ARONSZAJN TREES AND FAILURE OF THE SINGULAR CARDINAL HYPOTHESIS

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Abstract. The tree property at κ^+ states that there are no Aronszajn trees on κ^+, or, equivalently, that every κ^+ tree has a cofinal branch. For singular strong limit cardinals κ, there is tension between the tree property at κ^+ and failure of the singular cardinal hypothesis at κ; the former is typically the result of the presence of strongly compact cardinals in the background, and the latter is impossible above strongly compacts. In this paper we reconcile the two. We prove from large cardinals that the tree property at κ^+ is consistent with failure of the singular cardinal hypothesis at κ.

§1. Introduction. In 1989 Woodin and others, see Foreman [7, §2], asked whether failure of the singular cardinal hypothesis (SCH) at a cardinal κ of cofinality ω, implies the existence of an Aronszajn tree on κ^+. To motivate the question recall the following terminology and results. A κ^+-tree is a tree of height κ^+, with levels of size less than κ^+. The tree property at κ^+ states that every κ^+-tree has a cofinal branch. In contrast, an Aronszajn tree on κ^+ is a κ^+-tree with no cofinal branches. The tree property at κ^+, for κ a singular strong limit cardinal of cofinality ω, is typically a consequence of the existence of large cardinals, specifically strongly compact cardinals, in the background. The only known route to establishing the property in such situations goes through a theorem of Magidor–Shelah [12], that the tree property holds at κ^+ if κ is a limit of strongly compact cardinals. The singular cardinal hypothesis holds at such cardinals κ, by Solovay [15]. It is natural then to ask whether failure of the singular cardinal hypothesis at κ implies failure of the tree property at κ^+, or in other words, whether it implies the existence of an Aronszajn tree on κ^+.

The singular cardinal hypothesis is closely tied with PCF theory (possible cofinalities theory, see Shelah [14] or any of [1], [2], [10], and [11]). It therefore seemed reasonable that a positive answer, if possible, would be obtained by isolating some PCF property that follows from failure of SCH, and implies the existence of an Aronszajn tree. Several candidates were considered for the intermediate property, between failure of SCH and the existence of an Aronszajn tree. Many had to do with square principles, introduced in Schimmerling [13] and generalizing the original principles defined by Jensen, since it is known by work of Jensen that the existence of certain Aronszajn trees is equivalent to a weak square principle. This program of research toward a positive answer to

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Woodin's question was initiated by Cummings, Foreman, and Magidor. It led to a large body of work, particularly about square principles and connections between these principles and PCF theory, for example [4], [5], [6], and [8].

However prospects that the program would lead to a solution to Woodin's question dimmed, when Gitik–Sharon [9] showed that two of the key candidates for the intermediate between failure of SCH and existence of Aronszajn trees, specifically the approachability property and the weak square principle, do not in fact follow from failure of SCH. And indeed, the answer to Woodin's question is negative. We prove in this paper that failure of SCH at κ does not imply the existence of an Aronszajn tree on κ^+:

THEOREM 1.1. *Suppose there are ω supercompact cardinals. Then it is consistent that there is a cardinal κ so that:*

1. *κ is a strong limit cardinal of cofinality ω.*
2. *$2^\kappa = \kappa^{++}$, hence SCH fails at κ.*
3. *There are no Aronszajn trees on κ^+.*

Moreover, it is consistent with the above that there is both a very good scale and a bad scale on κ.

Scales are PCF objects. The existence of a bad scale implies failure of the approachability property, which in turn implies failure of weak square.

Theorem 1.1 relies heavily on the construction of Gitik–Sharon [9]. The simple outline of its proof is this: combine the construction of Gitik–Sharon [9] with the proof in Magidor–Shelah [12] that the tree property holds at successors of limits of strongly compact cardinals. Gitik and Sharon start with a model where κ is supercompact, force to make $2^\kappa = \nu^{++}$ where $\nu = \kappa^{+(\omega)}$, and then force further, with a diagonal Prikry poset, to add a sequence $g = \langle g(n) \mid n < \omega \rangle$ which collapses ν to κ and changes the cofinality of κ to ω. Here we assume that there are ω supercompact cardinals $\kappa = \kappa_0 < \kappa_1 < \ldots$, and modify the Gitik–Sharon poset to use $\nu = \sup_{n<\omega} \kappa_n$. Then the successor of κ in the extension is ν^+. By Magidor–Shelah [12], ν^+ has the tree property in V. All we have to do is show that it continues to have the tree property in the extension. This, of course, is easier said than done. But it is too early to go into further details. The proof that ν^+ continue to have the tree property in the extension is given in Section 3.

