Higher analog of the proper forcing axiom

Itay Neeman

Department of Mathematics
University of California Los Angeles
Los Angeles, CA 90095

www.math.ucla.edu/~ineeman

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Forcing axioms

Developed in late 1960s early 1970s, initially to crystalize center points for applications of iterated forcing.

Martin's axiom (MA, for ω_1 antichains): for any c.c.c. poset $\mathbb{P}$ and any collection F of ω_1 maximal antichains of $\mathbb{P}$, there is a filter on $\mathbb{P}$ which meets every antichain in F.

Obtained through an iteration of enough c.c.c. posets. Can then be used axiomatically as a starting point for consistency proofs that would otherwise require an iteration of c.c.c. posets.

Key points in proving consistency of MA:

(a) Finite support iteration of c.c.c. posets does not collapse ω_1, and in fact the iteration poset is itself c.c.c.

(b) Can "close off", that is reach a point where enough c.c.c. posets have been hit to ensure MA.

Proper forcing

There are classes of posets other than c.c.c. which also preserve ω_1.

Definition

Let $\mathbb{P}$ be a poset. Let κ be large enough that $\mathbb{P} \in H(\kappa)$. $p \in \mathbb{P}$ is a master condition for $M \prec H(\kappa)$ if

1. p forces that every maximal antichain A of $\mathbb{P}$ that belongs to M is met by the generic filter inside M.

Equivalently any of:

2. p forces that $\dot{G} \cap \check{M}$ is generic over M.
3. p forces that $M[\dot{G}] \prec H(\kappa)[\dot{G}]$ and $M[\dot{G}] \cap V = M$.

Definition

$\mathbb{P}$ is proper if for all large enough κ and all countable $M \prec H(\kappa)$, every condition in M extends to a master condition for M.

Proper posets do not collapse ω_1; immediate from (3).

PFA

Proper forcing axiom (PFA): the parallel of MA for proper posets. Again used axiomatically as a starting point for consistency proofs.

Key points in consistency proof of PFA:

(a) Countable support iteration of proper posets does not collapse ω_1, and is indeed proper.

(b) Can close off, assuming a supercompact cardinal.

For (b), fix a supercompact cardinal θ. Iterate up to θ hitting proper posets given by a Laver function. At stage θ, using properties of the Laver function and supercompactness, have covered enough posets to ensure PFA holds.

Note

In addition to (a) and (b), important also that iteration does not collapse θ, but this is clear.

Higher analogs

In the case of MA, the forcing axiom has higher analogs, and in fact strengthenings.

For example it is consistent that for all c.c.c. posets, all maximal antichain in families of size ω_2 can be simultaneously met by a filter.

Initial expectation was that similar analogs should exist for PFA.

Naive attempt: demand existence of master conditions also for models of size ω_1.

Posets in the resulting class preserve ω_1 and ω_2 (certainly a necessary property for a higher analog).

But preservation under iteration fails.

More on iteration

When iterating proper posets, countable support used to ensure properness of iteration at limits of countable cofinality. Basic idea for preservation, e.g. at stage ω, for iteration $\langle \mathbb{P}_\xi \mid \xi \leq \theta \rangle$ of posets $\langle \dot{\mathbb{Q}}_\xi \mid \xi < \theta \rangle$:

Let D_n enumerate all dense sets of $\mathbb{P}_\omega$ that belong to M. Diagonalize to create a condition $p \in \mathbb{P}_\omega$ which "almost meets" each of them, meaning that below p, D_n is reduced to a dense set in $\mathbb{P}_n$. This can be done extending only coordinates $\geq n$ when handling D_n, so that the construction converges to a condition p.

That p is a condition uses countable support.

Properness of the individual posets iterated then allows extending p to a master condition in $\mathbb{P}_\omega$.

Similar diagonalization used at all limits. For limits of cofinality ω, exactly same ideas. For limits of higher cofinality same idea because support is countable.

Higher analogs and iteration

For diagonalization process at a limit α over dense sets in M, important that both $\mathrm{cof}(\sup(M \cap \alpha))$ and support match $|M|$.

Breaks down already at $\alpha = \omega$ if $|M| \geq \omega_1$.

Seemingly a terminal barrier for higher analog of PFA.

Moreover, in contrast with MA, PFA actually implies that the continuum is ω_2. This is further evidence against higher analogs, though strictly speaking only implies that analog is not a *strengthening* of PFA.

Models as side conditions

Models are used as side conditions in several very nice applications of PFA.

For example, fix θ and consider the following posets $\mathbb{P}$.

Conditions are increasing finite sequences $M_0 \in M_1 \in \cdots \in M_n$ of countable Σ_1 elementary submodels of $H(\theta)$.

