# ARONSZAJN TREES AND FAILURE OF THE SINGULAR CARDINAL HYPOTHESIS

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**Abstract.** The tree property at $\kappa^+$ states that there are no Aronszajn trees on $\kappa^+$, or, equivalently, that every $\kappa^+$ tree has a cofinal branch. For singular strong limit cardinals $\kappa$, there is tension between the tree property at $\kappa^+$ and failure of the singular cardinal hypothesis at $\kappa$; the former is typically the result of the presence of strongly compact cardinals in the background, and the latter is impossible above strongly compacts. In this paper we reconcile the two. We prove from large cardinals that the tree property at $\kappa^+$ is consistent with failure of the singular cardinal hypothesis at $\kappa$.

**§1. Introduction.** In the early 1980s Woodin asked whether failure of the singular cardinal hypothesis (SCH) at $\aleph_\omega$ implies the existence of an Aronszajn tree on $\aleph_{\omega+1}$. More generally, in 1989 Woodin and others asked whether failure of the SCH at a cardinal $\kappa$ of cofinality $\omega$, implies the existence of an Aronszajn tree on $\kappa^+$, see Foreman [7, §2]. To understand the motivation for the question let us recall some results surrounding the SCH and trees in infinitary combinatorics.

The singular cardinal hypothesis, in its most specific form, states that $2^\kappa = \kappa^+$ whenever $\kappa$ is a singular strong limit cardinal. (There are several forms that are more general. For example the statement that $\kappa^{\mathrm{cof}(\kappa)} = \kappa^+$ whenever $\kappa$ is singular and $2^{\mathrm{cof}(\kappa)} < \kappa$. Or the statement that for every singular cardinal $\kappa$, $2^\kappa$ is as small as it can be, subject to two requirements: monotonicity, namely that $2^\kappa \geq \sup\{2^\delta \mid \delta < \kappa\}$; and König's theorem, which implies $\mathrm{cof}(2^\kappa) > \kappa$. Both these forms imply the specific form, that $2^\kappa = \kappa^+$ whenever $\kappa$ is a singular strong limit cardinal.)

Cohen forcing of course shows that the parallel hypothesis for regular cardinals is consistently false. Indeed it can be made to fail in any arbitrary way, subject to monotonicity and König's theorem. For a while after the introduction of forcing it was expected that the same should hold for the SCH, and that proving this was only a matter of discovering sufficiently sophisticated forcing notions. Some progress was made in this direction, and ultimately led to models with failure of the SCH described below. But it turned out that changing the power of a singular cardinal is much harder than changing the power of a regular cardinal, and in some cases it is outright impossible. The first indication of this was a theorem of Silver [25], that the continuum hypothesis cannot fail for the first time at a singular cardinal of uncountable cofinality. Another is a theorem of

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Solovay [27], that the SCH holds above a strongly compact. The most celebrated is a theorem of Shelah [23], that if $2^{\aleph_n} < \aleph_\omega$ for each $n < \omega$, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

Still, the SCH can be made to fail. One way to violate the hypothesis is to start with a measurable cardinal $\kappa$, make the continuum hypothesis fail at $\kappa$, for example by increasing the power set of $\kappa$ to $\kappa^{++}$—an easy task as $\kappa$ is regular, not singular—and then, assuming $\kappa$ remained measurable, use Prikry forcing to turn its cofinality to $\omega$. In the resulting model, $\kappa$ is a singular strong limit cardinal of cofinality $\omega$, and the singular cardinal hypothesis fails at $\kappa$.

To make sure that $\kappa$ remains measurable after its powerset is increased, one has to start with a stronger assumption on $\kappa$ than measurability. This was first done by Silver, who assumed $\kappa$ was supercompact. The assumption was reduced by Woodin to the existence of an elementary embedding $j: V \to M$ with critical point $\kappa$, so that $j(\kappa) > \kappa^{++}$ and $M^\kappa \subseteq M$. Gitik [9] constructed a model where $\kappa$ satisfies Woodin's assumption, starting from the assumption that the Mitchell order of $\kappa$ is $\kappa^{++}$. This assumption is thus sufficient for violating the SCH. By Mitchell [20] the assumption is necessary for failure of the continuum hypothesis at a measurable cardinal. By Gitik [10] it is in fact necessary for failure of the SCH.

Woodin's question was intended to test whether the method above, singularizing a measurable cardinal where the continuum hypothesis fails, is the only way to violate the SCH. To understand this we need a few facts about Aronszajn trees.

A $\kappa^+$-tree is a tree of height $\kappa^+$, with levels of size less than $\kappa^+$. The tree property at $\kappa^+$ states that every $\kappa^+$-tree has a cofinal branch. In contrast, an Aronszajn tree on $\kappa^+$ is a $\kappa^+$-tree with no cofinal branches. Aronszajn was the first to construct such a tree, on $\aleph_1$. More generally, if $\kappa^{<\kappa} = \kappa$, then there is an Aronszajn tree on $\kappa^+$. Moreover the tree is the union of $\kappa$ antichains (such trees are called special). Being special Aronszajn implies that the tree remains Aronszajn, and indeed special Aronszajn, in cardinal preserving generic extensions.

If $\kappa$ is measurable, then certainly $\kappa^{<\kappa} = \kappa$. Thus, if $\kappa$ is measurable, then there is an Aronszajn tree on $\kappa^+$, and the tree remains Aronszajn in cardinal preserving extensions where $\kappa$ is singularized.

We can now connect this with the general form of Woodin's question: does failure of the SCH at a cardinal $\kappa$ of cofinality $\omega$ imply the existence of an Aronszajn tree on $\kappa^+$. Remember that the question was intended to test whether the only way to violate the SCH is by singularizing a measurable cardinal. If every model where the SCH fails at $\kappa$ is obtained by singularizing from a model where $\kappa$ is measurable, then by the above, in every such model there is an Aronszajn tree on $\kappa^+$, and the answer to the question is positive.

It turns out that there are other ways to violate the SCH, see for example Gitik–Magidor [12, 13], and Gitik [11]. But these methods did not answer the test question, and over time the test question gained a life of its own. It can be rephrased, to ask whether the tree property at $\kappa^+$ implies the SCH at $\kappa$. The tree property is a reflection property, and at successors of singular strong limit cardinals it probably has substantial strength. (The only known route to establishing the property in such situations goes through a theorem of Magidor–Shelah

[19], which uses strongly compact cardinals.) $\mathsf{SCH}$ is sometimes a consequence of reflection that has substantial strength (though it is not a consequence of stationary reflection, by Sharon [22]). Perhaps the earliest example of this is Solovay's theorem that the $\mathsf{SCH}$ holds above a strongly compact. A more recent one is Viale's theorem [30] that the proper forcing axiom implies $\mathsf{SCH}$. An example more closely related to trees is Todorčević's theorem [29] that Rado's conjecture implies $\mathsf{SCH}$. (By Todorčević [28] Rado's conjecture is equivalent to the statement that a tree of height $\aleph_1$ is special iff all its subtrees of size $\aleph_1$ are special.) With the question persisting, it was natural to hope for a positive answer, adding another theorem to this list.

The singular cardinal hypothesis is closely tied with PCF theory (possible cofinalities theory, see Shelah [24] or any of [1], [2], [15], and [16]). It therefore seemed reasonable that a positive answer, if possible, would be obtained by isolating some PCF property that follows from failure of $\mathsf{SCH}$, and implies the existence of an Aronszajn tree. Several candidates were considered for the intermediate property, between failure of $\mathsf{SCH}$ and the existence of an Aronszajn tree. Many had to do with square principles, introduced in Schimmerling [21] and generalizing the original principles defined by Jensen, since it is known by work of Jensen that the existence of special Aronszajn trees is equivalent to a weak square principle. This program of research was initiated by Cummings, Foreman, and Magidor. It led to a large body of work, particularly about square principles and connections between these principles and PCF theory, for example [4], [5], [6], and [8].

However prospects that the program would lead to a solution to the general form of Woodin's question dimmed, when Gitik–Sharon [14] showed that two of the key candidates for the intermediate between failure of $\mathsf{SCH}$ and existence of Aronszajn trees, specifically the approachability property and the weak square principle, do not in fact follow from failure of $\mathsf{SCH}$. And indeed, the answer to the question is negative. We prove in this paper that failure of $\mathsf{SCH}$ at $\kappa$ does not imply the existence of an Aronszajn tree on $\kappa^+$:

THEOREM 1.1. *Suppose there are $\omega$ supercompact cardinals. Then it is consistent that there is a cardinal $\kappa$ so that:*

1. *$\kappa$ is a strong limit cardinal of cofinality $\omega$.*
2. *$2^\kappa = \kappa^{++}$, hence $\mathsf{SCH}$ fails at $\kappa$.*
3. *There are no Aronszajn trees on $\kappa^+$.*

*Moreover, it is consistent with the above that there is both a very good scale and a bad scale on $\kappa$.*

Scales are PCF objects. The existence of a bad scale implies failure of the approachability property, which in turn implies failure of weak square. In the Gitik–Sharon model there is both a very good scale and a bad scale on $\kappa$. The existence proofs for good and bad respectively are due to Gitik–Sharon [14] and Cummings–Foreman [3]. The existence proofs in our model are similar.