The assumption in Theorem 1.1 can be weakened to the existence of ω cardinals $\langle \kappa_n \mid n < \omega \rangle$, so that each κ_n is ν^+ supercompact, where $\nu = \sup_{n<\omega} \kappa_n$. The theorem obtains the most economical failure of the SCH: $2^\kappa = \kappa^{++}$. The proof adapts easily to produce arbitrary failures $2^\kappa = \lambda > \kappa^+$, but one has to increase the large cardinal assumption to λ supercompactness for $\lambda > (\nu^+)^V$. It remains open whether the result can be pushed down to smaller cardinals, and in particular it is not known whether failure of SCH at $\aleph_\omega$ implies the existence of an Aronszajn tree on $\aleph_{\omega+1}$.

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§2. The forcing notion. Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of supercompact cardinals. Without loss of generality assume the GCH above κ_0. Suppose that the supercompactness of κ_0 is indestructible under $<\kappa_0$ closed forcing. This can always be arranged using the Laver preparation, maintaining the GCH above κ_0.

let $\nu = \sup\{\kappa_n \mid n < \omega\}$. Let $\mathbb{A}$ be the poset adding ν^{++} subsets of κ_0, with conditions of size $< \kappa_0$. Let E be generic for $\mathbb{A}$ over V. For each $\xi < \nu^{++}$ let E_ξ be the ξth subset of κ added by E. E itself is the characteristic function of $\{\langle \xi, u \rangle \mid \xi < \nu^{++}, u \in E_\xi\}$. By indestructibility, κ_0 remains supercompact in $V[E]$. Moreover, there is in $V[E]$ a ν^+ supercompactness measure on κ_0, so that the elements of its ultrapower up to the image of κ_0 require only κ_0 for their support. This is due to Gitik–Sharon [9]. For completeness we give the proof:

Lemma 2.1. *There is a ν^+ supercompactness embedding $\pi\colon V[E] \to M$ in $V[E]$, with critical point κ_0, so that every element of $M \| \pi(\kappa_0)$ has the form $\pi(f)(\kappa_0)$.*

Proof. We work in $V[E]$. It is enough to construct π so that every ordinal $< \pi(\kappa_0)$ has the required form, since there is a bijection in $\pi'' V[E]$ between $\pi(\kappa_0)$ and $M \| \pi(\kappa_0)$. Ordinals below κ_0 clearly have the required form, as the critical point of π is κ_0. So it is enough to handle ordinals in the interval $[\kappa_0, \pi(\kappa_0))$.

Using the indestructibility of κ_0, fix in $V[E]$ a ν^+ supercompactness embedding $\tau\colon V[E] \to N[F]$. Let σ be the restriction of τ to V, so that $\sigma\colon V \to N$, and $\tau(\dot{x}[E]) = \sigma(\dot{x})[F]$ for each $\mathbb{A}$-name $\dot{x}$ in V. Let $a = \sigma'' \nu^+ = \tau'' \nu^+$. Condensing τ if needed, we may assume that every element x of $N[F]$ has the form $\tau(f)(a)$ with $f \in V[E]$. If $x \in \tau(\kappa_0)$ then x has the form $\tau(f)(a)$ with $f\colon \mathcal{P}_{\kappa_0}(\nu^+) \to \kappa_0$. Since the size of ${\kappa_0}^{\mathcal{P}_{\kappa_0}(\nu^+)}$ is ν^{++} in $V[E]$, the cardinality of $\tau(\kappa_0)$ in $V[E]$ is ν^{++}. The cardinality of the interval $[\kappa_0, \sigma(\kappa_0)) = [\kappa_0, \tau(\kappa_0))$ is the same. Let $\langle u_\xi \mid \xi < \nu^{++} \rangle$ enumerate the elements of this interval.

For each $\xi < \nu^{++}$ let $f_\xi \in V[E]$ be the function that assigns to each $\alpha < \kappa$, the αth element of E_ξ. We make some adjustments to F, to obtain a revised generic E^* from it, and an embedding $\pi\colon V[E] \to N[E^*]$ extending σ, so that $u_\xi = \pi(f_\xi)(\kappa)$. As $[\kappa_0, \sigma(\kappa_0)) = \{u_\xi \mid \xi < \nu^{++}\}$, this will complete the proof.

Define E^* through the conditions:

- $E^*_\zeta = F_\zeta$ for $\zeta \notin \sigma'' \nu^{++}$.
- $E^*_{\sigma(\xi)} = F_{\sigma(\xi)} - [\kappa_0, u_\xi) \cup \{u_\xi\}$ for $\xi < \nu^{++}$.