(Abusing notation slightly regard the condition as a set $s = \{M_0, \ldots, M_n\}$. No loss of information since the order of the sequence is determined from the models.)

Poset order is the natural one, reverse inclusion.

$\mathbb{P}$ is proper. For $\delta > \theta$ and $M^* \prec H(\delta)$, any condition s with $M = M^* \cap H(\theta) \in s$ is a master condition for M^*. In fact a strong master condition: forces that the generic filter for $\mathbb{P}$ is also generic (over V) for $\mathbb{P} \cap M$.

Models as side conditions (cont.)

This is among the simplest examples, and models are not needed.

Can cast the forcing in terms that use the ordinals $M_i \cap \omega_1$ instead of the models M_i.

Due to Baumgartner, adds a club in ω_1 with finite conditions.

Other, much more sophisticated uses of models as side conditions. Models used to enforce properness.

Clubs in ω_2

Around 2003, higher analog found for adding clubs with finite conditions.

Friedman, Mitchell independently force to add a club subset of ω_2, with finite conditions. Mitchell also adds club subsets to inaccessible θ, turning θ to ω_2.

Use countable models as side conditions to enforce properness (and in particular preservation of ω_1).

Proofs are quite complicated. Sequence of models is no longer increasing, and there is a careful agreement condition between countable models on the sequence.

Can be simplified substantially by explicitly adding models of greater size.

We illustrate in the case of adding a club subset to an inaccessible θ while converting it to ω_2. Similar definitions work for adding club subset of ω_2.

Adding club in θ with finite conditions

Definition

A node is an a model M of one of the following types:

1. $M \prec_1 H(\theta)$ is countable. (Countable type nodes.)
2. $M = H(\kappa) \prec_1 H(\theta)$ with κ of cofinality at least ω_1. (Rank type nodes.)

A side condition is an increasing sequence of nodes $M_0 \in M_1 \in \cdots \in M_n$ which is closed under intersections.

As before can regard the condition as a set $s = \{M_0, \ldots, M_n\}$ with no loss of information.

$\mathbb{P}_{\text{side}}$ is the poset of side conditions, ordered by reverse inclusion.

Adding club in θ with finite conditions (cont.)

Lemma

If s is a side condition and $Q \in s$, then s is a strong master condition for Q.

Sketch of proof.

Define $\mathrm{res}_Q(s)$, the residue of s in Q, to be $\{M \in s \mid M \in Q\}$.

Using closure of s under intersections can show $\mathrm{res}_Q(s)$ is increasing. It is also closed under intersections by closure of s and elementarity of Q. So $\mathrm{res}_Q(s)$ is a side condition.

Prove that any side condition $t \in Q$ which extends $\mathrm{res}_Q(s)$ is compatible with s. This is enough to establish lemma.

Proof of compatibility is straightforward if Q if of rank type, a bit more involved if Q is of countable type. $\square$

Adding club in θ with finite conditions (cont.)

Lemma holds, with same proof, if nodes are restricted to belong to a given class $\mathcal{C}$, so long as:

1. If $W \in M$ of rank and countable type respectively both belong to $\mathcal{C}$, then $M \cap W \in \mathcal{C}$.
2. In the situation of condition (1), $M \cap W \in W$.

(1) needed for closure of side condition under intersection to make sense. (2) clear when working in $H(\theta)$, but meaningful in parallel forcing to add club in ω_2.

For Lemma 4 to be useful also need $\mathcal{C}$ to be stationary in both $\mathcal{P}_{<\omega_1}(H(\theta))$ and $\mathcal{P}_{<\theta}(H(\theta))$. Can then use forcing to add clubs through stationary sets.

Another proof of the consistency of PFA

$\mathbb{P}_{\text{side}}$ can be used to reprove the consistency of PFA, with finite support.

Let θ be supercompact, f a Laver function, $\mathbb{P}_{\text{side}}$ the poset of side conditions with nodes elementary in $(H(\theta); f)$.

Definition

Conditions in $\mathbb{A}$ are pairs $\langle s, p\rangle$ where:

1. $s \in \mathbb{P}_{\text{side}}$.
2. p is a function with $\mathrm{dom}(p) \subseteq \{\kappa \mid H(\kappa) \in s\}$.
3. $p(\kappa)$ is defined only if
   - **3.1** $f(\kappa)$ is a name in the poset $\mathbb{A} \cap H(\kappa)$. Call it $\dot{\mathbb{Q}}_\kappa$.
   - **3.2** $\langle \emptyset, \emptyset\rangle$ forces in $\mathbb{A} \cap H(\kappa)$ that $\dot{\mathbb{Q}}_\kappa$ is proper.
4. When defined, $p(\kappa)$ is an $\mathbb{A} \cap H(\kappa)$-name, forced by $\langle s \cap H(\kappa), p \restriction \kappa\rangle$ to be (a) in $\dot{\mathbb{Q}}_\kappa$, (b) a master condition for each countable $M \in s$ with $\kappa \in M$.

