mean that $q \leq p$ and $\mathrm{stem}(q) = \mathrm{stem}(p)$. It will also be convenient to write $\vec{z} \leq \vec{y}$ to mean that $z_n \subseteq y_n$ for every $n < \omega$.

A generic filter $G \subseteq \mathbb{P}_U$ gives rise in a natural way to a generic real g, the range of the union of all the stems of conditions in G. Clause (v) of the definition ensures that a condition $\langle s, \vec{y} \rangle$ forces the generic real to almost-diagonalize $\vec{y}$.

There is a natural Ellentuck-type neighborhood associated to a condition $\langle s, \vec{y} \rangle$. Let's first define a version for finite sequences:

$$[\![s, \vec{y}]\!] := \{ t \in \omega^{<\omega} : t \subseteq s, \text{ or } s \subseteq t \text{ and } t(n+1) \in y_{t(n)} \text{ for every } n \geq \mathrm{lh}(s) \}.$$

[1] This follows from [7, Theorems 2.4, 4.1]. See [14, Theorem 2.10] for the Mathias property, and for a related result see [11, Theorem 1.20].

Informally, $[\![s, \vec{y}]\!]$ comprises all end-extensions of s that diagonalize $\vec{y}$ beyond s. Now define $[s, \vec{y}]$ to be the set of $x \in [\omega]^\omega$ such that s enumerates an initial segment of x and every initial segment of x belongs to $[\![s, \vec{y}]\!]$.

Remarks.

- $[\![s, \vec{y}]\!]$ is a subtree of $\omega^{<\omega}$, and $[s, \vec{y}]$ is the set of its branches.
- $[\emptyset, \vec{y}]$ is exactly the set of diagonalizations of $\vec{y}$.
- If $p = \langle s, \vec{y}\rangle$ is a condition and $t \in [\![s, \vec{y}]\!]$, then $[\![t, \vec{y}]\!]$ is a condition that extends p.

We have described a natural identification of conditions $\langle s, \vec{y}\rangle$ with subtrees $[\![s, \vec{y}]\!]$ of $\omega^{<\omega}$. A fusion argument (see the proof of Lemma 10) shows that these trees form a dense subposet of Laver forcing guided by U, for which many of the results in this section were proved by Judah and Shelah [11]. Nonetheless, our presentation of the poset allows us to characterize $\mathbb{P}_U$-genericity by a proof very similar to Mathias's original characterization of genericity for Mathias forcing. We will therefore reprove some of the results of [11] and translate them into our language.

Lemma 10. Let $O \subseteq [\omega]^\omega$ be an open set. There is a condition $\langle \emptyset, \vec{y}\rangle \in \mathbb{P}_U$ such that either $[\emptyset, \vec{y}] \subseteq O$ or $[\emptyset, \vec{y}] \subseteq O^c$.

Proof. Consider the following game. Players I & II play to create a sequence $\langle y_0, k_0, y_1, k_1, \dots\rangle$

I	y_0	$\supseteq$	y_1	$\supseteq$	y_2	$\supseteq$	$\cdots$
II		$k_0 \in y_0$		$k_1 \in y_1$		$k_2 \in y_2$	$\cdots$

such that, for every $n < \omega$,

- $y_n \in U$,
- $y_{n+1} \subseteq y_n$,
- $k_n \in y_n$, and
- $k_n < k_{n+1}$.

At the end of the game, Player I wins iff $\{k_0, k_1, \dots\} \in O$. By the determinacy of open games, one of the two players has a winning strategy σ.

Suppose first that σ is a winning strategy for Player I. Define

$$y_n = \bigcap \{\sigma(t) : t \text{ a sequence of II's moves consistent with } \sigma, \text{ last}(t) \leq n\}.$$

Any diagonalization x of the sequence $\vec{y}$ is a sequence of II's moves in a complete run of the game consistent with σ, and thus $x \in O$. That is, $[\emptyset, \vec{y}] \subseteq O$.

Suppose that, on the other hand, σ is a winning strategy for Player II.

Claim. The set $z_\emptyset$ of $k < \omega$ for which there is $w_k \in U$ satisfying $k = \sigma(\langle w_k \rangle)$ belongs to U.

Proof of Claim. Suppose not, so that $v := \{k : (\forall w \in U)\, k \neq \sigma(\langle w \rangle)\} \in U$. But this means that v is a valid first move for Player I, and then $\sigma(v) \in v$. This is impossible. ∎

We can repeat the argument of the Claim to obtain a tree $T \subseteq \omega^{<\omega}$ and, for each $t \in T$, sets $w_t, z_t \in U$ satisfying

(i) every $t \in T$ has $\mathrm{succ}_T(t) \in U$;
(ii) $w_t \subseteq w_s$ whenever $s \subseteq t$;
(iii) $z_t = \big\{k < \omega : k = \sigma\big(\big\langle w_\emptyset, \dots, w_t, w_{t^\frown\langle k \rangle}\big\rangle\big)\big\}$.

Put $y_0 = z_\emptyset$ and inductively define $y_n = y_{n-1} \cap \bigcap \{z_t : \mathrm{last}(t) = n,\, t \in T\}$ Clause (iii) and the choice of y_n imply that every $x \in [\emptyset, \vec{y}]$ is a sequence of II's plays consistent with the strategy σ. We conclude that $[\emptyset, \vec{y}] \subseteq O^c$, as desired. ∎

Definition 11. Say that a condition $\langle s, \vec{y} \rangle$ **captures** a dense set $D \subseteq \mathbb{P}_U$ if every $x \in [s, \vec{y}]$ has an initial segment whose enumeration $t \in [\![s, \vec{y}]\!]$ satisfies $\langle t, \vec{y} \rangle \in D$.

Compare with [14, Definition 2.2]. Informally, a condition captures D if to get into D one need only extend the *stem* of the condition, and the set of extensions that work is very rich, containing an initial segment of every diagonalization of $\vec{y}$.

If $\langle s, \vec{y} \rangle$ captures D, then $\langle s, \vec{y} \rangle$ still captures D in any outer model. To see this, observe that

$$\{x \in [\omega]^\omega : (\exists t,\ \text{an initial segment of } x)\ \langle t, \vec{y} \rangle \in D\}$$

is open in the product topology (since D is open in the forcing topology) and apply $\mathbf{\Pi}^1_1$-absoluteness. We will use this fact in the proof of Lemma 14.

Lemma 12. Let $p \in \mathbb{P}_U$ be a condition.

(a) For every dense open set $D \subseteq \mathbb{P}_U$, there is a condition $q \leq_0 p$ that captures D.
(b) For every countable family $\{D_n : n < \omega\}$ of dense open subsets of $\mathbb{P}_U$, there is a condition $q \leq_0 p$ that captures all of the D_n.

Proof. (a) Assume for convenience that p is the empty condition; the proof of the more general result is similar. For every increasing sequence $t \in \omega^{<\omega}$ choose (if possible) a sequence $\vec{y}^{\,t} \in U^\omega$ such that $\langle t, \vec{y}^{\,t} \rangle \in D$. (If no such sequence exists, just put $\vec{y}^{\,t} = \langle \omega, \omega, \dots \rangle$.) Construct a sequence $\vec{y}^{\,\infty}$ by defining

$$y^\infty_n = \bigcap_{\mathrm{last}(t) \leq n} y^t_n.$$

By the choice of $\vec{y}^t$, if there is any $\vec{z} \in U^\omega$ such that $\langle t, \vec{z}\rangle \in D$, then $\langle t, \vec{y}^\infty\rangle \in D$. (This uses that D is open.) Consider

$$O = \{x \in [\omega]^\omega : (\exists t, \text{ an initial segment of } x)\ \langle t, \vec{y}^\infty\rangle \in D\},$$

an open subset of $[\omega]^\omega$. By Lemma 10, there is a $\vec{y}^*$ (which can be taken to satisfy $\vec{y}^* \leq \vec{y}^\infty$) in U^ω such that either $[\emptyset, \vec{y}^*] \subseteq O$ or $[\emptyset, \vec{y}^*] \subseteq O^c$.

If $[\emptyset, \vec{y}^*] \subseteq O$, then $\langle \emptyset, \vec{y}^*\rangle$ captures D, and we're done.

Suppose that $[\emptyset, \vec{y}^*] \subseteq O^c$. Since D is dense, there is a condition $\langle s, \vec{z}\rangle \in D$ satisfying $\langle s, \vec{z}\rangle \leq \langle \emptyset, \vec{y}^*\rangle$. By choice of $\vec{y}^\infty$, $\langle s, \vec{y}^\infty\rangle \in D$, so $\langle s, \vec{y}^*\rangle \in D$ since D is open. But now $[s, \vec{y}^*] \subseteq O$, and $[s, \vec{y}^*]$ is nonempty, a contradiction.

Part (b) can be proved similarly to part (a), with an added fusion argument. Again assume for convenience that p is the empty condition. Define $\vec{y}^t$ so that for any $i \leq \text{last}(t)$, if there is $\vec{z}$ with $\langle t, \vec{z}\rangle \in D_i$, then $\langle t, \vec{y}^t\rangle \in D_i$. Set $y^\infty_n = \bigcap_{\text{last}(t)\leq n} y^t_n$.

Part (a) shows that there exists $\langle \emptyset, \vec{y}^{*n}\rangle \leq \langle \emptyset, \vec{y}^\infty\rangle$ that captures D_n. Indeed for any fixed $z_0, \ldots, z_{n-1} \in U$ one can find such $\vec{y}^{*n}$ with $y^{*n}_i = z_i$ for $i < n$. To see this, for any r of length $\leq n$ with $r(i+1) \in z_{r(i)}$ run the construction of $\vec{y}^*$ in part (a) below $\langle r, \vec{y}^\infty\rangle$. The option $[r, \vec{y}^*] \subseteq O^c$ is still impossible, because of the current definition of $\vec{y}^\infty$. So the construction produces $\vec{y}^{*r}$ with $[r, \vec{y}^{*r}] \subseteq O$. Set y^{*n}_i for $i \geq n$ to be $\bigcap_{\text{last}(r)\leq i} y^{*r}_i$, and $y^{*n}_i = z_i$ for $i < n$. Then for any $x \in [\emptyset, \vec{y}^{*n}]$, setting r to be the first initial segment of x with $\text{last}(r) \geq n$ we have $x \in [r, \vec{y}^{*r}] \subseteq O$.