Theorem 1.1 relies heavily on the construction of the Gitik–Sharon model. The simple outline of the proof of Theorem 1.1 is this: combine the construction of Gitik–Sharon [14] with the proof in Magidor–Shelah [19] that the tree property holds at successors of limits of strongly compact cardinals. Gitik and Sharon

start with a model where $\kappa$ is supercompact, force to make $2^\kappa = \nu^{++}$ where $\nu = \kappa^{+(\omega)}$, and then force further, with a diagonal Prikry poset, to add a sequence $g = \langle g(n) \mid n < \omega\rangle$ which collapses $\nu$ to $\kappa$ and changes the cofinality of $\kappa$ to $\omega$. Here we assume that there are $\omega$ supercompact cardinals $\kappa = \kappa_0 < \kappa_1 < \dots$, and modify the Gitik–Sharon poset to use $\nu = \sup_{n<\omega} \kappa_n$. Then the successor of $\kappa$ in the extension is $\nu^+$. By Magidor–Shelah [19], $\nu^+$ has the tree property in $V$. All we have to do is show that it continues to have the tree property in the extension. This, of course, is easier said than done. But it is too early to go into further details. The proof that $\nu^+$ continues to have the tree property in the extension is given in Section 3.

The assumption in Theorem 1.1 can be weakened to the existence of $\omega$ cardinals $\langle \kappa_n \mid n < \omega\rangle$, so that each $\kappa_n$ is $\nu^+$ supercompact, where $\nu = \sup_{n<\omega} \kappa_n$. The theorem obtains the most economical failure of the $\mathsf{SCH}$: $2^\kappa = \kappa^{++}$. The proof adapts easily to produce other failures $2^\kappa = \lambda > \kappa^+$, but one has to increase the large cardinal assumption to $\delta$ supercompactness for $\delta > (\nu^+)^V$. The work of Gitik–Sharon and Cummings–Foreman was generalized by Sinapova [26] to produce an extension with arbitrary cofinality for $\kappa$. It is likely, but not known, that similar generalizations are possible with our construction.

Combinatorial questions involving the $\mathsf{SCH}$ are of particular interest at $\aleph_\omega$, and Woodin emphasized this case in his question. In the past most forcing constructions violating the $\mathsf{SCH}$ at a large cardinal $\kappa$ could be combined with collapses, to turn $\kappa$ into a small cardinal, ideally $\aleph_\omega$. The techniques used to do this trace back to Magidor [17, 18]. Using a more elaborate collapsing technique due to Woodin, or the methods of Gitik–Magidor [12], one can also secure the $\mathsf{GCH}$ below $\kappa$. So, starting from a cardinal $\kappa$ with Mitchell order $\kappa^{++}$, one can force to obtain failure of the $\mathsf{SCH}$ at $\aleph_\omega$, with the $\mathsf{GCH}$ holding below $\aleph_\omega$.

In the case of the Gitik–Sharon theorem too the construction can be combined with collapses, but the combination requires some space, and so far it is only known how to turn $\kappa$ into $\aleph_{\omega^2}$, not $\aleph_\omega$. In the case of Theorem 1.1 it is not known whether such combinations can be made at all, and the specific form of Woodin's question, on $\aleph_\omega$, and even on $\aleph_{\omega^2}$, remains open.

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**§2. The forcing notion.** Let $\langle \kappa_n \mid n < \omega\rangle$ be an increasing sequence of supercompact cardinals. Without loss of generality assume the $\mathsf{GCH}$ above $\kappa_0$. Suppose that the supercompactness of $\kappa_0$ is indestructible under $<\!\kappa_0$ closed forcing. This can always be arranged using the Laver preparation, maintaining the $\mathsf{GCH}$ above $\kappa_0$.

let $\nu = \sup\{\kappa_n \mid n < \omega\}$. Let $\mathbb{A}$ be the poset adding $\nu^{++}$ subsets of $\kappa_0$, with conditions of size $< \kappa_0$. Let $E$ be generic for $\mathbb{A}$ over $V$. For each $\xi < \nu^{++}$ let $E_\xi$ be the $\xi$th subset of $\kappa$ added by $E$. $E$ itself is the characteristic function of $\{\langle \xi, u\rangle \mid \xi < \nu^{++}, u \in E_\xi\}$. By indestructibility, $\kappa_0$ remains supercompact in $V[E]$. Moreover, there is in $V[E]$ a $\nu^+$ supercompactness measure on $\kappa_0$, so that the elements of its ultrapower up to the image of $\kappa_0$ require only $\kappa_0$ for their support. This is due to Gitik–Sharon [14]. For completeness we give the proof:

LEMMA 2.1. *There is a $\nu^+$ supercompactness embedding $\pi\colon V[E] \to M$ in $V[E]$, with critical point $\kappa_0$, so that every element of $M \| \pi(\kappa_0)$ has the form $\pi(f)(\kappa_0)$.*

PROOF. We work in $V[E]$. It is enough to construct $\pi$ so that every ordinal $< \pi(\kappa_0)$ has the required form, since there is a bijection in $\pi''V[E]$ between $\pi(\kappa_0)$ and $M \| \pi(\kappa_0)$. Ordinals below $\kappa_0$ clearly have the required form, as the critical point of $\pi$ is $\kappa_0$. So it is enough to handle ordinals in the interval $[\kappa_0, \pi(\kappa_0))$.

Using the indestructibility of $\kappa_0$, fix in $V[E]$ a $\nu^+$ supercompactness embedding $\tau\colon V[E] \to N[F]$. Let $\sigma$ be the restriction of $\tau$ to $V$, so that $\sigma\colon V \to N$, and $\tau(\dot{x}[E]) = \sigma(\dot{x})[F]$ for each $\mathbb{A}$-name $\dot{x}$ in $V$. Let $a = \sigma''\nu^+ = \tau''\nu^+$. Condensing $\tau$ if needed, we may assume that every element $x$ of $N[F]$ has the form $\tau(f)(a)$ with $f \in V[E]$. If $x \in \tau(\kappa_0)$ then $x$ has the form $\tau(f)(a)$ with $f\colon \mathcal{P}_{\kappa_0}(\nu^+) \to \kappa_0$. Since the size of ${\kappa_0}^{\mathcal{P}_{\kappa_0}(\nu^+)}$ is $\nu^{++}$ in $V[E]$, the cardinality of $\tau(\kappa_0)$ in $V[E]$ is $\nu^{++}$. The cardinality of the interval $[\kappa_0, \sigma(\kappa_0)) = [\kappa_0, \tau(\kappa_0))$ is the same. Let $\langle u_\xi \mid \xi < \nu^{++}\rangle$ enumerate the elements of this interval.

For each $\xi < \nu^{++}$ let $f_\xi \in V[E]$ be the function that assigns to each $\alpha < \kappa$, the $\alpha$th element of $E_\xi$. We make some adjustments to $F$, to obtain a revised generic $E^*$ from it, and an embedding $\pi\colon V[E] \to N[E^*]$ extending $\sigma$, so that $u_\xi = \pi(f_\xi)(\kappa)$. As $[\kappa_0, \sigma(\kappa_0)) = \{u_\xi \mid \xi < \nu^{++}\}$, this will complete the proof.

Define $E^*$ through the conditions:

- $E^*_\zeta = F_\zeta$ for $\zeta \notin \sigma''\nu^{++}$.
- $E^*_{\sigma(\xi)} = F_{\sigma(\xi)} - [\kappa_0, u_\xi) \cup \{u_\xi\}$ for $\xi < \nu^{++}$.

The difference between $E^*$ and $F$ can be approximated inside $N$. Precisely, for each $\delta < \sigma(\nu^{++})$, there is a set $X$ of size less than $\sigma(\kappa_0)$ in $N$, so that $E^* \restriction \delta \times \kappa_0$ and $F \restriction \delta \times \kappa_0$ differ only on $X$, and $E^* \restriction X$ belongs to $N$ and is therefore a condition in $\sigma(\mathbb{A})$. (The set $X$ is the product $\Pi_{\sigma(\xi)<\delta}\{\sigma(\xi)\} \times [\kappa_0, u_\xi]$. It is a product of $\nu^+$ intervals, and belongs to $N$ because of $N$s closure. $E^* \restriction X$ belongs to $N$ by closure too.) It follows from all this, the genericity of $F$ over $N$, and the chain condition for $\sigma(\mathbb{A})$ (which implies that genericity of $E^*$ for $\sigma(\mathbb{A})$ is the same as genericity of $E^* \restriction \delta \times \kappa_0$ for $\sigma(\mathbb{A}) \restriction \delta \times \kappa_0$ for all $\delta < \sigma(\nu^{++})$), that $E^*$ is generic for $\sigma(\mathbb{A})$ over $N$.

It is clear from the definition that $E^* \restriction \mathrm{range}(\sigma) = F \restriction \mathrm{range}(\sigma)$, and since $\sigma''E \subseteq F$, it follows that $\sigma''E \subseteq E^*$. The embedding $\sigma\colon V \to N$ can therefore be extended to an embedding $\pi\colon V[E] \to N[E^*]$, setting $\pi(\dot{x}[E]) = \sigma(\dot{x})[E^*]$.

By elementarity, $\pi(f_\zeta)$ is the function that assigns to each $\alpha < \pi(\kappa_0)$, the $\alpha$th element of $E^*_\zeta$. For $\xi < \nu^{++}$ and $\zeta = \pi(\xi)$, the $\kappa_0$th element of $E^*_\zeta$ is $u_\xi$ by definition of $E^*$. So $\{\pi(f_\xi)(\kappa_0) \mid \xi < \nu^{++}\} = \{u_\xi \mid \xi < \nu^{++}\} = [\kappa_0, \sigma(\kappa_0)) = [\kappa_0, \pi(\kappa_0))$, as required. ⊣

Let $\pi$ be given by the last lemma. Let $\mathcal{U}$ be the $\nu^+$ supercompactness measure induced by $\pi$. Precisely, $\mathcal{U}$ measures sets in $\mathcal{P}_{\kappa_0}(\nu^+)$, and $\mathcal{U}(X) = 1$ iff $\pi''\nu^+ \in \pi(X)$. For each $n < \omega$, let $\mathcal{U}_n$ be the $\kappa_n$ supercompactness measure induced by $\pi$. $\mathcal{U}_n$ measures sets in $\mathcal{P}_{\kappa_0}(\kappa_n)$, and $\mathcal{U}_n(X) = 1$ iff $\pi''\kappa_n \in \pi(X)$.