$\langle s^*, p^*\rangle < \langle s, p\rangle$ iff $s^* \supseteq s$, and for each $\kappa \in \mathrm{dom}(p)$, $\langle s^* \cap H(\kappa), p^* \restriction \kappa\rangle$ forces that $p^*(\kappa)$ extends $p(\kappa)$.

Another proof of the consistency of PFA, comments

Formally this is a definition of $\mathbb{A} \cap H(\kappa)$ by induction on κ.

Note that $\mathrm{dom}(p)$ is finite.

Use of side conditions allows proving that $\mathbb{A}$ is proper. (Proof is again by induction on κ.) In particular ω_1 is preserved.

Must also show θ is preserved. Since $\mathbb{A}$ is not quite an iteration, this is not automatic. Use the fact that any s with $H(\kappa) \in s$ is a strong master condition for $H(\kappa)$ in $\mathbb{P}_{\mathrm{side}}$ to get preservation of θ.

Why?

Why bother with a finite support proof of the consistency of PFA?

Recall the question of higher analogs.

Impediment for higher analogs is need for support and cofinality of limits to match size of models. This need is eliminated through use of finite support and side conditions.

Get higher analog?

Not so fast....

In finite support proof of PFA, needed $\mathbb{P}_{\mathrm{side}}$ to preserve two cardinals, ω_1 and θ.

For a higher analog, need a poset of side conditions which preserves *three* cardinals, ω_1, ω_2, and θ.

Side conditions preserving three cardinals

A pre-cursor exists in Mitchell's proof that $I(\omega_2)$ can be trivial. This proof involves preservation of three cardinals: ω_1, a weakly compact cardinal κ which is turned into ω_2, and κ^+.

Need a different poset, to decouple the third cardinal from κ, so that the third preserved cardinal can be supercompact.

Can be done, but poset is quite complicated.

As expected involves nodes of three types, countable, ω_1, and rank type.

But not all nodes are elementary.

Side conditions preserving three cardinals (cont.)

The presence of non-elementary nodes causes substantial technical complications (including closure requirements beyond closure under intersections).

In resulting forcing axiom, countable non-elementary nodes translate to certain countable $\in$-linear sets of nodes of type ω_1.

Will present a "baby" version of forcing axiom, where many more countable $\in$-linear sets of nodes of type ω_1 are allowed, not only ones induced by the countable non-elementary nodes of the three-type side conditions.

This weakens the axiom, but simplifies it substantially.

Two-type side conditions with non-elementary nodes

Definition

$\mathcal{C}$ is suitable for two-type side conditions relative to H if:

1. Every $M \in \mathcal{C}$ belongs to H and satisfies one of:
   - **1.1** (Elementary countable type) $M \prec_1 H$ and $|M| = \omega$.
   - **1.2** (Type ω_1) $M \prec_1 H$, $|M| = \omega_1$, and $\omega_1 \subseteq M$.
   - **1.3** (Non-elementary countable type) $|M| \leq \omega$, $M \neq \emptyset$, M is linearly ordered by $\in$, and every element of M is a $\mathcal{C}$ node of type ω_1.
2. Let $M, N \in \mathcal{C}$ of ctbl and ω_1 type resp with $N \in M$.
   - **2.1** M elementary $\rightarrow M \cap N \in \mathcal{C}$, $M \cap N \in N$.
   - **2.2** M non-elementary and $M \cap N \neq \emptyset \rightarrow M \cap N \in \mathcal{C}$, $M \cap N \in N$.

Not always possible to arrange (2.1) for stationarily many N. But (2.1) holds automatically for N internal on a club in H, meaning $N = \bigcup_{\xi<\omega_1} X_\xi$, for $\{X_\xi\} \in H$ continuous increasing with $X_\xi \in N$ for each ξ. Typically restrict to such N, assume the set of such N is stationary.

Two-type side conditions with non-elem nodes (cont.)

Let H be transitive, correct about cardinality. Let $\mathcal{C}$ be suitable for two-type side conditions relative to H.