In particular we can obtain $\vec{y}^{*n}$ so that for all $i < n$, $y^{*n}_i = y^{*(n-1)}_i$. Set $y^*_n = \bigcap_{m\leq n} y^{*m}_n$. Then $\langle \emptyset, \vec{y}^*\rangle$ captures all D_n. ∎

Lemma 13 (Prikry property)**.** If $p = \langle s, \vec{y}\rangle \in \mathbb{P}_U$ is a condition and σ is a sentence in the forcing language, then there is a condition $p^* \leq_0 p$ that either forces σ or forces $\neg\sigma$.

Proof. Assume for convenience that $s = \emptyset$; the proof of the general case is similar. We can again assume, by shrinking the y_ns if necessary, that every sequence in $[\![\emptyset, \vec{y}]\!]$ is strictly increasing. Let D be the (dense, open) set of conditions that decide σ:

$$D = \{q \in \mathbb{P}_U : q \Vdash \sigma \text{ or } q \Vdash \neg\sigma\}.$$

By Lemma 12 there is a condition $\langle \emptyset, \vec{y}^*\rangle \leq \langle \emptyset, \vec{y}\rangle$ that captures D. Now define

$$B = \{x \in [\omega]^\omega : \langle t, \vec{y}^*\rangle \Vdash \sigma \text{ for some initial segment } t \text{ of } x\}$$

and

$$C = \{x \in [\omega]^\omega : \langle t, \vec{y}^*\rangle \Vdash \neg\sigma \text{ for some initial segment } t \text{ of } x\}.$$

Notice that both B and C are open subsets of $[\omega]^\omega$ and that $[\emptyset, \vec{y}^*] \subseteq B \cup C$. Apply Lemma 10 to get a condition $\langle \emptyset, \vec{y}^{**} \rangle \leq \langle \emptyset, \vec{y}^* \rangle$ such that $[\emptyset, \vec{y}^{**}] \subseteq B$ or $[\emptyset, \vec{y}^{**}] \subseteq C$.

We will show that $\langle \emptyset, \vec{y}^{**} \rangle \Vdash \sigma$ in the case when $[\emptyset, \vec{y}^{**}] \subseteq B$. (The other case is similar.) Suppose for a contradiction that we can find a condition $\langle s, \vec{z} \rangle \leq \langle \emptyset, \vec{y}^{**} \rangle$ that forces $\neg\sigma$. Choose any $x \in [s, \vec{z}] \subseteq [\emptyset, \vec{y}^{**}] \subseteq B$ and let t be an initial segment of x such that $\langle t, \vec{y} \rangle \Vdash \sigma$. This fact contradicts our assumption that $\langle s, \vec{z} \rangle \Vdash \neg\sigma$; indeed, $\langle t, \vec{y} \rangle$ and $\langle s, \vec{z} \rangle$ are compatible conditions, since one of s and t is an initial segment of the other. ∎

Lemma 14 (Characterization of $\mathbb{P}_U$-genericity, Judah–Shelah)**.** Let $V \subseteq W$ be models of set theory, let $U \in V$ be an ultrafilter in V, and let $\mathbb{P}_U \in V$ be the associated diagonalization forcing. For a real $g \in [\omega]^\omega \cap W$, the following are equivalent.

(a) g is a $\mathbb{P}_U$-generic real over V;
(b) for every decreasing sequence $\vec{y} \in V$ of sets $y_n \in U$, g almost-diagonalizes $\vec{y}$.

Proof. First, we assume that g is a $\mathbb{P}_U$-generic real over V, by which we mean that the set

$$G = \{\langle s, \vec{y} \rangle \in \mathbb{P}_U : g \in [s, \vec{y}]\}$$

is a generic filter. Fix a sequence $\vec{y}$ of sets $y_n \in U$ that belongs to the ground model V. The set D of conditions $\langle t, \vec{z} \rangle$ such that $\vec{z} \leq \vec{y}$ is a dense set in V, so genericity provides a condition $\langle t, \vec{z} \rangle \in G \cap D$. There is $n < \omega$ such that t is the enumeration of $g \cap n$. It follows that $g \smallsetminus n$ diagonalizes $\vec{z}$ and therefore also $\vec{y}$.

For the converse, suppose that (b) holds of g and let $D \in V$ be a dense subset of $\mathbb{P}_U$. We'll need a way to relate a condition p to a similar condition with a different stem. For any condition $p = \langle s, \vec{y} \rangle$ and any increasing sequence $a \in \omega^{<\omega}$, define $\mathrm{copy}_a(p)$ to be the condition $\langle a^\frown s, \vec{y} \rangle$. Likewise, define $\mathrm{copy}_a(D) = \{\langle t, \vec{z} \rangle : \mathrm{copy}_a(\langle t, \vec{z} \rangle) \in D\}$. Observe that $\mathrm{copy}_a(D)$ is a dense open subset of $\mathbb{P}_U$.

Apply Lemma 12 to the empty condition to get $p = \langle \emptyset, \vec{y} \rangle$ that captures the dense open set $\mathrm{copy}_a(D)$ for every increasing $a \in \omega^{<\omega}$. By assumption, there is $n < \omega$ such that $g \smallsetminus n$ diagonalizes $\vec{y}$. The fact that p captures $\mathrm{copy}_{g \cap n}(D)$ means that there is $k \geq n$ such that $\langle g \cap [n, k), \vec{y} \rangle \in \mathrm{copy}_{g \cap n}(D)$. (Here we are using the absoluteness of capturing from V to W; $g \smallsetminus n$ does not belong to V.) In other words, $\langle g \cap k, \vec{y} \rangle \in D$. And $g \in [g \cap k, \vec{y}]$, so we have shown the filter determined by g to be generic, as desired. ∎

The argument at the end of the proof of Lemma 14 patches a small error in the proof of [11], Lemma 1.12.

Corollary 15 (Mathias property)**.** If g is $\mathbb{P}_U$-generic over a model M and $g' \subseteq g$, then g' is $\mathbb{P}_U$-generic over M too.

4. The Solovay model

Let κ be an inaccessible cardinal, and let G be generic over V for the Levy collapse $\mathrm{Coll}(\omega, <\kappa)$. For our purposes, the Solovay model is $L(\mathbb{R})$ of the generic extension $V[G]$. In this section we will prove the following theorem.

Theorem 16. If $X \in L(\mathbb{R})^{V[G]}$ is a subset of $[\omega]^\omega$ and $H \in L(\mathbb{R})^{V[G]}$ is a happy family, then X is H-Ramsey.

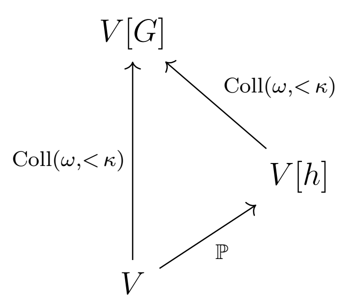
Remarks.

(i) When we say that "H is a happy family" and "X is H-Ramsey," we mean that these statements are true in $V[G]$, though they are certainly absolute between $L(\mathbb{R})^{V[G]}$ and $V[G]$.

(ii) Contrast this theorem with Mathias's theorem 7: Using a Mahlo cardinal, Mathias constructs a model in which all sets in $L(\mathbb{R})$ are H-Ramsey for *every* happy family H, whereas using only an inaccessible cardinal we get this strong Ramsey property for only the happy families that belong to $L(\mathbb{R})$.

We'll need the following lemma, an essential ingredient in Solovay's proof (see [20]) that all sets in the Solovay model are Lebesgue-measurable.

Lemma 17 (Factor Lemma, Solovay)**.** Suppose that κ is an inaccessible cardinal and that $\mathbb{P}$ is a poset of size $< \kappa$. Let G be generic over V for the Levy collapse $\mathrm{Coll}(\omega, <\kappa)$. If in $V[G]$ there is a filter $h \subseteq \mathbb{P}$ that is $\mathbb{P}$-generic over V, then there is $G^* \in V[G]$ that is $\mathrm{Coll}(\omega, <\kappa)$-generic over $V[h]$ and such that $V[h][G^*] = V[G]$.



Proof of Theorem 16. Let $H \in L(\mathbb{R})^{V[G]}$ be a happy family, and let $X \subseteq [\omega]^\omega$ be a set of reals that belongs to $L(\mathbb{R})^{V[G]}$. Every set in $L(\mathbb{R})$ is ordinal-definable from a real, so there are a formula $\phi = \phi(x, y, w)$, a real $r \in V[G]$, and a sequence α of ordinals such that

$$x \in X \text{ iff } V[G] \models \phi[x, r, \alpha].$$

Recall that κ is inaccessible and G is generic over V for the Levy collapse $\mathrm{Coll}(\omega, <\kappa)$. It follows from the chain-condition of the Levy collapse that there is an ordinal $\beta < \kappa$ such that the real parameter r belongs to $V[G \restriction \beta]$. Let $\phi^* = \phi^*(x, y, w)$ be the natural formula satisfying the equivalence

$$\phi^*(x, y, w) \iff \emptyset \Vdash_{\mathrm{Coll}(\omega, <\kappa)} \phi(x, \check{y}, \check{w}).$$

And let

$$\overline{X} = \{x \in [\omega]^\omega \cap V[G \restriction \beta] : V[G \restriction \beta] \models \phi^*[x, r, \alpha]\}.$$

Notice that $\overline{X}$ belongs to $V[G \restriction \beta]$; in fact, by the symmetry of the collapse, it is equal to $X \cap V[G \restriction \beta]$. Indeed, for $x \in [\omega]^\omega \cap V[G \restriction \beta]$,

$$\begin{aligned} x \in X &\iff V[G] \models \phi[x, r, \alpha] \\ &\iff \emptyset \Vdash_{\mathrm{Coll}(\omega, <\kappa)} \phi[x, r, \alpha] \\ &\iff x \in \overline{X}. \end{aligned}$$

By similar arguments, $\overline{H} := H \cap V[G \restriction \beta]$ belongs to $V[G \restriction \beta]$. Certainly $\overline{H}$ need not be closed under diagonalizations in $V[G \restriction \beta]$, so we cannot expect to find in $V[G \restriction \beta]$ a Ramsey ultrafilter $U \subseteq \overline{H}$ to guide Mathias forcing. It is for this reason that we will use the diagonalization forcing, since — unlike Mathias forcing — it has the Mathias and Prikry properties even when guided by a non-Ramsey ultrafilter.