We now force over $V[E]$ using the following poset $\mathbb{P}$, adapted from Gitik–Sharon [14]. Conditions are pairs $p = \langle g_p, A_p\rangle$ where:

- $g_p = \langle g_p(0), \dots, g_p(k-1)\rangle$, with $g_p(n) \in \mathcal{P}_{\kappa_0}(\kappa_n)$.

- $A_p = \langle A_p(n) \mid k \le n < \omega \rangle$, with $A_p(n) \subseteq \mathcal{P}_{\kappa_0}(\kappa_n)$ and $\mathcal{U}_n(A_p(n)) = 1$.
- We require the stem to be monotone increasing and nice, in the sense that $g_p(n+1) \supseteq g_p(n)$, $g_p(n) \cap \kappa_0$ is an inaccessible cardinal (by necessity $< \kappa_0$), and $g_p(n+1) \cap \kappa_0 > \mathrm{card}(g_p(n))$.

The poset belongs to the family of Prikry forcing notions, and conditions are ordered in the natural way: $q \le p$ iff $g_q$ extends $g_p$, $A_q(n) \subseteq A_p(n)$ for each $n \ge \mathrm{lh}(g_q)$, and $g_q(n) \in A_p(n)$ for each $n \in \mathrm{lh}(g_q) - \mathrm{lh}(g_p)$.

$\mathbb{P}$ is a variant of a poset introduced by Gitik–Sharon [14]. If instead of $\kappa_n$ we used $\kappa_0^{+(n)}$, we would have obtained precisely the Gitik–Sharon poset. Gitik–Sharon proved the Prikry property for their poset. Their argument, with trivial modification, gives:

Fact 2.2. $\mathbb{P}$ has the Prikry property. Precisely, let $\dot{x}_1, \ldots, \dot{x}_k \in V[E]$ be $\mathbb{P}$-names, let $\varphi$ be a formula, and let $p \in \mathbb{P}$. Then there is a condition $q \le p$ which decides $\varphi(\dot{x}_1, \ldots, \dot{x}_k)$, with $g_q = g_p$.

The part $g_p$ is called the stem of $p$. It is clear that two conditions with the same stem are compatible. Since the stems are finite sequences from $\bigcup_{n<\omega} \mathcal{P}_{\kappa_0}(\kappa_n)$, which has cardinality $\nu$ in $V[E]$, $\mathbb{P}$ has the $\nu^+$ chain condition.

For two conditions $p$ and $q$ with the same stem, we use $p \wedge q$ to denote the condition $r$ determined by $g_r = g_p = g_q$ and $A_r(n) = A_p(n) \cap A_q(n)$. $r$ is the weakest common extension of $p$ and $q$.

$\mathbb{P}$ is not even $\omega$ closed, but for each stem $h$, the collection of conditions with stem equal to $h$ is $<\kappa_0$ closed, since the measures $\mathcal{U}_n$ are $<\kappa_0$ complete.

Let $G$ be generic for $\mathbb{P}$ over $V[E]$. $G$ is completely determined by $g = \bigcup_{p \in G} g_p$. A condition $p$ belongs to $G$ iff $g_p \subseteq g$ and $g(n) \in A_p(n)$ for each $n \ge \mathrm{lh}(g_p)$.

The following properties of the generic extension $V[E][G]$ are clear:

1. $\kappa_0$ is a strong limit in $V[E][G]$, $(2^{\kappa_0})^{V[E][G]} = (\nu^{++})^V$ and the $\mathsf{GCH}$ holds in $V[E][G]$ from $\nu^+$ upward.
2. $\langle g(n) \cap \kappa_0 \mid n < \omega \rangle$ is cofinal in $\kappa_0$. In particular $\kappa_0$ is singular, of cofinality $\omega$, in $V[E][G]$.
3. $\nu$ is collapsed to $\kappa_0$ in the extension. Indeed, it is equal to $\bigcup_{n<\omega} g(n)$, a union of $\omega$ sets each of cardinality $< \kappa_0$.
4. No cardinals are collapsed below $\kappa_0$ (because of the closure of $\mathbb{A}$, the Prikry property for $\mathbb{P}$, and the closure for conditions with a fixed stem in $\mathbb{P}$), and no cardinals are collapsed above $\nu$ (because of the chain condition).
5. In fact cofinalities smaller than $\kappa_0$ and greater than $\nu$ are preserved. Cofinalities in the interval $[\kappa_0, \nu)$ are changed to cofinality $\omega$.

It follows in particular that the extension $V[E][G]$ satisfies $\neg\mathsf{SCH}_{\kappa_0}$. In the extension, $\kappa_0$ is a strong limit of cofinality $\omega$, and $2^{\kappa_0} = \kappa^{++}$.

Gitik–Sharon [14] introduced their poset so as to produce an extension with a singular strong limit $\kappa$ so that $\mathsf{SCH}_\kappa$ fails, there is a very good scale at $\kappa$, and yet the approachability property fails at $\kappa$, and in particular so does the weak square property. Cummings–Foreman [3] showed that there is also a bad scale on $\kappa$ in the Gitik–Sharon extension, and this implies the failure of the approachability property. All these results adapt with little change to our extension, yielding:

FACT 2.3. In $V[E][G]$, there is a very good scale on $\kappa_0$, and there is a bad scale on $\kappa_0$.

Let $\tau^n = g(n) \cap \kappa_0$, so that $\langle \tau^n \mid n < \omega \rangle$ is increasing and cofinal in $\kappa_0$. Using the property of $\pi$ given by Lemma 2.1, fix for each $\alpha < \nu^+$ a function $f_\alpha \colon \kappa \to \kappa$ in $V[E]$ so that $\pi(f_\alpha)(\kappa) = \alpha$. Let $\tau_i^n = f_{\kappa_i}(\tau^n)$, and let $\mu^n = f_\nu(\tau^n)$. By assuming that the generic $G$ contains a condition $p_0 = \langle \emptyset, A_0 \rangle$ with an appropriately restricted sequence of measure one sets $A_0(n)$, we can reflect the fact that $\langle \kappa_i \mid i < \omega \rangle$ is increasing, and arrange that $\langle \tau_i^n \mid i < \omega \rangle$ is increasing for each $n$. Reflecting the fact that $\sup\{\kappa_i \mid i < \omega\} = \nu$ we may arrange that $\sup\{\tau_i^n \mid i < \omega\} = \mu^n$ for each $n$. Making sure that each of the elements of the measure one set $A_0(n+1)$ is sufficiently closed, we can arrange that $\tau^{n+1} > \mu^n$. Finally, by taking $f_{\kappa_0}$ to be the identity, and adjusting each $f_\alpha$ on a measure zero set, we may assume that $\tau_0^n = \tau^n$, and that $f_\alpha(\tau^n) < (\mu^n)^+$ for each $\alpha < \nu^+$ and each $n$.

The very good scale mentioned in Fact 2.3 is the sequence $\langle \varphi_\alpha \mid \alpha < \nu^+ \rangle$ defined by $\varphi_\alpha(n) = f_\alpha(\tau^n)$. It is a scale on $\Pi_{n<\omega}(\mu^n)^+$ in $V[E][G]$.

Fix, in $V[E]$, a scale $\langle \psi_\alpha^* \mid \alpha < \nu^+ \rangle$ on $\Pi_{n<\omega}{\kappa_n}^+$. Since $\nu$ is above a supercompact, $\langle \psi_\alpha^* \mid \alpha < \nu^+ \rangle$ is a bad scale on $\nu$ in $V[E]$. The bad scale mentioned in Fact 2.3 is the sequence $\langle \psi_\alpha \mid \alpha < \nu^+ \rangle$ defined by $\psi_\alpha(n) = f_{\psi_\alpha^*(n)}(\tau^n)$. It is a scale on $\Pi_{n<\omega}(\tau_n^n)^+$ in $V[E][G]$.

The proofs that $\vec{\varphi}$ and $\vec{\psi}$ are very good, and bad, scales respectively are direct adaptations of the corresponding proofs in [14] and [3], and we do not include them here. We proceed now to prove the extra property for which we created the extension: that in $V[E][G]$, ${\kappa_0}^+$ has the tree property.

**§3. The tree property.** Fix, in $V[E]$, a $\mathbb{P}$-name $\dot{T}$ which is forced by the empty condition to be a tree on $(\nu^+)^V = ({\kappa_0}^+)^{V[E][G]}$, with levels of size at most $\kappa_0$. We work to prove that in $V[E][G]$ there is a branch through $T = \dot{T}[G]$. Our argument is inspired by the proof in Magidor–Shelah [19] that the tree property holds at successors of limits of $\omega$ supercompact cardinals. In our context the Magidor–Shelah proof shows that the tree property at $\nu^+$ holds in $V$. We shall have to do additional work to account for the move to an extension by $\mathbb{A} \times \mathbb{P}$. Much of the difficulty is in dealing with $\mathbb{A}$, but of course we could not have defined $\mathbb{P}$ without first forcing with $\mathbb{A}$, since the definition of $\mathbb{P}$ uses the measures given by Lemma 2.1, and the lemma relies on the addition of subsets of $\kappa_0$.

