Forcing with ultrafilters

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Outline

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For κ singular strong limit ($\tau < \kappa \to 2^\tau < \kappa$), $2^\kappa = \kappa^+$.

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A scale of length τ^+ in $\prod_{i<\omega} \delta_i$ is a sequence $\langle f_\xi \mid \xi < \tau^+ \rangle$ which is $<^*$-increasing and cofinal.

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A scale of length τ^+ is good if it has a club of good points. Otherwise the scale is bad.

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Requires some assumptions on $\mathbb{P}$ and G: $\mathbb{P}$ is a partially ordered set, G is a filter, and G meets every dense open subset of $\mathbb{P}$ in M. (Such G are called generic over M.)

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Prove $(\aleph_2)^{M[G]} = (\aleph_2)^M$. Here fundamental theorem of forcing is crucial. Allows reasoning in M about $f : (\aleph_1)^M \rightarrow (\aleph_2)^M$ that belong to $M[G]$. Using properties of $\mathbb{P}$ in M show such f are not onto.

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Further, M and $M[G]$ have the same cardinals. Argument again uses fundamental theorem of forcing. Relies on the closure of $\mathbb{P}$ to show cardinals $\leq \kappa$ are preserved.

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Theorem (Shelah)

Failures of SCH *at* $\aleph_\omega$ *are limited: if* $2^{\aleph_n} < \aleph_\omega$ *for each n, then* $2^{\aleph_\omega} < \aleph_{\omega_4}$.

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Fix ω measurable cardinals κ_n, with ultrafilters $\mathcal{U}_n$.

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Magidor forcing is a generalization of Prikry forcing, designed to change cofinalities to $\lambda > \omega$.

Requires a measurable cardinal κ with many measurable cardinals below it. Will add, for example, $g = \{\alpha_\xi \mid \xi < \omega_1\}$ increasing and cofinal.

Uses many local instances of Prikry forcing. In example above, $\sup\{\alpha_\xi \mid \xi < \omega\}$ is measurable, and $\{\alpha_\xi \mid \xi < \omega\}$ Prikry generic for it. Similarly above ω.

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Adds a cofinal subset $\{\alpha_i \mid i < \omega\}$ of $\sup\{\kappa_n \mid n < \omega\}$, and does nothing else.

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Question on PFA answered in the positive.

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(Assuming the existence of a supercompact cardinal.) There is a model with a cardinal κ so that:

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Gitik–Sharon showed it does not follow from failure of SCH.

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Try to preserve Magidor–Shelah result.

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Try to preserve Magidor–Shelah result. (Two levels of preservation. First, ${}^\kappa 2 = \tau^{++}$, second G. Both pose difficulties.)

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Question

Can this be done with additional collapsing, so that κ becomes $\aleph_\omega$? Or even $\aleph_{\omega^2}$?