Because H is a coideal in $V[G]$, its intersection $\overline{H}$ with $V[G \restriction \beta]$ must be a coideal in $V[G \restriction \beta]$. Working in $V[G \restriction \beta]$, we can extend the filter dual to the ideal $\overline{H}^c$ to obtain an ultrafilter $U \subseteq \overline{H}$.

Claim 1. Let $g \in V[G]$ be a real that is generic over $V[G \restriction \beta]$ for a poset $\mathbb{P} \in V[G \restriction \beta]$. Then $g \in X$ iff $V[G \restriction \beta][g] \models \phi^*[g, r, \alpha]$.

Proof of Claim 1. This is just an application of the Factor Lemma 17. Work in $V[G]$. Because $\mathrm{Coll}(\omega, <\beta) * \dot{\mathbb{P}}$ is a poset in V of size $< \kappa$, we can apply the Factor Lemma to find a filter G^* that is $\mathrm{Coll}(\omega, <\kappa)$-generic over $V[G \restriction \beta][g]$ and such that

$$V[G] = V[G \restriction \beta][g][G^*].$$

Suppose first that $g \in X$, so $V[G] \models \phi[g, r, \alpha]$. The symmetry of the Levy collapse guarantees that in $V[G \restriction \beta][g]$ the empty condition forces $\phi(\dot{g}, \check{r}, \check{\alpha})$. That is,

$$V[G \restriction \beta][g] \models \phi^*[g, r, \alpha].$$

For the converse, suppose that $V[G \restriction \beta][g] \models \phi^*[g, r, \alpha]$. Now $V[G] = V[G \restriction \beta][g][G^*]$ must satisfy $\phi[g, r, \alpha]$, since the empty condition in $\mathrm{Coll}(\omega, <\kappa)$ forces it and G^* is $\mathrm{Coll}(\omega, <\kappa)$-generic over $V[G \restriction \beta]$. That is, $g \in X$. ∎

Claim 2. In $V[G]$, for every $\langle \emptyset, \vec{y} \rangle \in \mathbb{P}_U$ there is a real $g \in H$ that is $\mathbb{P}_U$-generic below $\langle \emptyset, \vec{y} \rangle$ over $V[G \restriction \beta]$.

Proof of Claim 2. Work in $V[G]$. By the characterization of $\mathbb{P}_U$-genericity (Lemma 14), we need only ensure that g almost-diagonalizes every $\subseteq$-decreasing sequence of sets in U that belongs to $V[G \upharpoonright \beta]$. But $[\omega]^\omega \cap V[G \upharpoonright \beta]$ is countable, so we can find a single sequence $\vec{z} = \langle z_n : n < \omega \rangle$ of subsets of U such that any real that almost-diagonalizes $\vec{z}$ also almost-diagonalizes every sequence in $U^\omega \cap V[G \upharpoonright \beta]$. Moreover, we can arrange that $\vec{z} \leq \vec{y}$.

It remains to show that there is a real $g \in H$ that diagonalizes the sequence $\vec{z}$, but this follows immediately from the definition of *happy family*, since each z_n belongs to H. ∎

Using the Prikry property of $\mathbb{P}_U$, find a condition $\langle \emptyset, \vec{y} \rangle \in V[G \upharpoonright \beta]$ that forces either $\phi^*(\dot{g}, \check{r}, \check{\alpha})$ or its negation. (Here $\dot{g}$ is the natural name for the generic real g.) Now the real $g \in [\emptyset, \vec{y}] \cap H$ provided by Claim 2 will witness that X is H-Ramsey. The argument is symmetric in X and X^c, so we may assume without loss of generality that $g \in X$. It follows by Claim 1 that $\langle \emptyset, \vec{y} \rangle$ forces $\phi^*(\dot{g}, \check{r}, \check{\alpha})$, not its negation. Working in $V[G]$, consider any real $g' \in [g]^\omega$. Since $\mathbb{P}_U$ has the Mathias property, g' is also $\mathbb{P}_U$-generic over $V[G \upharpoonright \beta]$. And g diagonalizes $\vec{y}$, so g' does also. That is, $\langle \emptyset, \vec{y} \rangle$ belongs to the $\mathbb{P}_U$-generic associated to g'. It follows that

$$V[G \upharpoonright \beta][g'] \models \phi^*[g', r, \alpha],$$

since $\langle \emptyset, \vec{y} \rangle$ forces it. Now apply Claim 1 to conclude that $g' \in X$. This completes the proof that X is H-Ramsey. ∎

By taking $X = H$ in Theorem 16 and applying Lemma 5, we obtain the following corollary.

Corollary 18. If $A \in V[G]$ is an infinite mad family, then A does not belong to the Solovay model $L(\mathbb{R})^{V[G]}$, and neither does the ideal that it generates.

5. Meager filters

To answer Törnquist's question, whether AD implies that there are no infinite mad families, one might hope to show that the ideal generated by an infinite mad family fails to have a regularity property that all sets enjoy under AD. Unfortunately, the Baire property is inadequate for this approach. It can be seen directly from the definition that the ideal generated by an infinite mad family has the Baire property; in this section we give a game-theoretic proof that all ideals are meager under AD.

Notice first that an ideal on ω has the Baire property if and only if it is meager; this is a well known application of 0–1 laws. See Oxtoby [17, Thm. 21.4]. (Likewise, an ideal is measurable if and only if it has measure zero.)

The reader is directed to Bartoszyński and Scheepers [1] for another characterization of meager filters using a integer game and for some applications.

For a real $x \in [\omega]^\omega$, write e_x for the unique increasing function $\omega \to \omega$ that enumerates x. We say a filter F on ω is *bounded* if the family $\{e_x : x \in F\}$ is bounded in the eventual-domination ordering on ω^ω.

Recall Talagrand's characterization of filters on ω with the Baire property.

Theorem (Talagrand [23])**.** A filter on ω is meager iff it is bounded.

The proof of Talagrand's theorem is very nearly contained in Blass [2]; for the reader's convenience, we give the rest of the details here.

Claim. Let F be a filter on ω containing every cofinite set. The following are equivalent.

(i) F is meager.
(ii) There is an interval partition $(I_0, I_1, \dots)$ of ω such that every set in F meets all but finitely many of the pieces I_n.
(iii) There is an increasing function $f \in \omega^\omega$ such that every set in F meets the interval $[n, f(n))$ for all but finitely many n.
(iv) F is bounded.

Proof of claim. The equivalence of (i) and (ii) is established in Proposition 9.4 of [2], so it suffices to prove the implications (ii) $\leftrightarrow$ (iii), (ii) $\to$ (iv), and (iv) $\to$ (i).

(ii) $\to$ (iii). Define $f(n)$ to be $\max(I_{k+1}) + 1$ for the unique k such that $n \in I_k$. Now let $x \in F$ and let n be large enough that $x \cap I_{k+1} \neq \emptyset$, where $n \in I_k$. Notice that $I_{k+1} \subseteq [n, f(n))$, so x meets $[n, f(n))$.

(iii) $\to$ (ii). Put $I_n = [f^n(0), f^{n+1}(0))$. Then every $x \in F$ meets I_n for sufficiently large n.

(ii) $\to$ (iv). Define $f(n) = \max I_{2n}$. Let $x \in F$ and choose k large enough that x meets I_m for all $m \geq k$. In particular, x meets each of the intervals $I_k, I_{k+1}, \dots, I_{2k}$, so $|x \cap \max I_{2k}| \geq k$. This implies that $e_x(k) \leq \max I_{2k} = f(k)$, from which we conclude that $e_x \leq^* f$.

(iv) $\to$ (i). Suppose that $f \in \omega^\omega$ dominates the enumerating function of every member of F. This means that F is included in the set

$$\{x \in 2^\omega : \forall^\infty k \ |x \cap f(k)| \geq k\} = \bigcup_m \{x \in 2^\omega : \forall k \geq m \, |x \cap f(k)| \geq k\}.$$

It's easy to see that for each m the set $\{x \in 2^\omega : \forall k \geq m \, |x \cap f(k)| \geq k\}$ is a nowhere-dense subset of 2^ω, so we conclude that F is meager. ∎

The game we study here is already well known. It provides a convenient proof, apparently part of the folklore, that there are no nonprincipal ultrafilters under AD.

Players I & II alternate playing nonempty intervals $I_k \in [\omega]^{<\omega}$

I	I_0		I_2		I_4		$\cdots$
II		I_1		I_3		I_5	$\cdots$

subject to the following rules.

(i) Each play I_k is a nonempty finite interval, and
(ii) $\min(I_{k+1}) = \max(I_k) + 1$.

So the game board looks like this:

I	$[0, n_0]$		$[n_1+1, n_2]$		$\cdots$
II		$[n_0+1, n_1]$		$[n_2+1, n_3]$	$\cdots$

We call an interval partition $(I_0, I_1, \dots)$ a **run** of the game, and an interval partition of some initial segment $[0, n]$ of ω is a **partial run** of the game. If, in a run of the game, $n \in I_{2k}$, then we say Player I **claims** n, and similarly Player II **claims** every member of each interval I_{2k+1}.

To determine the winner of a run of the game, we require a filter F on ω. Player I wins the game iff $\bigcup_{n\in\omega} I_{2n} \in F$. That is, Player I wins iff he claims a measure-one set of integers.

It's clear that Player I has a winning strategy if F is a principal ultrafilter. Conversely, if Player I has a winning strategy, then a strategy-stealing argument shows that the filter is principal. A strategy-stealing argument also shows that Player II cannot have a winning strategy if F is an ultrafilter and $\{0\} \notin F$. (If $\{0\} \in F$, then Player II still does not have a winning strategy since Player I does.) This gives an easy proof that there are no nonprincipal ultrafilters on ω under AD.

Proposition 19. A filter F on ω is meager iff Player II has a winning strategy for the partition game for F.