Recall that $\dot{T} \in V[E]$ is a $\mathbb{P}$-name for $T$. Without loss of generality suppose that the nodes on level $\alpha$ of $T$ are the elements of $\{\alpha\} \times \kappa_0$, and that this is forced by the empty condition in $\mathbb{P}$.

For every $\alpha < \beta < \nu^+$ there are $\xi, \zeta < \kappa_0$, and $k < \omega$, so that $\langle \alpha, \xi \rangle \; T \; \langle \beta, \zeta \rangle$, and so that this is forced by a condition with stem of length $k$. We begin by finding a cofinal set $I \subseteq \nu^+$ on which $k$ can be fixed.

LEMMA 3.1. *There is $\bar{k} < \omega$, and a cofinal $I \subseteq \nu^+$ in $V[E]$, so that for all $\alpha, \beta \in I$, there exists $\xi, \zeta < \kappa_0$ and $p \in \mathbb{P}$, so that* $\mathrm{lh}(g_p) = \bar{k}$ *and* $p \Vdash \langle \check{\alpha}, \check{\xi} \rangle \; \dot{T} \; \langle \check{\beta}, \check{\zeta} \rangle$.

Proof. We work in $V[E]$. Recall that $\pi\colon V[E] \to M$ is a $\nu^+$ supercompactness embedding. Let $G^*$ be generic for $\pi(\mathbb{P})$ over $M$. Using the fact that $\nu^+$ is a discontinuity point of $\pi$ (which follows from the closure of $M$ under $\nu^+$ sequences) fix $\gamma$ between $\sup(\pi''\nu^+)$ and $\pi(\nu^+)$. Fix a node $u$ of $\pi(\dot{T})[G^*]$ on level $\gamma$ of the tree, and a name $\dot{u}$ for this node.

For every $\alpha < \nu^+$, there is $\xi_\alpha$ so that $\langle \pi(\alpha), \xi_\alpha\rangle$ is a node of $\pi(\dot{T})[G^*]$ on level $\pi(\alpha)$, and is below $u$ in the tree order. Let $p_\alpha \in G^*$ force that $\langle \pi(\check{\alpha}), \check{\xi}_\alpha\rangle \, \pi(\dot{T}) \, \dot{u}$. Let $k_\alpha = \mathrm{lh}(g_{p_\alpha})$. Since $\pi''\nu^+$ belongs to $M$, all this can be done inside $M[G^*]$. Since $\nu^+$ is a regular cardinal in $M[G^*]$, there is a fixed $\bar{k}$, so that $k_\alpha = \bar{k}$ for cofinally many $\alpha < \nu^+$.

Let $h^*$ be the restriction to $\bar{k}$ of the stem of some (equivalently all) condition in $G^*$ with stem of length $> \bar{k}$. Let $I \subseteq \nu^+$ be the set of all $\alpha$ so that there is a condition $r \in \pi(\mathbb{P})$ with stem $h^*$, and an ordinal $\zeta < \pi(\kappa_0)$, so that $r \Vdash \langle \pi(\check{\alpha}), \check{\zeta}\rangle \, \pi(\dot{T}) \, \dot{u}$. The definition of $I$ is made with reference to $\pi$ but without reference to $G^*$. So $I$ belongs to $V[E]$. By the way we fixed $\bar{k}$ in the last paragraph, $I$ is cofinal in $\nu^+$.

Suppose now that $\alpha < \beta$ both belong to $I$. By the definition of $I$ there are conditions $r_\alpha, r_\beta \in \pi(\mathbb{P})$, both with stem $h^*$, and ordinals $\zeta_\alpha, \zeta_\beta$, so that $r_\alpha \Vdash \langle \pi(\check{\alpha}), \check{\zeta}_\alpha\rangle \, \pi(\dot{T}) \, \dot{u}$ and $r_\beta \Vdash \langle \pi(\check{\beta}), \check{\zeta}_\beta\rangle \, \pi(\dot{T}) \, \dot{u}$

Since $r_\alpha$ and $r_\beta$ have the same stem $h^*$, $r_\alpha \wedge r_\beta$ is defined and is a common extension of the conditions, again with stem $h^*$. It forces that both $\langle \pi(\check{\alpha}), \check{\zeta}_\alpha\rangle$ and $\langle \pi(\check{\beta}), \check{\zeta}_\beta\rangle$ are below $\dot{u}$ in $\pi(\dot{T})$. Since $\pi(\dot{T})$ is forced to be a tree, it follows that the condition forces the two nodes to be compatible, i.e., $r_\alpha \wedge r_\beta \Vdash \langle \pi(\check{\alpha}), \check{\zeta}_\alpha\rangle \pi(\dot{T}) \langle \pi(\check{\beta}), \check{\zeta}_\beta\rangle$. By elementarity of $\pi$ then, there exists $p \in \mathbb{P}$ with stem of length $\bar{k}$, and $\zeta, \zeta' < \kappa_0$, so that $p \Vdash \langle \check{\alpha}, \check{\zeta}\rangle \, \dot{T} \, \langle \check{\beta}, \check{\zeta}'\rangle$. ⊣

Having fixed length, we now proceed to fix the stem itself, and also the nodes of $\dot{T}$ involved.

Lemma 3.2. *There is in $V[E]$ a cofinal $J \subseteq \nu^+$, a map $\alpha \mapsto \xi_\alpha$ ($\alpha \in J$) and a stem $\bar{h}$ of length $\bar{k}$, so that for every $\alpha < \beta$ both in $J$, there is a condition $p$ with stem $\bar{h}$, that forces $\langle \check{\alpha}, \check{\xi}_\alpha\rangle \, \dot{T} \, \langle \check{\beta}, \check{\xi}_\beta\rangle$.*

Proof. Let $\sigma\colon V \to N$ be a $\nu^+$ supercompactness embedding with critical point $\kappa_{\bar{k}+1}$. Let $\mathbb{B}$ be the poset for adding $\sigma(\nu^{++})$ subsets of $\kappa_0$ with conditions of size $< \kappa_0$, and let $F$ be generic for $\mathbb{B}$ over $V[E]$. Since $\sigma(\mathbb{A})$ is itself the poset for adding $\sigma(\nu^{++})$ subsets of $\kappa_0$, we can in $V[E][F]$ combine $\sigma''E$ and $F$ to find $E^* \supseteq \sigma''E$ which is generic for $\sigma(\mathbb{A})$ over $N$. The embedding $\sigma$ then extends to an embedding $\sigma^*\colon V[E] \to N[E^*]$. We have $\sigma^* \in V[E][F]$.

$\nu^+$ is a discontinuity point of $\sigma$, and $I$ is cofinal in $\nu^+$, so we can find $\gamma > \sigma''\nu^+$ with $\gamma \in \sigma^*(I)$.

Using the conclusion of the previous lemma, shifted to $N[E^*]$ via the elementary embedding $\sigma^*$, we can find for each $\alpha \in I$ some $\xi_\alpha, \zeta_\alpha < \kappa_0 = \sigma(\kappa_0)$ and $p_\alpha \in \sigma^*(\mathbb{P})$ so that $p_\alpha \Vdash \langle \sigma(\check{\alpha}), \check{\xi}_\alpha\rangle \, \sigma^*(\dot{T}) \, \langle \check{\gamma}, \check{\zeta}_\alpha\rangle$, and so that the length of $g_{p_\alpha}$ is $\bar{k}$. Since $\sigma''\nu^+$ belongs to $N$, all this can be done inside $N[E^*]$. Continuing to work inside $N[E^*]$, we can find some fixed stem $\bar{h}$, some fixed $\zeta$, and a cofinal $J \subseteq I$, so that for every $\alpha \in J$, $g_{p_\alpha} = \bar{h}$, and $\zeta_\alpha = \zeta$.

Suppose now that $\alpha < \beta$ both belong to $J$. Then $p_\alpha \wedge p_\beta$ is a condition with stem $\bar{h}$, which forces both $\langle \sigma(\check{\alpha}), \check{\xi}_\alpha \rangle$ and $\langle \sigma(\check{\beta}), \check{\xi}_\beta \rangle$ to be below $\langle \check{\gamma}, \zeta \rangle$ in the tree order $\sigma^*(\dot{T})$, and therefore forces them to be comparable in the tree order. Pulling back to $V[E]$ using the elementarity of $\sigma^*$, it follows that there is a condition $p \in \mathbb{P}$, with stem equal to $\bar{h}$, forcing over $V[E]$ that $\langle \check{\alpha}, \check{\xi}_\alpha \rangle \, \dot{T} \, \langle \check{\beta}, \check{\xi}_\beta \rangle$. Note that $\bar{h}$, $\xi_\alpha$, and $\xi_\beta$ are not affected by the pull back to $V[E]$, as they are all below the critical point of $\sigma^*$. $\xi_\alpha$ and $\xi_\beta$ are smaller than $\kappa_0$, and $\bar{h}$ is a finite sequence from $\mathcal{P}_{\kappa_0}(\kappa_{\bar{k}})$, while the critical point of $\sigma^*$ is $\kappa_{\bar{k}+1}$.

So far we found $J$, $\bar{h}$, and a map $\alpha \mapsto \xi_\alpha$ satisfying the condition in the claim, except that $J$ and the map $\alpha \mapsto \xi_\alpha$ belong to $V[E][F]$, not to $V[E]$. It remains to see that we can find similar objects in $V[E]$.

Let $Z \in V[E]$ be the set of tuples $\langle \bar{h}, \alpha, \xi, \alpha', \xi' \rangle$ so that there exists a condition $p \in \mathbb{P}$ with $g_p = \bar{h}$ forcing $\langle \check{\alpha}, \check{\xi} \rangle \, \dot{T} \, \langle \check{\alpha}', \check{\xi}' \rangle$ over $V[E]$.