Definition

A sequence $s = \langle M_i \mid i < n \rangle$ is a two-type side condition relative to $\mathcal{C}$ if:

1. Each M_i is an element of $\mathcal{C}$.
2. $\langle M_i \mid i < n \rangle$ is $\in$-increasing, meaning $M_i \in M_{i+1}$.
3. Let $M, N \in s$ of ctbl and ω_1 type resp, with $N \in M$.
   - 3.1 If M is elementary, then $M \cap N \in s$.
   - 3.2 If M is non-elementary and $M \cap N \neq \emptyset$ then $\exists$ ctbl non-elementary $\bar{M} \in s$ with $\bar{M} \in N$, $\bar{M} \supseteq M \cap N$.

Similar to previous definition, replacing transitive nodes by nodes of type ω_1.Residue and strong properness unaffected by this change.

Main change is allowing non-elementary nodes, and the addition of condition (3.2).

Higher analog of properness

Definition

A poset $\mathbb{P}$ is two-type proper on $\mathcal{C}$ if there exists a function mc on $\mathcal{C}$ so that (for all nodes and side conditions):

1. $\mathrm{mc}(M)$ is an open subset of $\mathbb{P}$.
2. If M is elementary, then every $p \in \mathrm{mc}(M)$ is a master condition for M in $\mathbb{P}$.
3. For $N \in M$ of type ω_1 and elementary ctbl resp, $\mathrm{mc}(M \cap N) \supseteq \mathrm{mc}(M) \cap \mathrm{mc}(N)$.
4. For $M_1 \subseteq M_2$ non-elementary ctbl, $\mathrm{mc}(M_1) \supseteq \mathrm{mc}(M_2)$.
5. For M non-elementary ctbl and $N \in M$ (so N of type ω_1), $\mathrm{mc}(N)$ is dense in $\mathrm{mc}(M)$.
6. For every two-type side condition s and every elementary $Q \in s$, if $p \in Q$ and $p \in \bigcap_{M \in \mathrm{res}_Q(s)} \mathrm{mc}(M)$ then p extends to a condition $q \in \bigcap_{M \in s} \mathrm{mc}(M)$.

(Take $\bigcap_{M \in \mathrm{res}_Q(s)} \mathrm{mc}(M) = \mathbb{P}$ if $\mathrm{res}_Q(s) = \emptyset$.)

Higher analog of properness (cont.)

Think of $\mathrm{mc}(M)$ as a set of distinguished master conditions for M.

Conditions in definition spell out properties of the distinguished master conditions.

The condition that actually gives existence of distinguished master conditions is (6).

If we restrict to only countable elementary nodes throughout, and $\mathbb{P}$ is proper in the ordinary sense, then the conditions hold with the function assigning each M the set of master conditions for M.

Note in particular that (6) in this case follows from the fact that any $p \in M$ extends to a master condition for M.

But for side conditions with nodes of two types, existence of master conditions for individual models does not imply (6).

Higher analog of properness (cont.)

Definition

For regular θ and $f\colon H(\theta)^{<\omega} \to H(\theta)$, let $\mathcal{C}(\theta, f)$ be the set of all M such that one of the following holds:

1. $|M| = \omega$, $M \prec_1 H(\theta)$, and M is closed under f.
2. $|M| = \omega_1$, M is internal on a club, $M \prec_1 H(\theta)$, and M is closed under f.
3. $|M| \leq \omega$, $M \neq \emptyset$, M is linearly ordered by $\in$, every $N \in M$ satisfies (2), and $(\forall N \in M)(M \cap N \in N)$.

Definition

A poset $\mathbb{P}$ is baby $\{\omega, \omega_1\}$-proper if there is θ and $f\colon H(\theta)^{<\omega} \to H(\theta)$ so that $\mathbb{P} \in H(\theta)$ and $\mathbb{P}$ is two-type proper on $\mathcal{C}(\theta, f)$.

Why baby? Allowed many non-elementary node. More refined version allows only non-elementary nodes obtained in very specific ways from partially elementary substructures of an inner model.

Higher analog of PFA

Definition

The baby $\{\omega, \omega_1\}$-proper forcing axiom states that for every baby $\{\omega, \omega_1\}$-proper poset $\mathbb{P}$, and every collection $\mathcal{F}$ of ω_2 maximal antichains of $\mathbb{P}$, there is a filter G on $\mathbb{P}$ which meets every antichain in $\mathcal{F}$.

Theorem

Assume θ is supercompact. Then the baby $\{\omega, \omega_1\}$-proper forcing axiom holds in a forcing extension of V.

Fairly broad. include all c.c.c. posets, and posets to collapse cardinals to ω_2 (but with finite conditions). Closed under compositions.

Will talk about applications in next workshop. Applications mostly use forcing axiom plus additional principles that hold in the extension giving the axiom.