Proof. This can be proved directly, but it will be easier to go through Talagrand's characterization of meager filters. Suppose first that F is meager, hence bounded. Let $g \in \omega^\omega$ be a function that dominates e_x for every $x \in F$; we can assume without loss that g is increasing. We describe a strategy σ for Player II. Suppose it is Player II's turn and that Player I has claimed exactly k integers so far. Player II simply plays the shortest nonempty interval I such that $\max(I) \geq g(k+1)$. (Notice that $g(k+1)$ may already have been claimed.) Explicitly,

$$\sigma(I_0, \dots, I_{2n}) = I_{2n+1} := \begin{cases} [\max(I_{2n}) + 1, g(k+1)] & \text{if } g(k+1) \geq \max(I_{2n}) + 1 \\ \{\max(I_{2n}) + 1\} & \text{if } g(k+1) \leq \max(I_{2n}) \end{cases}.$$

Now consider a run $(I_0, I_1, I_2, \dots)$ of the game according to σ, and put $x = I_0 \cup I_2 \cup \cdots$, the set of integers claimed by Player I. If Player I has claimed exactly k integers before at the beginning of II's turn $2n+1$, then we have

$$g(k+1) \leq \max I_{2n+1} < \min I_{2n+2} = e_x(k+1).$$

We have found infinitely many k for which $g(k+1) < e_x(k+1)$; this means that g does not dominate e_x. (Our infinitely k are those of the form $k = |I_0| + |I_2| + \cdots + |I_{2n}|$.) Because g dominates the enumerating function of every member of F, x cannot belong to F. So Player I loses, and σ is a winning strategy for Player II.

For the converse, suppose that F is unbounded, and we'll show that Player I can defeat any strategy σ for Player II. For a fixed n, there are only finitely many partial runs $(I_0, \dots, I_k)$ of the game such that both

- $I_0 \cup I_1 \cup \cdots \cup I_k = [0, n]$, and
- k is even, so it is Player II's turn to play after the partial run.

Therefore, there are only finitely many intervals Player II might play immediately after such a partial run, if Player II plays according to σ. With this in mind, let $f(n)$ be the largest m contained in any such interval. This definition guarantees that in any run $(I_n)_{n\in\omega}$ of the game according to σ, $\max(I_{2n+1}) \leq f(\max(I_{2n}))$ for every n.

Since F is unbounded, there is some set $x \in F$ whose enumerating function e_x is not dominated by the function $n \mapsto f^{2n}(0)$. (The exponent here indicates how many times f should be iterated.) That is, there are infinitely many $n \in \omega$ such that $|f^{2n}(0) \cap x| < n$. For such n, the set x meets fewer than n of the $2n$ intervals $(f^k(0), f^{k+1}(0)]$, $0 \leq k < 2n$. It follows that there are infinitely many k for which

$$x \cap (f^k(0), f^{k+1}(0)] = \emptyset. \qquad (*)$$

On his turn, Player I should claim all integers up to $f^k(0)$ for some k satisfying $(*)$. Player II, if using her strategy σ, must respond by playing an interval $I \subseteq (f^k(0), f(f^k(0))]$. Therefore Player II never claims any members of x, so Player I claims all members of x. Player I claims a set in F and wins, so σ is not a winning strategy for Player II. ∎

6. AD^+ and large cardinals

This section includes a proof of our partial answer to Törnquist's question. We begin by clarifying some relevant parts of the literature.

In [7] Farah isolates the weakest analogue of *happy family* for which the key facts of Mathias's original paper [14] remain true.

Definition 20. Let H be a coideal on ω. If $\langle D_n \rangle_{n<\omega}$ is a sequence of dense subsets of the poset $\langle H, \subseteq \rangle$, then a set $x \in [\omega]^\omega$ is a **diagonalization** of

$\langle D_n\rangle_{n<\omega}$ if there exists a sequence of sets $d_n \in D_n$ so that x is a diagonalization of $\langle d_n : n < \omega\rangle$, meaning that $f(n+1) \in d_{f(n)}$ for every $n < \omega$, where $f\colon \omega \to \omega$ is the increasing enumeration of x. We say that H is **semiselective** if for every sequence $\langle D_n\rangle_{n<\omega}$ of dense subsets and every $y \in H$, there is $x \in H \cap [y]^\omega$ that diagonalizes the sequence $\langle D_n\rangle_{n<\omega}$.

When guided by a happy family, Mathias forcing is σ-closed $*$ ccc, whereas when guided by a semiselective coideal it is σ-distributive $*$ ccc.

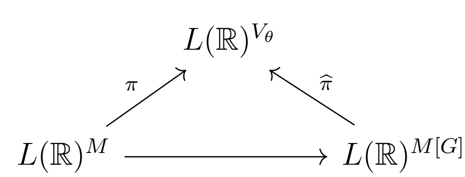
In [7, §4] Farah presents a theorem of Todorcevic, that, in the presence of a supercompact cardinal, every set in $L(\mathbb{R})$ is H-Ramsey for every semiselective coideal H. Lemma 4.3 in that paper is not true as stated, but it is also not used in Todorcevic's argument; rather Todorcevic used a connection between the Ramsey property and the Ellentuck topology (see also Todorcevic [24, Ch. 7]) and the conclusion of Lemma 4.4 of the paper, which is true assuming the existence of a supercompact cardinal, by Theorem 4.1 of Feng–Magidor–Woodin [8].

We prove Todorcevic's theorem by different methods, from weaker large cardinals, in the region of Woodin cardinals. Our large cardinal assumptions are weak enough to follow from determinacy assumptions. This allows us to then prove a version of the theorem under determinacy. Our methods also give a correct version of Lemma 4.3 of [7]. The restriction of the lemma to proper posets, or more generally to the class of reasonable posets of Foreman and Magidor [9], is true and follows from Theorem 22 below.

Theorem 22 is a triangular version of the following Embedding Theorem of Neeman–Zapletal [15] and [16].

Theorem 21 (Neeman–Zapletal [16])**.** (Large cardinals) Suppose that $\mathbb{P}$ is a proper (or reasonable) poset and that G is $\mathbb{P}$-generic over V. There is an elementary embedding $j\colon L(\mathbb{R})^V \to L(\mathbb{R})^{V[G]}$ that fixes every ordinal.

Theorem 22. (Large cardinals, see remarks following Corollary 24 for the exact assumptions) Suppose M is a countable transitive model that embeds into a sufficiently large rank-initial segment of V, say by $\pi\colon M \xrightarrow{\sim} V_\theta$. If G is $\mathbb{P}$-generic over M for some proper (or reasonable) poset $\mathbb{P}$ in M, then there is an embedding $\widehat{\pi}\colon L(\mathbb{R})^{M[G]} \to L(\mathbb{R})^{V_\theta}$ that agrees with π on ordinals and reals, and completes a commuting system of elementary embeddings:



Using Theorem 22, we prove Todorcevic's theorem by imitating Mathias's proof [14, paragraph after 2.9] that there are no analytic infinite mad families.

We will use $\mathbb{M}_H$ to denote Mathias forcing guided by H.

Theorem 23. In the presence of large cardinals (specifically $|\mathbb{R}|^+ + 1$-iterable model with the sharp of ω Woodin cardinals), every set $X \subseteq [\omega]^\omega$ that belongs to $L(\mathbb{R})$ is H-Ramsey for every happy family H.

Proof. Let $X \in L(\mathbb{R})$ be a subset of $[\omega]^\omega$ and let H be any happy family. Find a formula ϕ, a real parameter u, and a sequence α of ordinal parameters such that

$$x \in X \iff L(\mathbb{R}) \models \phi[x, \alpha, z].$$

Let Z be a countable elementary submodel of a sufficiently large rank-initial segment of V containing z, α, and H, and let M be the transitive collapse of Z. Let $\pi\colon M \to Z$ be the anti-collapse map. Put $\overline{H} = \pi^{-1}(H)$, and $\overline{\alpha} = \pi^{-1}(\alpha)$. Notice that $\overline{H} = H \cap Z$, and $\overline{H}$ is a happy family in M. It follows that Mathias forcing $\mathbb{M}^M_{\overline{H}}$ has the Prikry and Mathias properties in M. (For these properties of Mathias forcing guided by a happy family, see [14, Theorems 2.9, 4.11, & 4.12].) So there is a condition $\langle 0, x\rangle \in M$ that either forces (over M) $\phi^{L(\mathbb{R})}(\dot{g}, \overline{\alpha}, z)$ or forces $\neg\phi^{L(\mathbb{R})}(\dot{g}, \overline{\alpha}, z)$. (Here $\dot{g}$ is the usual name for the generic real.) For any real g that is $\mathbb{M}^M_{\overline{H}}$-generic over M below $\langle 0, x\rangle$, we have the following equivalence.

$$\begin{aligned} g \in X &\iff L(\mathbb{R}) \models \phi[g, \alpha, z] \\ &\iff M[g] \models \phi^{L(\mathbb{R})}[g, \overline{\alpha}, z] \\ &\iff \langle 0, x\rangle \Vdash_{\mathbb{M}^M_{\overline{H}}} \phi^{L(\mathbb{R})}[\dot{g}, \overline{\alpha}, z]. \end{aligned} \tag{$*$}$$

The second equivalence is a consequence of Theorem 22.

Since $\overline{H}$ is countable, there is a real $g \in H$ that is $\mathbb{M}^M_{\overline{H}}$-generic over M below $\langle 0, x\rangle$. (To see this, argue exactly as we did in the proof of Claim 2 while proving Theorem 16.) Since $\mathbb{M}^M_{\overline{H}}$ has the Mathias property in M, every subset $g' \subseteq g$ is generic too. Suppose without loss of generality that $g \in X$. Then the $\Longrightarrow$ direction of $(*)$ implies that $\langle 0, x\rangle \Vdash \phi^{L(\mathbb{R})}[\dot{g}, \overline{\alpha}, z]$ over M, and the $\Longleftarrow$ direction of $(*)$ implies that $g' \in X$ for every $g' \subseteq g$. ∎

We apply Lemma 5 to obtain

Corollary 24. In the presence of large cardinals (specifically $|\mathbb{R}|^+ + 1$-iterable model with the sharp of ω Woodin cardinals), there are no infinite mad families in $L(\mathbb{R})$.

Remarks.

(1) The large cardinal assumption we use to prove Theorem 22 is the existence of $M^\sharp_\omega$, the minimal iterable model for the sharp of ω Woodin cardinals, and its $|\pi(\mathbb{P})|^+ + 1$-iterability.

(2) Theorem 21 applies not only to proper posets, but to the larger class of so-called *reasonable* posets (introduced by Foreman and Magidor [9]), which includes posets of the form σ-distributive $*$ ccc. Our triangular version, Theorem 22, also applies to this broader class. We therefore obtain Todorcevic's theorem for semiselective coideals, not just happy families.
(3) Schindler [19] has computed the consistency strength of Theorem 21 for proper posets to be exactly the existence of a *remarkable* cardinal, which is stronger than an ineffable cardinal but weaker than the existence of $0^\sharp$. The consistency strength of the version for reasonable posets remains open.