Let $\theta(Z, \bar{h}, J, f, \nu^+)$ be the statement that $J$ is cofinal in $\nu^+$, $\bar{h}$ has length $\bar{k}$, and for every $\alpha < \beta$ both in $J$, $\langle \bar{h}, \alpha, f(\alpha), \beta, f(\beta) \rangle \in Z$. We proved that $V[E][F] \models (\exists \bar{h}, J, f)\, \theta(Z, \bar{h}, J, f, \nu^+)$. (Take $f$ to be the function $\alpha \mapsto \xi_\alpha$ obtained above, with the objects $J$ and $\bar{h}$ obtained above.) To obtain $\bar{h}$, $J$, and $f$ inside $V[E]$, we simply use the fact that $Z$ can be coded by a subset of $\nu^+$, and $\theta$ is absolute.

Precisely, let $H$ be a $<\!\kappa_0$ closed elementary model of a sufficiently large rank initial segment of $V[E]$, with $\mathrm{card}(H)^{V[E]} = \nu^+$ and $\nu^+ \cup \{\nu^+, Z, \mathbb{B}\} \subseteq H$. Let $Q$ be the transitive collapse of $H$, and let $c\colon H \to Q$ be the collapse embedding. Let $\delta = H \cap \nu^{++}$. Then $\delta < \nu^{++}$, and $Q$ has the form $R[E \restriction \delta]$, where by $E \restriction \delta$ we mean the part of $E$ adding the sets $E_\xi$, $\xi < \delta$. Since $Z$ can be coded by a subset of $\nu^+$, it is not moved by $c$. Nor is $\nu^+$ itself moved. The poset $c(\mathbb{B})$ is $\mathrm{Add}(\kappa_0, c(\sigma(\nu^{++})))$, and $c(\sigma(\nu^{++}))$ is smaller than $\nu^{++}$, since $\mathrm{card}(H) < \nu^{++}$. So $E \restriction [\delta, \nu^{++})$ supplies more than enough subsets of $\kappa_0$ that are generic over $Q = R[E \restriction \delta]$, to construct $\bar{F} \in V[E]$ which is generic for $c(\mathbb{B})$ over $Q$.

By the elementarity of the anticollapse embedding, $Q[\bar{F}]$ satisfies $(\exists \bar{h}, J, f)$ $\theta(Z, \bar{h}, J, f, \nu^+)$. Since $Q[\bar{F}]$ belongs to $V[E]$, we can find $\bar{h}$, $J$, and $f$ in $V[E]$ so that $Q[\bar{F}] \models \theta(Z, \bar{h}, J, f, \nu^+)$. Now since $\theta$ is absolute, $V[E] \models \theta(Z, \bar{h}, J, f, \nu^+)$. $\dashv$

Remark 3.3. For any condition $q \in \mathbb{P}$, Lemmas 3.1 and 3.2 can be strengthened to give $\bar{h} = g_r$ for some $r \leq q$, simply by restricting attention to conditions stronger than $q$. It follows that the set of conditions $r$ so that $J$, $\alpha \mapsto \xi_\alpha$, and $\bar{h}$ as in Lemma 3.2 can be found, with $\bar{h} = g_r$, is dense in $\mathbb{P}$. The generic $G$ meets every dense set below every $q_0 \in G$. So we may assume, for an arbitrary $q_0 \in G$, that $\bar{h} = g_r$ for some $r \in G$ stronger than $q_0$.

We continue to work with $J$ and the map $\alpha \mapsto \xi_\alpha$ for the rest of the section. We know that for $\alpha < \beta$ both in $J$, there is a condition with stem $\bar{h}$ forcing $\langle \check{\alpha}, \check{\xi}_\alpha \rangle \, \dot{T} \, \langle \check{\beta}, \check{\xi}_\beta \rangle$. Our next goal is to find a map $\alpha \mapsto p_\alpha$, so that for all $\alpha < \beta$ both in $J$, the condition $p_\alpha \wedge p_\beta$ forces this compatibility. We do this in stages. We shall set $p_\alpha = \langle \bar{h}, A^\alpha \rangle$, and we work recursively on $n \geq \bar{k}$ to define $A^\alpha(n)$.

One of the biggest problems we face is obtaining the sets $A^\alpha(n)$ inside $V[E]$ (as opposed to a generic extension of this model). The next claim will come in handy, though at face value it does not seem to handle the kind of sets we need.

Claim 3.4. *Let $S$ be a tree of height $\theta$ in a model $M$ of* ZFC*. Let $\mathbb{B} \in M$ be a poset, and suppose that, in $M$, $\mathbb{B} \times \mathbb{B}$ has the* $\mathrm{cof}(\theta)$ *chain condition. Suppose further that a power $\mathbb{B}^{|S|^+}$ does not collapse $|S|^+$. (Which power is used, meaning which support is used to form the power, is irrelevant to the claim, so long as the resulting power does not collapse $|S|^+$.) Then $\mathbb{B}$ does not add new branches to $S$. Precisely, if $F$ is generic for $\mathbb{B}$ over $M$, and $b \in M[F]$ is a branch of $S$, then $b \in M$.*

Proof. We work over $M$. Replacing $\theta$ by $\mathrm{cof}(\theta)$, and replacing $S$ by a restriction of $S$ to nodes on levels in a set of order type $\mathrm{cof}(\theta)$ cofinal in $\theta$, we may assume that $\theta$ is regular. Let $\dot{b}$ name a branch of $S$, and suppose that it is forced that $\dot{b}$ does not belong to $M$. Let $\mathbb{B}^*$ denote the power $\mathbb{B}^{|S|^+}$, and let $F^* = \Pi_{\delta < |S|^+} F_\delta$ be generic for $\mathbb{B}^*$ over $M$. Let $b_\delta = \dot{b}[F_\delta]$.

We shall work with the product $\mathbb{B} \times \mathbb{B}$, and with generics $F_{\delta_1} \times F_{\delta_2}$, $\delta_1 \neq \delta_2$, for this product. We use $b_{\mathrm{left}}$ and $b_{\mathrm{right}}$ to refer to the branches $\dot{b}[F_{\delta_1}]$ and $\dot{b}[F_{\delta_2}]$ in the extension $M[F_{\delta_1} \times F_{\delta_2}]$.

Let $H$ be an elementary submodel of a sufficiently large rank initial segment of $M$, with $\mathrm{card}(H) < \theta$, $H \cap \theta$ an ordinal, and $\{\theta, S, \dot{b}, \mathbb{B}, \mathbb{B}^*\} \subseteq H$. Let $\eta = H \cap \theta < \theta$.

Because $H$ is elementary, $H \cap \theta$ is an ordinal, and $\mathbb{B}$ has the $\theta$ chain condition, every antichain of $\mathbb{B}$ that belongs to $H$ is contained in $H$. It follows that $H[F_\delta]$ is an elementary substructure of (a rank initial segment of) $M[F_\delta]$ for each $\delta$, and $H[F_\delta] \cap M = H$. Similarly $H[F_{\delta_1} \times F_{\delta_2}]$ is an elementary substructure of (a rank initial segment of) $M[F_{\delta_1} \times F_{\delta_2}]$ for $\delta_1 \neq \delta_2$, and $H[F_{\delta_1} \times F_{\delta_2}] \cap M = H$.

For each $\delta < |S|^+$, let $\beta_\delta$ be the node of $b_\delta$ of height $\eta$. As $\beta_\delta \in S$ for each $\delta < |S|^+$, and $|S|^+$ is not collapsed in $M[F^*]$, there must be $\delta_1 \neq \delta_2$ so that $\beta_{\delta_1} = \beta_{\delta_2}$. By elementarity of $H[F_{\delta_1}]$, $b_{\delta_1} \cap H$ consists of all nodes of $b_{\delta_1}$ of height $< \eta$. Thus $b_{\delta_1} \cap H$ is equal to the set of nodes of $S$ below $\beta_{\delta_1}$. Similar reasoning applies to $b_{\delta_2} \cap H$, and since $\beta_{\delta_1} = \beta_{\delta_2}$ it follows that $b_{\delta_1} \cap H = b_{\delta_2} \cap H$.

Consider now the situation in $M[F_{\delta_1} \times F_{\delta_2}]$. From the conclusion of the previous paragraph we get $b_{\delta_1} \cap H[F_{\delta_1} \times F_{\delta_2}] = b_{\delta_2} \cap H[F_{\delta_1} \times F_{\delta_2}]$, in other words $(b_{\mathrm{left}} = b_{\mathrm{right}})^{H[F_{\delta_1} \times F_{\delta_2}]}$. By elementarity of $H[F_{\delta_1} \times F_{\delta_2}]$ in $M[F_{\delta_1} \times F_{\delta_2}]$ it follows that $\dot{b}_{\mathrm{left}}[F_{\delta_1} \times F_{\delta_2}] = \dot{b}_{\mathrm{right}}[F_{\delta_1} \times F_{\delta_2}]$. From this, standard arguments produce a condition in $\mathbb{B}$ forcing $\dot{b}$ to belong to $M$. ⊣

Recall that we are working with $J$ and $\alpha \mapsto \xi_\alpha$ $(\alpha \in J)$ so that for $\alpha < \beta$ both in $J$, there is a condition with stem $\bar{h}$ forcing $\langle \check{\alpha}, \check{\xi}_\alpha \rangle \, \dot{T} \, \langle \check{\beta}, \check{\xi}_\beta \rangle$. We aim to find a map $\alpha \mapsto p_\alpha$, so that for all $\alpha < \beta$ both in $J$, the condition $p_\alpha \wedge p_\beta$ forces this compatibility.