We proceed now to results under determinacy. We assume ZF throughout; theorems listed below under AD^+ are theorems of $\mathsf{ZF}+\mathsf{AD}^+$. For the precise definition of AD^+ and a thorough discussion of it, the reader is encouraged to consult [3, §§1–3]. AD^+ implies AD, and it is open whether the two are equivalent.

A version of Theorem 23 holds under AD^+, and implies a version of Corollary 24 under AD^+. These are proved using an AD^+ version of the triangular embedding theorem. The theorem applies to posets in the following class:

Definition 25. Call poset $\mathbb{P}\subseteq\mathbb{R}$ **absolutely proper** (respectively **absolutely reasonable**) if there exists a club $C\subseteq\mathcal{P}_{<\omega_1}(\mathbb{R})$ and $A\subseteq\mathbb{R}$ so that for every $U\in C$ and every transitive countable model N of a large enough fragment of ZFC with $\mathbb{R}^N=U$ and $\mathbb{P}\cap U, A\cap U\in N$, $\mathbb{P}\cap U$ is proper (respectively reasonable) in N.

Note that absolute properness (reasonableness) is a Σ_1 property in $\{\mathbb{R}\}$ over $\mathbb{N}\cup\mathbb{R}\cup\mathcal{P}(\mathbb{R})$, with predicates for membership and coding finite and countable sequences of reals by reals, in other words it is a Σ^2_1 statement. In particular it reflects up from transitive models containing $\{\mathbb{R}\}$. It is phrased without reference to countable elementary substructures of initial segments of the universe, which need not exist in contexts where DC fails. Posets which are provably proper (reasonable) under AC by sufficiently absolute arguments are often absolutely proper (reasonable). For example:

Lemma 26. Let H be a happy family. Then Mathias's forcing $\mathbb{M}_H$ (with conditions coded as reals in a natural way) is absolutely proper.

Proof. Let C be the club of countable $U\subseteq\mathbb{R}$ which are elementary under the coding predicates for countable sequences and for conditions in $\mathbb{M}_H$, and so that $U\cap H$ satisfies the requirements in the definition of a happy family with upward closure restricted to $y\in U$ and selectivity restricted to descending sequences that are coded by a real in U. Then for any N as in Definition 25,

$U \cap H$ is a happy family in N. Hence in particular $\mathbb{M}^N_{U\cap H}$ is proper in N. Using elementarity under coding, $\mathbb{M}^N_{U\cap H} = \mathbb{M}_H \cap U$. ∎

Theorem 27. (AD^+) For every $\alpha < \Theta$, for every $A \subseteq \mathbb{R}$, and for stationarily many countable $Z \preceq L_\alpha(\mathbb{R}, A)$, if M is the transitive collapse of Z, $\pi\colon M \xrightarrow{\sim} L_\alpha(\mathbb{R}, A)$ is the anti-collapse embedding, $\overline{\alpha} = M \cap \mathrm{On}$, $\overline{A} = \pi^{-1}(A)$, and $\mathbb{P}$ is absolutely proper (or absolutely reasonable) in M, then there exists a countable transitive model N of ZFC with $\mathbb{R}^N = \mathbb{R} \cap M$ and $\overline{\alpha}, \overline{A} \in N$, and there exists a $\mathbb{P}$-name $\dot{A}^* \in N$, so that for every G which is $\mathbb{P}$-generic over N, there is an embedding $\widehat{\pi}\colon L_{\overline{\alpha}}(\mathbb{R}^{N[G]}, \dot{A}^*[G]) \to L_\alpha(\mathbb{R}, A)$ that agrees with π on ordinals and reals, maps $\dot{A}^*[G]$ to A, and completes a commuting system of elementary embeddings:

$$\begin{array}{ccc} & L_\alpha(\mathbb{R}, A) & \\ {\scriptstyle\pi}\nearrow & & \nwarrow{\scriptstyle\widehat{\pi}} \\ M & \longrightarrow & L_{\overline{\alpha}}(\mathbb{R}^{N[G]}, \dot{A}^*[G]) \end{array}$$

Theorem 28. (AD^+) Every set $X \subseteq [\omega]^\omega$ is H-Ramsey for every happy family H. Consequently, there are no infinite mad families.

Proof. Let $A \subseteq \mathbb{R}$ code X and H. Fix a countable Z elementary in $L_\omega(\mathbb{R}, A)$ for which Theorem 27 holds, with $X, H, A \in Z$. Let M and π be as in the theorem. By Lemma 26, $\mathbb{M}^M_H$ is absolutely proper in M. Now proceed as in the proof of Theorem 23, but using Theorem 27 over N instead of Theorem 22, and forcing a truth value to $\phi^{L_\omega(\mathbb{R},\dot{A}^*)}(\dot{g}, \dot{A}^*)$, where ϕ is a formula so that $\phi(x, A)$ holds in $L_\omega(\mathbb{R}, A)$ iff $x \in X$. ∎

Remarks. (1) Since Theorem 27 applies to absolutely reasonable posets, the last theorem can be strengthened to cover absolutely semiselective coideals, where absoluteness is meant in the manner of Definition 25, replacing proper there with semiselective throughout.

(2) Some contexts where semiselectivity is absolute were obtained by Larson and Raghavan [12]. For example the notion asserting that player II wins the strategic selectivity game there is clearly absolute, and is equivalent under $\mathsf{AD}_\mathbb{R}$ to semiselectivity that persists to some outer model of choice with the same reals, by [12, Proposition 1.4].

(3) Versions of Theorem 27 that provide less elementarity can be proved with less than full determinacy. For example, the restriction of the theorem to projective sets A, where we weaken stationarity to existence, weaken elementarity throughout to just Σ_1 elementarity in the structure $(\mathbb{R}, A)$, and remove the club in the definition of absolute properness requiring the conditions of the definition for all Σ_1 elementary countable substructures in $(\mathbb{R}, A)$, is provable under projective determinacy.

The argument is less involved than the proof of the full theorem, simply using the facts that projective truth is absolute to iterable models with a large enough finite number of Woodin cardinals, the ability to iterate to make any real generic for the collapse of the first Woodin cardinal, and the existence of these models under projective determinacy.

(4) This projective version of Theorem 27 is enough to run the proof of Theorem 28 for projective sets H and X. In particular projective determinacy implies that there are no projective infinite mad families. This was proved independently by Karen Haga.

The AD^+ triangular embedding theorem has additional applications. For example it can be used to obtain results similar to these obtained under $\mathsf{AD}_{\mathbb{R}}+\mathsf{DC}$ for proper ideals in Chan–Magidor [4], working instead under the more general AD^+, but restricting to absolutely proper ideals:

Theorem 29. (AD^+) Let I be a σ-ideal on ω^ω, so that $\mathbb{P}_I$ is absolutely proper. Let $\boldsymbol{\Gamma}$ be a pointclass closed under Borel substitutions, with a universal set. Let E be an equivalence relation whose equivalence classes are all in $\boldsymbol{\Gamma}$ (respectively in both $\boldsymbol{\Gamma}$ and $\check{\boldsymbol{\Gamma}}$). Then there exist densely many Borel sets C in I^+ so that $E \restriction C$ is in $\boldsymbol{\Gamma}$ (respectively in both $\boldsymbol{\Gamma}$ and $\check{\boldsymbol{\Gamma}}$).

Recall that $\mathbb{P}_I$, studied in Zapletal [26], is the poset of Borel sets in I^+, ordered by reverse inclusion mod I. Zapletal presented many of the standard cardinal invariants forcing notions in this form. For many of the ideals he considered the posets $\mathbb{P}_I$ are provably proper, and by methods similar to the proof of Lemma 26 one can show these specific posets are also absolutely proper. In the context of absolute properness we assume some coding of Borel sets to view conditions in $\mathbb{P}_I$ as reals.

Proof of Theorem 29. We prove that there exists C as in the theorem. Working throughout below an arbitrary $B \in I^+$ the same argument would show there exist densely many such C. We prove only the case of membership in $\boldsymbol{\Gamma}$. The proof handling both $\boldsymbol{\Gamma}$ and $\check{\boldsymbol{\Gamma}}$ simultaneously is similar.

Let $U \subseteq \mathbb{R} \times \mathbb{R}$ be universal for $\boldsymbol{\Gamma}$. Let $A \subseteq \mathbb{R}$ code I, U, E, and a club witnessing that $\mathbb{P}_I$ is absolutely proper, so that the poset is absolutely proper in $L_\omega(\mathbb{R}, A)$. Note that $\mathbb{P}_I$ is also proper in $L_\omega(\mathbb{R}, A)$. Otherwise there is a stationary set of substructures without master conditions. But this can be reflected into a model N as in Definition 25, contradicting properness in N.

Fix a countable Z elementary in $L_\omega(\mathbb{R}, A)$ for which Theorem 27 holds, with $I, U, E, A \in Z$. Since $\mathbb{P}_I$ is proper we can pick Z for which there are master conditions in $\mathbb{P}_I$. Let M and π be as in the theorem. Let $\overline{A} = \pi^{-1}(A)$, $\overline{I} = \pi^{-1}(I)$, $\overline{U} = \pi^{-1}(U)$, $\overline{E} = \pi^{-1}(E)$. Let N and $\dot{A}^*$ be as in Theorem 27 for the forcing $\mathbb{P}^M_{\bar{I}}$. Let $\dot{U}^*$ name the set coded by $\dot{A}^*$ in the way U is coded by A, and similarly with $\dot{E}^*$.

Recall from [26, Proposition 2.1.2] that if G is $\mathbb{P}^M_{\bar{I}}$ generic over N then $\bigcap G$ is a singleton real. Let $\dot{x}$ name this real. The E-equivalence class of $\dot{x}[G]$ is in $\mathbf{\Gamma}$. By universality it is equal to the section U_y for some real y. Using the elementarity given by Theorem 27 it follows that there is a name $\dot{y} \in N$ so that over N the $\dot{E}^*$-equivalence class of $\dot{x}$ is forced to be equal to the section $\dot{U}^*_{\dot{y}}$. Again by the theorem, for any generic G, the E-equivalence class of $\dot{x}[G]$ is then equal to $U_{\dot{y}[G]}$.

Let $C = \{\dot{x}[G] \colon G \text{ is } \mathbb{P}^M_{\bar{I}}\text{-generic over } N\}$. Since N is countable, C is non-empty. In fact since there is a master condition for Z in $\mathbb{P}_I$, $C \in I^+$, see the proof of [26, Proposition 2.2.2]. Each $x \in C$ uniquely determines the generic which induces it, denoted G_x, through the condition that $B \in G_x$ iff $B \in \mathbb{P}^M_{\bar{I}} \wedge x \in B$.

We claim that $E \restriction C$ is in $\mathbf{\Gamma}$. To see this, note simply that for any $x_1, x_2 \in C$, $x_1 \mathrel{E} x_2$ iff $x_2 \in U_{\dot{y}[G_{x_1}]}$ iff $\langle \dot{y}[G_{x_1}], x_2 \rangle \in U$. The condition on the right is in $\mathbf{\Gamma}$ using closure under Borel substitutions since N is countable. ∎

The rest of the section is dedicated to the proof of the triangular embedding theorems. We derive both versions from the same lemma.

Call $\langle Q, \Sigma, \delta \rangle$ a **germ** which **captures** $F \subseteq \mathbb{R}$ just in case that:

(1) Q is a model of a large enough fragment of ZFC with infinitely many Woodin cardinals, whose supremum is δ.
(2) Σ is an $\omega_1 + 1$ iteration strategy for Q.
(3) (Condensation) If $\mathcal{T}$ is an iteration tree by Σ, and $\pi \colon N \to N^*$ is an elementary embedding, with N, N^* transitive models of enough of ZFC and $\mathcal{T} \in \operatorname{range}(\pi)$, then $\pi^{-1}(\mathcal{T})$ is according to Σ.
(4) (Capturing) For every iteration map $j \colon Q \to Q^*$ by Σ, there is a $\operatorname{Coll}(\omega, j(\delta))$-name $\dot{F}^* \in Q^*$, so that for every generic H for $\operatorname{Coll}(\omega, j(\delta))$ over Q^*, every $\eta < j(\delta)$, and every $x \in Q^*[H \restriction \eta]$, $x \in \dot{F}^*[H]$ iff $x \in F$.

Condition (3) typically holds for uniquely iterable fine structural models. In particular it holds for $Q = M^\sharp_\omega$, the minimal model for the sharp of ω Woodin cardinals, and its unique iteration strategy.

Let $j \colon Q \to Q^*$ be according to Σ. Let H be generic over Q^* for $\operatorname{Coll}(\omega, < j(\delta))$. The **derived model** of Q^* using H is $L_\alpha(\mathbb{R}^*, F^*)$ where $\alpha = Q^* \cap \mathrm{On}$, $\mathbb{R}^*$ consists of the reals in the extensions of Q^* by strict initial segments of H, and $F^* = F \cap \mathbb{R}^*$. We refer to $\mathbb{R}^*$ as the **derived reals**. By the capturing condition, F^* belongs to $Q^*[H]$, and in fact there is a name in Q^* which produces the right F^* independently of H.

By a **composed Σ-iteration of Q with direct limit** $j \colon Q \to Q^*$ we mean a sequence of embeddings $j_n \colon Q_n \to Q_{n+1}$ so that $Q_0 = Q$, each j_n is the embedding of a countable iteration tree $\mathcal{T}_n$ on Q_n with final model Q_{n+1}, for each n the composition of the trees $\mathcal{T}_0, \dots, \mathcal{T}_n$ is according to Σ, Q^* is the direct limit of the models Q_n, and j is the direct limit embedding. We write $j_{n,m}\colon Q_n \to Q_m$ for the composed embeddings, and $j_{n,\infty}\colon Q_n \to Q^*$ for the direct limit embedding from Q_n.

If the sequence of trees belongs to V then the composition of all ω trees is by Σ, and hence $j\colon Q \to Q^*$ is itself an iteration embedding by Σ. But we will use the definition in models which do not know enough of Σ to recognize the entire sequence as an iteration by Σ.

Lemma 30 (by Neeman–Zapletal [16])**.** Let N be a countable transitive model of a large enough fragment of ZFC. Let $\mathbb{P}$ be a proper (or reasonable) poset in N. Let $\langle Q, \Sigma, \delta\rangle$ be a germ. Suppose that $Q \in N$ and that the restriction of Σ to trees of length $\leq |\mathbb{P}|^+$ in N belongs to N. Let G be $\mathbb{P}$-generic over N. Then there exists a composed Σ-iteration $\langle j_n \colon n < \omega\rangle$ of Q with direct limit $j\colon Q \to Q^*$ so that both $\mathbb{R}^N$ and $\mathbb{R}^{N[G]}$ can be realized as the derived reals of derived models of Q^*. Moreover such $\langle j_n \colon n < \omega\rangle$ can be found inside any forcing extension of $N[G]$ by $\mathrm{Coll}(\omega, \mathbb{R}^N) \times \mathrm{Coll}(\omega, \mathbb{R}^{N[G]})$, the finite restrictions of the sequence belong to N, and the sequence of critical points of the maps $j_{n,\infty}$ is increasing and cofinal in $j(\delta)$.

Proof. This follows by the construction in Section 2 of Neeman–Zapletal [16], building iterates of the model Q of the current lemma using the iteration strategy Σ, and working over the model N of the current lemma rather than V. We only note that the use of unique iterability in Lemma 3 of [16] is replaced here by a use of the condensation assumption on Σ, which implies condensation in N for the restriction of Σ to trees in N. ∎

Proof of Theorem 22. Fix the relevant objects. It is enough to prove for every $\beta \in M \cap \mathrm{On}$, every $\vec{\gamma} \in \beta^{<\omega}$, every $\vec{y} \in (M[G] \cap \mathbb{R})^{<\omega}$, and every formula ϕ, that $L_\beta(\mathbb{R}^{M[G]}) \models \phi(\vec{\gamma}, \vec{y})$ iff $L_{\pi(\beta)}(\mathbb{R}) \models \phi(\pi(\vec{\gamma}), \vec{y})$. Since every element of $L(\mathbb{R})$ is definable from ordinals and reals, one can then set $\widehat{\pi}$ to map elements definable from $\vec{\gamma}, \vec{y}$ in $L_\beta(\mathbb{R})^{M[G]}$ to elements definable in the same way from $\pi(\vec{\gamma}), \vec{y}$ in $L_{\pi(\beta)}(\mathbb{R})$, define the horizontal embedding $k\colon L(\mathbb{R})^M \to L(\mathbb{R})^{M[G]}$ to map elements definable from $\vec{\gamma}, \vec{y}$ in $L_\beta(\mathbb{R})^M$ to elements definable in the same way from $\vec{\gamma}, \vec{y}$ in $L_\beta(\mathbb{R})^{M[G]}$, and verify that the two embeddings, which obviously commute with π, are elementary.

Fix β, $\vec{\gamma}$, $\vec{y}$, and ϕ. Suppose that $L_\beta(\mathbb{R}^{M[G]}) \models \phi(\vec{\gamma}, \vec{y})$. We prove that $L_{\pi(\beta)}(\mathbb{R}) \models \phi(\pi(\vec{\gamma}), \vec{y})$.

The large cardinal assumption we use is the existence of $M_\omega^\sharp$, and its $|\pi(\mathbb{P})|^+ + 1$-iterability. Let $Q = M_\omega^\sharp$, let δ be the supremum of the Woodin cardinals of Q, and let Σ be the unique iteration strategy for Q. Then $\langle Q, \Sigma, \delta\rangle$ is a germ (capturing, for example, the empty set). By the elementarity of π, Q belongs to M, and so does the restriction of Σ to trees of length $\leq |\mathbb{P}|^+$ in M.

Let $j_n\colon Q_n \to Q_{n+1}$ and $j\colon Q \to Q^*$ be given by Lemma 30 applied with $N = M$. Let Q^{**} be obtained by iterating the sharp of Q^* past β. Q^* does not belong to M, but it belongs to $M[G][h]$ for some generic h for $\mathrm{Coll}(\omega, \mathbb{R}^M) \times \mathrm{Coll}(\omega, \mathbb{R}^M[G])$ over $M[G]$. Q^{**} then also belongs to $M[G][h]$. Let $\dot{j}_{n,m}$, $\dot{Q}_n$, $\dot{Q}^*$, $\dot{Q}^{**}$, and $\dot{j}_{n,\infty}$ in M name the corresponding objects in $M[G][h]$.

Fix H witnessing that $\mathbb{R}^{M[G]}$ can be realized as the derived reals of Q^{**}. Let $\eta < j_{0,\infty}(\delta)$ be large enough that $\vec{y}$ belongs to $Q^{**}[H \restriction \eta]$. Fix a name $\dot{u} \in Q^{**}$ so that $\dot{u}[H \restriction \eta] = \vec{y}$. Using the symmetry of the collapse we can assume that it is outright forced in $\mathrm{Coll}(\omega, j_{0,\infty}(\delta))$ over Q^{**} that $L_\beta(\dot{\mathbb{R}}) \models \phi(\vec{\gamma}, \dot{u})$, where $\dot{\mathbb{R}}$ names the derived reals.

Let m be large enough that the critical point of $j_{m,\infty}$ is above $(2^\eta)^{Q^{**}}$. The statements above all hold in $M[G][h]$, and $Q_m, j_{0,m}$ belong to M. Let $q \in G*h$ force these statements, and force the values of $\dot{Q}_m$ and $j_{0,m}$.

We now apply π to shift the statements from M to V_θ. Let $G' * h'$ be generic for $\pi(\mathbb{P}) * (\mathrm{Coll}(\omega, \mathbb{R}) \times \mathrm{Coll}(\omega, \mathbb{R}^{V[\dot{G}]}))$ over V, below $\pi(q)$. Let $Q' = \pi(\dot{Q}^{**})[G'][h']$, $Q'_n = \pi(\dot{Q}_n)[G'][h']$, $j'_{0,n} = \pi(\dot{j}_{0,n})[G'][h']$, $j'_{n,\infty} = \pi(\dot{j}_{n,\infty})[G'][h']$. Note that $Q'_m = Q_m$, $j'_{0,m} = j_{0,m}$, and the critical point of $j'_{m,\infty}$ is above $(2^\eta)^{Q'}$, because $G' * h'$ is below q. In particular $H \restriction \eta$ is generic for $\mathrm{Coll}(\omega, \eta)$ over Q'.