For a stem $h$, we write that $h \Vdash \varphi$ iff there is a condition $p \in \mathbb{P}$ with $g_p = h$ so that $p \Vdash \varphi$. Note that if $h \Vdash \varphi$, then any condition $q$ with $g_q = h$ either forces $\varphi$, or does not decide $\varphi$. This is because any two conditions $p, q$ with the same stem are compatible. Note further that, by the Prikry property, for every $\varphi$ and any stem $h$, either $h \Vdash \varphi$ or $h \Vdash \neg\varphi$. It is important to emphasize though, that even if $h \Vdash \varphi$, there may well be stems $h'$ extending $h$ so that $h' \Vdash \neg\varphi$. The fact that $h \Vdash \varphi$ merely implies that there are not very many such $h'$.

Lemma 3.5. *Let $h$ be a stem of length $k$ extending $\bar{h}$. Let $J^h \subseteq J$ be unbounded in $\nu^+$, and suppose that for all $\alpha < \beta$ both in $J^h$, $h \Vdash \langle \check{\alpha}, \check{\xi}_\alpha \rangle \, \dot{T} \, \langle \check{\beta}, \check{\xi}_\beta \rangle$. Then there is $\rho^h < \nu^+$, and a map $\alpha \mapsto A^h_\alpha$ ($\alpha \in J^h - \rho^h$) in $V[E]$, so that:*

1. *$\mathcal{U}_k(A^h_\alpha) = 1$ for each $\alpha$.*
2. *For every $\alpha < \beta$ both in $J^h$ and greater than $\rho^h$, and for every $x \in A^h_\alpha \cap A^h_\beta$, $h \frown \langle x \rangle \Vdash \langle \check{\alpha}, \check{\xi}_\alpha \rangle \, \dot{T} \, \langle \check{\beta}, \check{\xi}_\beta \rangle$.*

Proof. Let $\sigma \colon V \to N$ be a $\nu^+$ supercompactness embedding with critical point $\kappa_{k+1}$. Let $\mathbb{B} = \mathrm{Add}(\kappa_0, \sigma(\nu^{++}))$, and let $F$ be generic for $\mathbb{B}$ over $V[E]$. In $V[E][F]$ we can combine $\sigma'' E$ and $F$ to find $E^* \supseteq \sigma'' E$ which is generic for $\sigma(\mathbb{A})$ over $N$. The embedding $\sigma$ then extends to an embedding $\sigma^* \colon V[E] \to N[E^*]$. We have $\sigma^* \in V[E][F]$.

$\nu^+$ is a discontinuity point of $\sigma$, and $J^h$ is cofinal in $\nu^+$, so we can find $\gamma > \sigma'' \nu^+$ with $\gamma \in \sigma^*(J^h)$. Let $\Xi$ denote the function $\alpha \mapsto \xi_\alpha$ and let $\zeta = \sigma^*(\Xi)(\gamma)$.

Note that $\xi_\alpha < \kappa < \mathrm{crit}(\sigma)$, so $\sigma(\xi_\alpha) = \xi_\alpha$ for all $\alpha$. We use this implicitly below, writing $\xi_\alpha$ where more directly we should write $\sigma(\xi_\alpha)$.

Claim 3.6. *There is, in $V[E][F]$, a map $\alpha \mapsto A^*_\alpha$ ($\alpha \in J^h$) so that:*

- *$A^*_\alpha$ has $\sigma^*(\mathcal{U}_k)$ measure one.*
- *For each $\alpha \in J^h$ and $x \in A^*_\alpha$, $h \frown \langle x \rangle \Vdash \langle \sigma(\check{\alpha}), \check{\xi}_\alpha \rangle \, \sigma^*(\dot{T}) \, \langle \check{\gamma}, \check{\zeta} \rangle$.*

Proof. By assumption of the lemma and the elementarity of $\sigma^*$, $h = \sigma^*(h)$ forces in $\sigma^*(\mathbb{P})$ over $N[E^*]$ that $\langle \sigma(\check{\alpha}), \check{\xi}_\alpha \rangle \, \sigma^*(\dot{T}) \, \langle \check{\gamma}, \check{\zeta} \rangle$. Fix a condition $r_\alpha$ with stem $h$ forcing this, and let $A^*_\alpha = A_{r_\alpha}(k)$. $\dashv$

It is tempting to think that we can set $A^h_\alpha = A^*_\alpha$, and use a trick similar to that in the proof of the previous lemma to pull the existence of the resulting map back to $V[E]$. Unfortunately the sets $A^*_\alpha$ are given measure one not by $\mathcal{U}_k$ but by $\sigma^*(\mathcal{U}_k)$. Both measures are on $\mathcal{P}_{\kappa_0}(\kappa_k)$, and this domain is not moved by $\sigma^*$ whose critical point is $\kappa_{k+1}$. Under $\mathsf{GCH}$ $\sigma^*$ would not affect the measures either. But we do not have the $\mathsf{GCH}$, and since $2^{\kappa_0} = \nu^{++} > \mathrm{crit}(\sigma^*)$, there are more subsets of $\mathcal{P}_{\kappa_0}(\kappa_k)$ in $N[E^*]$ than in $V[E]$. The measures $\mathcal{U}_k$ and $\sigma^*(\mathcal{U}_k)$ are different, and the sets $A^*_\alpha$ we obtained above need not even belong to the domain of $\mathcal{U}_k$, let alone have $\mathcal{U}_k$ measure one.

Our biggest problem in proving the lemma is to produce sets $A^h_\alpha$ which belong to $V[E]$, so that they are measured by $\mathcal{U}_k$. The next claim provides our initial tool in pulling existence of sets from $V[E][F]$, back to $V[E]$. Unfortunately it handles the wrong sets—"vertical" subsets of $\nu^+$ rather than "horizontal" subsets of $\mathcal{P}_{\kappa_0}(\kappa_k)$—but we shall deal with that problem later.

Claim 3.7. *Let $h^*$ be a stem of length $k+1$ extending $h$. Suppose that $J^* \in V[E][F]$ is a subset of $J^h$ so that:*

1. *$J^*$ is unbounded in $\nu^+$.*
2. *For $\alpha < \beta$ both in $J^h$, with $\beta \in J^*$, we have $\alpha \in J^*$ iff $h^* \Vdash \langle \check{\alpha}, \check{\xi}_\alpha \rangle \dot{T} \langle \check{\beta}, \check{\xi}_\beta \rangle$.*

*Then $J^*$ belongs to $V[E]$.*

Proof. By condition (2), knowledge that $\beta \in J^*$ is sufficient to completely determine $J^* \cap \beta$, in $V[E]$. Thus $J^*$ is a branch through a tree in $V[E]$. We can use Claim 3.4 on this tree to conclude that the branch must also belong to $V[E]$.

Let us be more precise. Let $M = V[E]$, let $\theta = \nu^+$, and let $S \in M$ be the tree of attempts to construct an increasing function $b\colon \nu^+ \to J^h$, so that condition (2) in the claim holds with $J^*$ replaced by range$(b)$. (A node in $S$ is an initial segment of $b$.)

By condition (2), every strict initial segment of $J^*$ belongs to $M$. So the function enumerating $J^*$ in increasing order is a branch of $S$. By Claim 3.4, the function belongs to $M$, and therefore so does $J^*$. The claim is applied with the poset $\mathbb{B}$, whose $\kappa_0$ support powers have the ${\kappa_0}^+$ chain condition. In particular they have the $\nu^+ = \theta$ chain condition, and do not collapse $|S|^+ = \nu^{++}$. ⊣

For every $x \in \mathcal{P}_{\kappa_0}(\kappa_k)$, let $h_x$ be the stem $h ^\frown \langle x \rangle$ of length $k+1$. Let $J_x$ be the set of $\alpha \in J^h$ so that $h_x \Vdash \langle \sigma(\check{\alpha}), \check{\xi}_\alpha \rangle \; \sigma^*(\dot{T}) \; \langle \check{\gamma}, \check{\zeta} \rangle$. Let $\dot{J}_x \in V[E]$ be a $\mathbb{B}$ name for the set defined this way. $J_x$ is defined in $V[E][\hat{F}]$, where we have access to $\sigma^*$. Still, using the previous claims, we get:

Claim 3.8. *If $J_x$ is unbounded in $\nu^+$, then it belongs to $V[E]$.*

Proof. We check that $h_x$ and $J_x$ satisfy the conditions in Claim 3.7, and then appeal to the claim. Condition (1) is clear as we explicitly assume that $J_x$ is unbounded. As for condition (2), if $h_x \Vdash \langle \sigma(\check{\beta}), \check{\xi}_\beta \rangle \; \sigma^*(\dot{T}) \; \langle \check{\gamma}, \check{\zeta} \rangle$, then for $\alpha < \beta$ (in $J^h$, so that $\xi_\alpha$ is defined),

$$\begin{aligned} h_x \Vdash \langle \alpha, \xi_\alpha \rangle \; \dot{T} \; \langle \beta, \xi_\beta \rangle &\Leftrightarrow h_x \Vdash \langle \sigma(\check{\alpha}), \check{\xi}_\alpha \rangle \; \sigma^*(\dot{T}) \; \langle \sigma(\check{\beta}), \check{\xi}_\beta \rangle \\ &\Leftrightarrow h_x \Vdash \langle \sigma(\check{\alpha}), \check{\xi}_\alpha \rangle \; \sigma^*(\dot{T}) \; \langle \check{\gamma}, \check{\zeta} \rangle \\ &\Leftrightarrow \alpha \in J_x. \end{aligned}$$

The first equivalence uses the elementarity of $\sigma^*$ and the fact that $\xi_\alpha$ and $\xi_\beta$ are not moved by the map. The second equivalence uses the fact that $\sigma^*(\dot{T})$ is forced by the empty condition to be a tree order. The third is by definition.