Using the elementarity of π, and since $\mathbb{R}^M$ can be realized as the derived reals of Q^{**}, $\mathbb{R} = \mathbb{R}^{V_\theta}$ can be realized as the derived reals of Q'. Let E be generic witnessing this. Since $H \restriction \eta$ can be coded by an element of $\mathbb{R}^V$, it belongs to the derived model of Q' using E. By standard arguments E can then be rearranged to get $E \restriction \eta = H \restriction \eta$ without changing the derived reals. Then $\dot{u}[E] = \vec{y}$. By the elementarity of π, it is forced in $\mathrm{Coll}(\omega, j'_{0,\infty}(\delta))$ over Q' that $L_{\pi(\beta)}(\dot{\mathbb{R}}) \models \phi(\pi(\vec{\gamma}), \dot{u})$. Interpreting this using E it follows that $L_{\pi(\beta)}(\mathbb{R}) \models \phi(\pi(\vec{\gamma}), \vec{y})$. ∎

To prove Theorem 27 we need the following fact from inner model theory.

Fact. (AD^+) Every true Σ^2_1 statement has a witness F which is captured by a germ.

Proof sketch. Let ϕ have only bounded quantifiers, and suppose that $(\exists F \subseteq \mathbb{R})\phi(F, \mathbb{R}, \mathbb{N})$ is true. By Woodin's basis theorem for Σ^2_1, see Steel [21, Theorem 9.10], there is then a witness F which belongs to $\mathbf{\Delta}_0 = \mathbf{\Gamma}_0 \cap \check{\mathbf{\Gamma}}_0$, for a pointclass $\mathbf{\Gamma}_0$ which is good in the sense of Steel [22, Definition 3.1]. Among other things this means that there is a universal set U for $\mathbf{\Gamma}_0$, and a $\mathbf{\Gamma}_0$ scale on this set. Since U is universal, there is a real z so that $F = U_z$.

These facts themselves can be witnessed using $\mathbf{\Gamma}_0$ and U which belong to $\mathbf{\Delta}_1 = \mathbf{\Gamma}_1 \cap \check{\mathbf{\Gamma}}_1$ for a good $\mathbf{\Gamma}_1$, so that $\mathbf{\Gamma}_0 \subseteq \mathbf{\Delta}_1$, and similarly we can further arrange to have a good $\boldsymbol{\Gamma}_2$ so that $\boldsymbol{\Gamma}_1 \subseteq \boldsymbol{\Delta}_2 = \boldsymbol{\Gamma}_2 \cap \check{\boldsymbol{\Gamma}}_2$. By [22, Lemma 3.13] there is then a coarse $\boldsymbol{\Gamma}_0$-Woodin mouse P, with $z \in P$, and an iteration strategy Σ_P for P so that all iterates P^* of P by Σ_P are themselves $\boldsymbol{\Gamma}_0$-Woodin structures. This means that the ordinal height of P^* is a Woodin cardinal in $L(P^* \cup \{T, P^*\})$, where T is the tree of a $\boldsymbol{\Gamma}_0$ scale on U, see [22, Definition 3.11, Theorem 3.4]. Moreover the proof is such that P is not $\boldsymbol{\Gamma}_1$-Woodin, and Σ_P is the unique strategy which correctly moves witnesses for this. This in particular gives branch condensation for Σ_P, a property similar to our condensation condition above but holding for more general hulls.

Now by an argument similar to the proof of Sargsyan-Steel [18, Lemma 2.5], there exists a (P, Σ_P) mouse Q satisfying an arbitrarily large finite fragment of ZFC, with ω Woodin cardinals. In particular this means that a real coding P belongs to Q, Q is closed under Σ_P and has a predicate for Σ_P, and Q is iterable by a strategy Σ which moves Σ_P correctly. This last property determines Σ uniquely, and implies that Σ has our condensation property. The argument for obtaining Q and Σ is similar to the construction of fine structure uniquely iterable models with infinitely many Woodin cardinals under AD, but replacing closure under ordinary constructibility with closure under constructibility *and* Σ_P. The branch condensation properties of Σ_P are used to see this hierarchy has condensation properties similar to the hierarchy of constructibility.

Let δ be the supremum of the Woodin cardinals of Q. It remains to prove that $\langle Q, \Sigma, \delta\rangle$ captures F. Fix an iteration map $j\colon Q \to Q^*$ by Σ. Recall that $P \in Q^*$ is $\boldsymbol{\Gamma}_0$-Woodin, U is a universal set for $\boldsymbol{\Gamma}_0$, z is a real so that $F = U_z$, and T is a $\boldsymbol{\Gamma}_0$ scale on U. The iteration strategy Σ_P is moved correctly by j, and in particular Q^* is closed under Σ_P and has a predicate for Σ_P. Working in Q^*, let $\sigma\colon P \to P^*$ be an iteration of P by Σ_P to reach a model $L(P^* \cup \{P^*, T\})$ so that every real generic for $\mathrm{Coll}(\omega, \eta)$ over Q^* for $\eta < j(\delta)$ is generic for Woodin's extender algebra at $P^* \cap \mathrm{On}$ over $L(P^* \cup \{P^*, T\})$. This is a variant of Woodin's second genericity iteration used in [16, §2], but working with many names for reals simultaneously. The named reals can be made generic for Woodin's extender algebra because (for any iterate P^*), the ordinal height of P^* is Woodin in $L(P^* \cup \{P^*, T\})$.

Let A be a maximal antichain of conditions in Woodin's extender algebra in $L(P^* \cup \{P^*, T\})$ which force the generic real to belong to $p[T]_z$. Note $A \in L(P^* \cup \{P^*, T\})$. Using the $P^* \cap \mathrm{On}$-chain condition for the extender algebra, A is bounded in $P^* \cap \mathrm{On}$. In particular $A \in P^*$, and hence $A \in Q^*$. Recall that conditions in A are identities in an infinitary language, and if x is generic for the extender algebra then the generic induced by x, call it W_x, consists of the identities satisfied by x. Now for any real x in a generic extension of Q^* by $\mathrm{Coll}(\omega, \eta)$ for $\eta < j(\delta)$ we have $x \in F$ iff $x \in p[T]_z$ iff $(\exists a \in A) a \in W_x$ iff x satisfies an identity $a \in A$. Letting $\dot{F}^* \in Q^*$ be a

$\mathrm{Coll}(\omega, j(\delta))$-name for the set of reals in small extensions of Q^* which satisfy an identity in A, it follows that $\langle Q, \Sigma, \delta\rangle$ captures F. ∎

Proof of Theorem 27. Given $A \subseteq \mathbb{R}$ and a prewellorder $\preceq$ on $\mathbb{R}$ of ordertype α, let $\psi_{A,\preceq}\colon \mathbb{R} \to L_\alpha(\mathbb{R}, A)$ be a standard surjection defined uniformly in A and $\preceq$. This can be done since every element of $L_\alpha(\mathbb{R}, A)$ is definable in the structure from reals and ordinals below α. Let $T_{A,\preceq}$ be the truth predicate for $L_\alpha(\mathbb{R}, A)$ in the codes, meaning that $\langle n, x_0, \ldots, x_i\rangle \in T_{A,\preceq}$ iff the nth formula in a standard enumeration holds of $\psi_{A,\preceq}(x_0), \ldots, \psi_{A,\preceq}(x_i)$ in $L_\alpha(\mathbb{R}, A)$. Given a function $e\colon L_\alpha(\mathbb{R}, A)^{<\omega} \to L_\alpha(\mathbb{R}, A)$ let $e_{A,\preceq} = \{\langle x_0, \ldots, x_i, y\rangle \colon \psi_{A,\preceq}(y) = e(\psi_{A,\preceq}(x_0), \ldots, \psi_{A,\preceq}(x_i))\}$.

If Theorem 27 fails then there exists $\alpha < \Theta$, $A \subseteq \mathbb{R}$, and a function $e\colon L_\alpha(\mathbb{R}, A)^{<\omega} \to L_\alpha(\mathbb{R}, A)$, so that the conclusion of the theorem fails for every Z which is closed under e, and every such Z is elementary in $L_\alpha(\mathbb{R}, A)$. Using the coding in the previous paragraph this can be expressed as a Σ^2_1 statement, quantifying over $\preceq$ and $e_{A,\preceq}$ instead of α and e. By Woodin's Σ^2_1 basis theorem, if the theorem fails then there are witnesses A, $\preceq$, $e_{A,\preceq}$ as above in a good pointclass, and in particular in a pointclass satisfying uniformization. So there exists $\widehat{e}_{A,\preceq}\colon \mathbb{R}^{<\omega} \to \mathbb{R}$ uniformizing $e_{A,\preceq}$ in its last coordinate. For $x_0, \ldots, x_i \in \mathbb{R}$ then $\widehat{e}_{A,\preceq}(x_0, \ldots, x_i)$ is an element of $\mathbb{R}$, and we view it as a function from ω to ω. Let $\widehat{e}^r_{A,\preceq}$ be the relation $\{\langle x_0, \ldots, x_i, n, k\rangle \colon \widehat{e}_{A,\preceq}(x_0, \ldots, x_i)(n) = k\}$.

Suppose for contradiction that Theorem 27 fails. By the previous fact from inner model theory, we can find a germ $\langle Q, \Sigma, \delta\rangle$ capturing a set F that codes A, $\preceq$, and $\widehat{e}^r_{A,\preceq}$ witnessing the failure in the sense of the previous paragraph, and codes the truth predicate $T_{A,\preceq}$.

Let $N = L_\theta(u)[\Sigma]$ where u is a real coding Q and θ is least so that $N \models \mathsf{ZFC}$. By $L_\theta(u)[\Sigma]$ we mean the model obtained by constructing over u relative to the predicate $\{\langle \mathcal{T}, \xi\rangle \colon \mathcal{T}$ is by Σ and ξ belongs to cofinal branch through $\mathcal{T}$ given by $\Sigma\}$. Note $\theta < \omega_1$ since ω_1 is measurable under AD. N is a model of choice, closed under Σ, and the restrictions of Σ to strict initial segments of N are in N.

Let α be the ordertype of $\preceq$, and let $e\colon L_\alpha(\mathbb{R}, A)^{<\omega} \to L_\alpha(\mathbb{R}, A)$ be the function coded by $\widehat{e}^r_{A,\preceq}$. Let $Z = \psi''_{A,\preceq}\mathbb{R} \cap N$. We will prove that Z is closed under e and that the conclusion of Theorem 27 holds for Z, contradicting the fact that A, $\preceq$, and $\widehat{e}^r_{A,\preceq}$ form a counterexample to the theorem.