Now Claim 3.7 yields $J_x \in V[E]$. ⊣

Let $K_x$ be the set of $C \subseteq J^h$ in $V[E]$ so that $C$ is unbounded in $\nu^+$ and there is $b \in \mathbb{B}$ forcing $\dot{J}_x = \check{C}$. $K_x$ and the map $x \mapsto K_x$ belong to $V[E]$. Since $\mathbb{B}$ has the ${\kappa_0}^+$ chain condition, card$(K_x) \leq \kappa_0$ in $V[E]$. By the last claim, $J_x$, if unbounded, belongs to $K_x$. But we cannot in $V[E]$ tell which element of $K_x$ it is.

Claim 3.9. *Suppose $C \in K_x$. Then for $\alpha < \beta$ both in $J^h$, with $\beta \in C$, we have $\alpha \in C$ iff $h_x \Vdash \langle \check{\alpha}, \check{\xi}_\alpha \rangle \; \dot{T} \; \langle \check{\beta}, \check{\xi}_\beta \rangle$.*

Proof. Since $C$ can be a value of $\dot{J}_x$, the calculation ending the proof of the previous claim applies (with $C$ for $J_x$), yielding the current claim. ⊣

Claim 3.10. *Suppose that $C$ and $C'$ are two distinct elements of $K_x$. Then they are disjoint on a tail-end of $\nu^+$.*

Proof. If $\beta \in C \cap C'$, then by the previous claim $C \cap \beta = C' \cap \beta$. As $C \neq C'$ there is some $\alpha < \nu^+$ which belongs to one but not the other. Then for any $\beta > \alpha$, $\beta \notin C \cap C'$. ⊣

Fix for each $x \in \mathcal{P}_{\kappa_0}(\kappa_k)$ and each $C, C' \in K_x$ with $C \neq C'$ an ordinal $\rho_{x,C,C'} < \nu^+$ so that $C$ and $C'$ are disjoint above $\rho_{x,C,C'}$. Let $\rho^h$ be the supremum

of the ordinals $\rho_{x,C,C'}$. Since $\mathcal{P}_{\kappa_0}(\kappa_k)$ and $K_x$ have cardinalities smaller than $\mathrm{cof}(\nu^+)$, $\rho^h < \nu^+$.

We have that for every $x$ and $\alpha \in J^h - \rho^h$, $\alpha$ belongs to at most one $C \in K_x$. Define a function $f$ on $\mathcal{P}_{\kappa_0}(\kappa_k) \times (J^h - \rho^h)$ letting $f(x, \alpha)$ be the unique $C \in K_x$ so that $\alpha \in C$ if there is such a $C$, and leaving $f(x, \alpha)$ undefined otherwise. The function is defined in $V[E]$.

Claim 3.11. *Let $\alpha \in J^h - \rho^h$. Then $\{x \in \mathcal{P}_{\kappa_0}(\kappa_k) \mid f(x, \alpha)$ is defined$\}$ is given measure one by $\mathcal{U}_k$.*

Proof. Note that the set belongs to $V[E]$. Let $Y$ be its complement, namely the set $\{x \in \mathcal{P}_{\kappa_0}(\kappa_k) \mid f(x, \alpha)$ is not defined$\}$. Suppose for contradiction that $\mathcal{U}_k(Y) = 1$.

We intend to find $x \in Y$ so that $J_x$ is unbounded in $\nu^+$ and $\alpha \in J_x$. Since $J_x \in K_x$ it follows then that $f(x, \alpha)$ is defined (and equal to $J_x$), contradicting the fact that $x \in Y$.

We work with the sets given by Claim 3.6. The set $A^*_\alpha$ is given measure one by $\sigma^*(\mathcal{U}_k)$, and so is each of the sets $A^*_\beta$ for $\beta \in J^h$. As $Y \subseteq \mathcal{P}_{\kappa_0}(\kappa_k)$, $Y$ is not moved by $\sigma^*$. So $\sigma^*(\mathcal{U}_k)(Y) = \sigma^*(\mathcal{U}_k)(\sigma^*(Y)) = \sigma^*(\mathcal{U}_k(Y))$, which again is one. The intersection $A^*_\alpha \cap A^*_\beta \cap Y$ of these three $\sigma^*(\mathcal{U}_k)$ measure one sets is non-empty. So we can fix for each $\beta \in J^h$ some $x_\beta \in A^*_\alpha \cap A^*_\beta \cap Y$.

Since $J^h$ is unbounded in $\nu^+$, and $\nu^+$ has cofinality greater than ${\kappa_k}^+ = \mathrm{card}(\mathcal{P}_{\kappa_0}(\kappa_k))$, there is $x \in \mathcal{P}_{\kappa_0}(\kappa_k)$ and an unbounded $U \subseteq J^h$, so that $x_\beta = x$ for all $\beta \in U$.

By Claim 3.6 and since $x \in A^*_\alpha$, $h^\frown\langle x\rangle \Vdash \langle \sigma(\check{\alpha}), \check{\xi}_\alpha\rangle\ \sigma^*(\dot{T})\ \langle \check{\gamma}, \check{\zeta}\rangle$. It follows by the definition of $J_x$ that $\alpha \in J_x$. Similarly, since $x = x_\beta \in A^*_\beta$ for each $\beta \in U$, $\beta \in J_x$ and therefore $U \subseteq J_x$. Thus $x \in Y$ is such that $\alpha \in J_x$ and $J_x$ is unbounded in $\nu^+$, as required. $\dashv$

Claim 3.12. *Let $\alpha, \alpha' \in J^h - \rho^h$. Then $\{x \in \mathcal{P}_{\kappa_0}(\kappa_k) \mid f(x, \alpha) = f(x, \alpha')\}$ is given measure one by $\mathcal{U}_k$.*

Proof. Again the set belongs to $V[E]$. Let $Y$ be its complement, namely the set $\{x \in \mathcal{P}_{\kappa_0}(\kappa_k) \mid f(x, \alpha) \neq f(x, \alpha')\}$ and suppose for contradiction that $\mathcal{U}_k(Y) = 1$.

An argument similar to that in the proof of the previous claim, adding $A^*_{\alpha'}$ to the list of sets in the intersection, produces $x \in Y$ so that $J_x$ is unbounded, and both $\alpha$ and $\alpha'$ belong to $J_x$. Then both $f(x, \alpha) = J_x$ and $f(x, \alpha') = J_x$, so $f(x, \alpha) = f(x, \alpha')$, contradicting the fact that $x \in Y$. $\dashv$

Remark 3.13. For each $\alpha \in J^h - \rho^h$, the function $x \mapsto f(x, \alpha)$ belongs to $V[E]$. The proofs of the last two claims show that it agrees with the function $x \mapsto J_x$ on a $\sigma^*(\mathcal{U}_k)$ measure one set. But of course $x \mapsto J_x$ does not belong to $V[E]$, nor does the measure $\sigma^*(\mathcal{U}_k)$.

We are ready now to complete the proof of Lemma 3.5. Let $\alpha_0$ be the first element of $J^h$ above $\rho^h$. Define $A^h_\alpha$ for $\alpha \in J^h - \rho^h$ to be the set of $x$ so that $f(x, \alpha)$ is defined and equal to $f(x, \alpha_0)$. By the last two claims, $\mathcal{U}_k(A^h_\alpha) = 1$, so condition (1) in Lemma 3.5 holds. As for condition (2): Suppose $\alpha < \beta$ both belong to $J^h - \rho^h$, and $x \in A^h_\alpha \cap A^h_\beta$. Then $f(x, \alpha)$ and $f(x, \beta)$ are both defined

and both are equal to $f(x,\alpha_0)$. Let $C = f(x,\alpha_0)$. Then $C \in K_x$ and since both $f(x,\alpha)$ and $f(x,\beta)$ are equal to $C$, both $\alpha$ and $\beta$ belong to $C$. Using Claim 3.9 it follows that $h_x \Vdash \langle\check{\alpha},\check{\xi}_\alpha\rangle\ \dot{T}\ \langle\check{\beta},\check{\xi}_\beta\rangle$. This gives condition (2) of Lemma 3.5, completing its proof. $\dashv$

Lemma 3.14. *There is $\rho < \nu^+$, and a map $\alpha \mapsto p_\alpha$ ($\alpha \in J - \rho$) in $V[E]$, so that:*

1. *$p_\alpha \in \mathbb{P}$, with stem equal to $\bar{h}$.*
2. *For any $\alpha < \beta$ both in $J - \rho$, $p_\alpha \wedge p_\beta \Vdash \langle\check{\alpha},\check{\xi}_\alpha\rangle\ \dot{T}\ \langle\check{\beta},\check{\xi}_\beta\rangle$.*

Proof. We intend to set $p_\alpha = \langle\bar{h}, A_\alpha\rangle$, defining $A_\alpha(k)$ for $k \geq \bar{k}$ by recursion on $k$. We also define an increasing sequence of ordinals $\rho_k < \nu^+$. $A_\alpha(k)$ will be defined for all $\alpha \in J - \rho_k$. We work in $V[E]$ throughout. We shall define the sets $A_\alpha(k)$ by taking diagonal intersections of sets $A^h_\alpha$ given by the previous lemma.

A stem $h$ of length $k$ can be *prepended* to $x \in \mathcal{P}_{\kappa_0}(\kappa_k)$ if $h(n) \subseteq x$ and $\operatorname{card}(h(n)) < x \cap \kappa_0$ for each $n < k$. Note that if $h^\frown\langle x\rangle$ is a stem, then $h$ can be prepended to $x$. This follows from the third condition in the definition of $\mathbb{P}$.