We will make several uses of Lemma 30. First, we use it to see that $F \cap N$ belongs to N. By the lemma, with the trivial poset $\mathbb{P}$, inside every extension $N[h]$ of N by $\mathrm{Coll}(\omega, \mathbb{R})^N$ there is an iteration map $j\colon Q \to Q^*$ by Σ so that $\mathbb{R}^N$ can be realized as the derived reals of Q^*. Since $\langle Q, \Sigma, \delta\rangle$ captures F it follows using the capturing condition that $F \cap \mathbb{R}^N$ belongs to $N[h]$. This is true for all h, so $F \cap \mathbb{R}^N \in N$.

It follows that $\preceq \cap N$, $A \cap N$, $\widehat{e}^r_{A,\preceq} \cap N$, and $T_{A,\preceq} \cap N$ all belong to N. From the membership of $\widehat{e}^r_{A,\preceq} \cap N$ in N it follows that $\mathbb{R}^N$ is closed under $\widehat{e}_{A,\preceq}$. This in turn implies that Z is closed under e. Let M be the transitive collapse of Z. M can be determined from $T_{A,\preceq} \cap N$, and therefore belongs to N. In particular $\mathbb{R} \cap Z = \mathbb{R} \cap M \subseteq \mathbb{R}^N$. Without loss of generality we can assume the coding $\psi_{A,\preceq}$ is such that for every real x, $\psi_{A,\preceq}(0^\frown x)$ is a code for x. Then since $\mathbb{R}^N$ is closed under prepending 0, it follows that $\mathbb{R}^N \subseteq \mathbb{R} \cap M$. So $\mathbb{R}^N = \mathbb{R} \cap M$.

Let $\pi \colon M \to L_\alpha(\mathbb{R}, A)$ be the anticollapse embedding, $\overline{A} = \pi^{-1}(A)$, and $\overline{\alpha} = M \cap \mathrm{On}$. π is elementary since $Z = \mathrm{range}(\pi)$ is closed under e. Let $\mathbb{P}$ be absolutely proper in M. Then $\pi(\mathbb{P})$ is absolutely proper in $L_\alpha(\mathbb{R}, A)$, and moreover a club witnessing this belongs to Z, hence in particular $M \cap \mathbb{R} = N \cap \mathbb{R}$ belongs to this club. By definition it follows that $\mathbb{P} = \pi(\mathbb{P}) \cap (\mathbb{R} \cap M)$ is proper in N. Similarly if $\mathbb{P}$ is absolutely reasonable in M then it is reasonable in N.

Let G be $\mathbb{P}$-generic over N. By Lemma 30, inside every extension of $N[G]$ by $\mathrm{Coll}(\omega, \mathbb{R}^N) \times \mathrm{Coll}(\omega, \mathbb{R}^{N[G]})$, an iteration map $j \colon Q \to Q^*$ by Σ so that both $\mathbb{R}^N$ and $\mathbb{R}^{N[G]}$ can be realized as the derived reals of Q^*. Arguing as above but over $N[G]$ instead of N it follows that $F \cap \mathbb{R}^{N[G]}$ belongs to $N[G]$. This in turn implies, again arguing as above, that $\mathbb{R}^{N[G]}$ is closed under $\widehat{e}_{A,\preceq}$, that $Z^* := \psi''_{A,\preceq}\mathbb{R} \cap N[G]$ is elementary in $L_\alpha(\mathbb{R}, A)$, and that $Z^* \cap \mathbb{R} = \mathbb{R}^{N[G]}$. Let M^* be the transitive collapse of Z^*, and let $\widehat{\pi}$ be the anticollapse embedding. By elementarity M^* has the form $L_\nu(\mathbb{R}^{N[G]}, A \cap \mathbb{R}^{N[G]})$.

Note that $Z^* \supseteq Z$ so we can define an elementary embedding $k = \widehat{\pi}^{-1} \circ \pi \colon M \to L_\nu(\mathbb{R}^{N[G]}, A \cap \mathbb{R}^{N[G]})$. We claim that $\nu = \overline{\alpha}$ and k is the identity on ordinals. For both statements it is enough to show that $Z^* \cap \mathrm{On} \subseteq Z \cap \mathrm{On}$. Without loss of generality, through an appropriate choice of $\psi_{A,\preceq}$, we can assume that $Z \cap \mathrm{On} = \varphi''\mathbb{R}^N$ and $Z^* \cap \mathrm{On} = \varphi''\mathbb{R}^{N[G]}$ where φ is the norm associated to $\preceq$. It is then enough to show that every real in $N[G]$ has a $\preceq$-equivalent real in N.

Fix H and H^* witnessing that $\mathbb{R}^N$ and $\mathbb{R}^{N[G]}$ respectively can be realized as the derived reals of Q^*. Using the capturing condition, and since F codes $\preceq$, we have a name $\dot{r} \in Q^*$ so that $\dot{r}[E]$ is equal to the intersection of $\preceq$ with the derived reals of Q^* using E, for every generic E for $\mathrm{Coll}(\omega, \delta)$ over Q^*. Let $\dot{\varphi}$ name the norm associated to $\dot{r}$. Fix $x \in N[G]$ and let $\mu = \dot{\varphi}[H^*](x)$. Fix $\eta < j(\delta)$ so that $x \in Q^*[H^* \restriction \eta]$, and a $\mathrm{Coll}(\omega, \eta)$-name $\dot{u} \in Q^*$ so that $\dot{u}[H^* \restriction \eta] = x$. Using the symmetry of the collapse we can assume it is outright forced in $\mathrm{Coll}(\omega, j(\delta))$ that $\dot{\varphi}(\dot{u}) = \mu$. Let $y = \dot{u}[H] \in N$. Since $H \restriction \eta$ can be coded by a real in $N \subseteq N[G]$, it belongs to the derived model of Q^* by H^*. Using standard arguments one can then rearrange H^* to a generic E^* which produces the same derived reals as H^*, but with $E^* \restriction \eta = H \restriction \eta$. The former fact implies that $\dot{r}[E^*] = \dot{r}[H^*]$ and hence $\dot{\varphi}[E^*](x) = \dot{\varphi}[H^*](x) = \mu$, while the latter implies that $\dot{\varphi}[E^*](y) = \mu$. Putting the two together it follows that

x and y belong to the same $\dot{r}[E^*]$-equivalence class, and hence to the same $\preceq$-equivalence class.

We now have the elementarity and agreement requirements of Theorem 27. It remains to find $\dot{A}^* \in N$ so that for every $\mathbb{P}$-generic G, $\dot{A}^*[G] = A \cap \mathbb{R}^{N[G]}$.

Let δ_0 be the first Woodin cardinal of Q. Working in N find an iteration $\sigma\colon Q \to Q'$ of Q so that every real in any generic extension of N by $\mathbb{P}$ is generic over Q' for Woodin's extender algebra at $\sigma(\delta_0)$. This is a variant of Woodin's second genericity iteration used in [16, §2], but working with all names for reals simultaneously. In N, the iteration terminates at some length $< |\mathbb{P} \times U|^+$ where U is the set of canonical $\mathbb{P}$-names for reals.

Using the capturing condition, and since F codes A, there is a name $\dot{A}' \in Q'$ so that for every generic E for $\mathrm{Coll}(\omega, \sigma(\delta))$, and any x added by a strict initial segment of E, $x \in A$ iff $x \in \dot{A}'[E]$. Woodin's extender algebra at $\sigma(\delta_0)$ is subsumed by $\mathrm{Coll}(\omega, \sigma(\delta))$. From this using the symmetry of the collapse it follows that there is an extender algebra name $\dot{A}'' \in Q'$ so that for every generic g for the extender algebra, $\dot{A}''[g] = A \cap Q'[g]$. From every real generic for the extender algebra one can definably construct a generic filter g_x for the algebra so that $x \in Q'[g_x]$. Now, working in N, let $\dot{A}^*$ be a $\mathbb{P}$-name for the set of reals x (in the extension by $\mathbb{P}$) so that $x \in \dot{A}''[g_x]$. Then for every x in an extension $N[G]$ of N by $\mathbb{P}$, $x \in \dot{A}^*[G]$ iff $x \in \dot{A}''[g_x]$ iff $x \in A$, with the last equivalence using the fact that by construction x is generic for the extender algebra over Q'. ∎

7. Open questions

Much remains open about how the existence of infinite mad families relates to the existence of other pathological sets of reals. Mathias's original conjecture is probably the most well known example.

Question 1 (Mathias [14])**.** (In ZF+DC) If every set has the Ramsey property, does it follow that there are no infinite mad families?

Theorem (Mathias [13])**.** If every set has the Ramsey property, then every filter is meager.

Question 2. If there are no infinite mad families, does it follow that every filter is meager?

Question 3 (Törnquist [25])**.** Does AD imply that there are no infinite mad families?

To establish Theorem 16 we needed $\mathbb{P}_U$, a poset that has the Mathias property even when there are no Ramsey ultrafilters.

Question 4. Is there a nontrivial definable (e.g. analytic) ccc forcing with the Mathias property?

Question 5. Is it consistent relative to ZFC alone that every set of reals in $L(\mathbb{R})$ is H-Ramsey for every $H \in L(\mathbb{R})$?

A positive answer to Question 5 would also answer the longstanding open question whether "every set has the Ramsey property" is consistent relative to ZFC alone. A negative answer might indicate something special about $[\omega]^\omega$, even among definable happy families.

There has been some further study of semiselective coideals since Farah's original paper.

Theorem (Di Prisco–Mijares–Uzcátegui [5])**.** Suppose that G is generic over V for the Levy collapse of a weakly compact cardinal. Then, in $V[G]$, every set $X \subseteq [\omega]^\omega$ that belongs to $L(\mathbb{R})$ is H-Ramsey for every semiselective coideal H.

Question 6 (Di Prisco–Mijares–Uzcátegui)**.** Is the weakly compact in the theorem necessary?

Many questions remain open about how the existence of infinite mad families relates to other weak choice principles. Here are two examples.

Question 7. Is it consistent with $\mathsf{ZF}+\mathsf{DC}$ that there are infinite mad families but there is an almost-disjoint family that doesn't extend to a mad family?

Call a family $A \subseteq [\omega]^\omega$ **nearly mad** if any two members of A are either almost equal (their symmetric difference is finite) or almost disjoint.

Question 8. Is it consistent with $\mathsf{ZF}+\mathsf{DC}$ that infinite nearly mad families exist but infinite mad families do not?

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