For a set $H$ of stems of length $k$ and a mapping $h \mapsto Z^h$, the set $D = \{x \in \mathcal{P}_{\kappa_0}(\kappa_k) \mid x \in Z^h$ for every $h$ which can be prepended to $x\}$ is the *diagonal intersection* of the sets $Z^h$, $h \in H$. If each of the sets $Z^h$ has $\mathcal{U}_k$ measure one, then their diagonal intersection too has $\mathcal{U}_k$ measure one. The proof of this fact is standard. Let us just comment that it uses the restriction that $(\forall n < k)(h(n) \subseteq x \wedge \operatorname{card}(h(n)) < x \cap \kappa_0)$ of the previous paragraph, to make sure that for each individual $x$, not too many $h \in H$ are involved in determining whether $x \in D$.

A stem $h \supseteq \bar{h}$ of length $k \geq \bar{k}$ is said to *fit* $\alpha \geq \rho_k$ if $h(n) \in A_\alpha(n)$ for $\bar{k} \leq n < k$. The concept assumes that $\rho_k$ has already been defined, and that $A_\alpha(n)$ has already been defined for $\bar{k} \leq n < k$ and $\alpha \geq \rho_k$. Let $J^h$ be the set of $\alpha \geq \rho_k$ so that $h$ fits $\alpha$.

During the recursive definition of $\rho_k$ and $A_\alpha(k)$ we intend to make sure that:

(i) If $\alpha < \beta$ both belong to $J^h$ (equivalently, $h$ fits both ordinals), then $h \Vdash \langle\check{\alpha},\check{\xi}_\alpha\rangle\ \dot{T}\ \langle\check{\beta},\check{\xi}_\beta\rangle$.

(ii) $\bigcup_{h\supseteq\bar{h},\operatorname{lh}(h)=k} J^h = J - \rho_k$.

The sets $J^h$ need not be disjoint.

Set to start $\rho_{\bar{k}} = 0$, and $J^{\bar{h}} = J$. That condition (ii) holds is clear. Condition (i) holds as $J$ and $\alpha \mapsto \xi_\alpha$ were given by Lemma 3.2.

Suppose $\rho_k$ and $A_\alpha(n)$ for $\alpha \in J - \rho_k$ and $\bar{k} \leq n < k$ have been defined, and conditions (i) and (ii) hold for stems of length $k$.

For each stem $h \supseteq \bar{h}$ of length $k$ so that $J^h$ is bounded in $\nu^+$, let $\rho^h < \nu^+$ be a bound for $J^h$.

For every other stem $h \supseteq \bar{h}$ of length $k$, let $\rho^h$ and $A^h_\alpha$ ($\alpha \in J^h - \rho^h$) be given by Lemma 3.5. Note that the assumptions of the lemma are satisfied, because $J^h$ is unbounded, and because of condition (i) above.

Let $\rho_{k+1} = \sup\{\rho^h \mid h \supseteq \bar{h}, \operatorname{lh}(h) = k\}$. The supremum is taken over a set of size $< \nu^+$, so $\rho_{k+1} < \nu^+$.

Let $H_\alpha(k)$ be the set of stems $h \supseteq \bar{h}$ of length $k$ which fit $\alpha$. $H_\alpha(k)$ is non-empty for $\alpha \in J - \rho_{k+1}$ (in fact even $\alpha \in J - \rho_k$), by condition (ii) for $k$. If $J^h$

has elements above $\rho_{k+1}$ then it is unbounded in $\nu^+$, by definition of $\rho_{k+1}$. So $h \in H_\alpha(k)$ and $\alpha > \rho_{k+1}$ implies that $A^h_\alpha$ is defined using Lemma 3.5, and in particular it has $\mathcal{U}_k$ measure one. For each $\alpha \in J - \rho_{k+1}$ define $A_\alpha(k)$ to be the diagonal intersection of the sets $A^h_\alpha$, $h \in H_\alpha(k)$.

This completes the recursive definition. Note that $A_\alpha(k)$ has $\mathcal{U}_k$ measure one, because it is a diagonal intersection of measure one sets. Condition (ii) above holds for $k+1$, since $J^{h\frown\langle x\rangle} = \{\alpha \in J^h \mid x \in A_\alpha(k)\}$, and for every $\alpha \in J^h - \rho_{k+1}$, there are $x \in A_\alpha(k)$ which can be appended to $h$ (measure one many in fact). Condition (i) for $k+1$ follows from condition (1) in Lemma 3.5. If $h\frown\langle x\rangle$ fits both $\alpha$ and $\beta$, then $h$ belongs to both $H_\alpha(k)$ and $H_\beta(k)$, and $x$ belongs to both $A_\alpha(k)$ and $A_\beta(k)$. By the definition of $A_\alpha(k), A_\beta(k)$, and the definition of diagonal intersection, $x$ belongs to both $A^h_\alpha$ and $A^h_\beta$. By the use of Lemma 3.5 then, $h\frown\langle x\rangle \Vdash \langle\check\alpha, \check\xi_\alpha\rangle\, \dot T\, \langle\check\beta, \check\xi_\beta\rangle$.

We have now defined $\rho_k < \nu^+$, and $A_\alpha(k)$ for $k \geq \bar k$ and $\alpha \in J - \rho_k$. The definition is such that conditions (i) and (ii) above hold, and $\mathcal{U}_k(A_\alpha(k)) = 1$ for each $\alpha$ and each $k$.

Let $\rho = \sup\{\rho_k \mid \bar k \leq k < \omega\}$. For $\alpha \in J - \rho$, let $A_\alpha = \langle A_\alpha(k) \mid \bar k \leq k < \omega\rangle$, and let $p_\alpha = \langle \bar h, A_\alpha\rangle$. To complete the proof of the lemma, suppose $\alpha < \beta$ both belong to $J - \rho$. We have to show that $p_\alpha \wedge p_\beta \Vdash \langle\check\alpha, \check\xi_\alpha\rangle\, \dot T\, \langle\check\beta, \check\xi_\beta\rangle$.

Suppose $q \in \mathbb{P}$ is stronger than $p_\alpha \wedge p_\beta$. It is enough to show that $q$ does not forces $\langle\check\alpha, \check\xi_\alpha\rangle$ and $\langle\check\beta, \check\xi_\beta\rangle$ to be incomparable in $\dot T$.

Let $h = g_q$. Since $q \leq p_\alpha \wedge p_\beta$, $h \supseteq \bar h$ and $h(n)$ belongs to both $A_\alpha(n)$ and $A_\beta(n)$ for $\bar k \leq n < \mathrm{lh}(h)$. In other words, $h$ fits both $\alpha$ and $\beta$. By condition (i) above, $h$ forces $\langle\check\alpha, \check\xi_\alpha\rangle\, \dot T\, \langle\check\beta, \check\xi_\beta\rangle$. By definition this means that there is a condition with stem $h$ forcing the statement. Since $q$ has stem $h$, it cannot force the negation of the statement, in other words it cannot force $\langle\check\alpha, \check\xi_\alpha\rangle$ and $\langle\check\beta, \check\xi_\beta\rangle$ to be incomparable in $\dot T$. ⊣

We finally have the tools necessary to show that the tree $T = \dot T[G]$ has a branch in $V[E][G]$. We just have to show that enough of the conditions $p_\alpha$ given by the previous lemma belong to $G$.

Claim 3.15. *If the set $\{\alpha \in J - \rho \mid p_\alpha \in G\}$ is unbounded in $\nu^+$, then $T$ has a branch.*

Proof. Let $B = \{\alpha \in J - \rho \mid p_\alpha \in G\}$. If $\alpha < \beta$ both belong to $B$, then $p_\alpha \wedge p_\beta \in G$ and therefore $\langle\alpha, \xi_\alpha\rangle\, T\, \langle\beta, \xi_\beta\rangle$ (the condition $p_\alpha \wedge p_\beta$ forces this, by the last lemma). So the set $\{\langle\alpha, \xi_\alpha\rangle \mid \alpha \in B\}$, if unbounded, generates a branch of $T$. ⊣

Lemma 3.16. *$T = \dot T[G]$ has a branch.*

Proof. Suppose not, and let $q_0 \in G$ force that there are no branches through $\dot T$. By Remark 3.3, and strengthening $q_0$ if needed, we may assume that $\bar h = g_{q_0}$.

By the last claim, $B = \{\alpha \in J - \rho \mid p_\alpha \in G\}$ must be bounded in $\nu^+$, and this must be forced by $q_0$. Since $\mathbb{P}$ has the $\nu^+$ chain condition, $q_0$ must in fact force a specific bound, $\delta < \nu^+$, for the set.

Let $\alpha \in J - \rho$ be greater than $\delta$. The conditions $q_0$ and $p_\alpha$ have the same stem, $\bar h$. They are therefore compatible. Let $r$ be a common extension of these

conditions. Then $r$ forces $\check{p}_\alpha \in \dot{G}$, since $r \leq p_\alpha$. On the other hand $r$ forces $\check{p}_\alpha \notin \dot{G}$, since $r \leq q_0$ and $q_0$ forces $\alpha$, and indeed all ordinals above $\delta$, to not belong to $B$. This contradiction completes the proof. ⊣

Recall that in the extension $V[E][G]$, SCH fails at $\kappa_0$ ($2^{\kappa_0} = {\kappa_0}^{++}$ in the extension), there is a bad scale on $\kappa_0$, and there is a very good scale on $\kappa_0$. We proved that every $({\kappa_0}^+)^{V[E][G]}$-tree $T \in V[E][G]$ has a branch. Thus, in the extension, ${\kappa_0}^+$ has the tree property. This completes the proof of Theorem 1.1.

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