The difference between E^* and F can be approximated inside N. Precisely, for each $\delta < \sigma(\nu^{++})$, there is a set X of size less than $\sigma(\kappa_0)$ in N, so that $E^* \restriction \delta \times \kappa_0$ and $F \restriction \delta \times \kappa_0$ differ only on X, and $E^* \restriction X$ belongs to N and is therefore a condition in $\sigma(\mathbb{A})$. (The set X is the product $\Pi_{\sigma(\xi) < \delta} \{\sigma(\xi)\} \times [\kappa_0, u_\xi]$. It is a product of ν^+ intervals, and belongs to N because of Ns closure. $E^* \restriction X$ belongs to N by closure too.) It follows from all this, the genericity of F over N, and the chain condition for $\sigma(\mathbb{A})$ (which implies that genericity of E^* for $\sigma(\mathbb{A})$ is the same as genericity of $E^* \restriction \delta \times \kappa_0$ for $\sigma(\mathbb{A}) \restriction \delta \times \kappa_0$ for all $\delta < \sigma(\nu^{++})$), that E^* is generic for $\sigma(\mathbb{A})$ over N.

It is clear from the definition that $E^*\restriction\text{range}(\sigma) = F\restriction\text{range}(\sigma)$, and since $\sigma''E \subseteq F$, it follows that $\sigma''E \subseteq E^*$. The embedding $\sigma\colon V \to N$ can therefore be extended to an embedding $\pi\colon V[E] \to N[E^*]$, setting $\pi(\dot{x}[E]) = \sigma(\dot{x})[E^*]$.

By elementarity, $\pi(f_\zeta)$ is the function that assigns to each $\alpha < \pi(\kappa_0)$, the αth element of E^*_ζ. For $\xi < \nu^{++}$ and $\zeta = \pi(\xi)$, the κ_0th element of E^*_ζ is u_ξ by definition of E^*. So $\{\pi(f_\xi)(\kappa_0) \mid \xi < \nu^{++}\} = \{u_\xi \mid \xi < \nu^{++}\} = [\kappa_0, \sigma(\kappa_0)) = [\kappa_0, \pi(\kappa_0))$, as required. $\dashv$

Let π be given by the last lemma. Let $\mathcal{U}$ be the ν^+ supercompactness measure induced by π. Precisely, $\mathcal{U}$ measures sets in $\mathcal{P}_{\kappa_0}(\nu^+)$, and $\mathcal{U}(X) = 1$ iff $\pi''\nu^+ \in \pi(X)$. For each $n < \omega$, let $\mathcal{U}_n$ be the κ_n supercompactness measure induced by π. $\mathcal{U}_n$ measures sets in $\mathcal{P}_{\kappa_0}(\kappa_n)$, and $\mathcal{U}_n(X) = 1$ iff $\pi''\kappa_n \in \pi(X)$.

We now force over $V[E]$ using the following poset $\mathbb{P}$, adapted from Gitik–Sharon [9]. Conditions are pairs $p = \langle g_p, A_p\rangle$ where:

- $g_p = \langle g_p(0), \dots, g_p(k-1)\rangle$, with $g_p(n) \in \mathcal{P}_{\kappa_0}(\kappa_n)$.
- $A_p = \langle A_p(n) \mid k \leq n < \omega\rangle$, with $A_p(n) \subseteq \mathcal{P}_{\kappa_0}(\kappa_n)$ and $\mathcal{U}_n(A_p(n)) = 1$.
- We require the stem to be monotone increasing and nice, in the sense that $g_p(n+1) \supseteq g_p(n)$, $g_p(n) \cap \kappa_0$ is an inaccessible cardinal (by necessity $< \kappa_0$), and $g_p(n+1) \cap \kappa_0 > \text{card}(g_p(n))$.

The poset belongs to the family of Prikry forcing notions, and conditions are ordered in the natural way: $q \leq p$ iff g_q extends g_p, $A_q(n) \subseteq A_p(n)$ for each $n \geq \text{lh}(g_q)$, and $g_q(n) \in A_p(n)$ for each $n \in \text{lh}(g_q) - \text{lh}(g_p)$.

$\mathbb{P}$ is a variant of a poset introduced by Gitik–Sharon [9]. If instead of κ_n we used $\kappa_0^{+(n)}$, we would have obtained precisely the Gitik–Sharon poset. Gitik–Sharon proved the Prikry property for their poset. Their argument, with trivial modification, gives:

Fact 2.2. $\mathbb{P}$ has the Prikry property. Precisely, let $\dot{x}_1, \dots, \dot{x}_k \in V[E]$ be $\mathbb{P}$-names, let φ be a formula, and let $p \in \mathbb{P}$. Then there is a condition $q \leq p$ which decides $\varphi(\dot{x}_1, \dots, \dot{x}_k)$, with $g_q = g_p$.

The part g_p is called the stem of p. It is clear that two conditions with the same stem are compatible. Since the stems are finite sequences from $\bigcup_{n<\omega} \mathcal{P}_{\kappa_0}(\kappa_n)$, which has cardinality ν in $V[E]$, $\mathbb{P}$ has the ν^+ chain condition.

For two conditions p and q with the same stem, we use $p \wedge q$ to denote the condition r determined by $g_r = g_p = g_q$ and $A_r(n) = A_p(n) \cap A_q(n)$. r is the weakest common extension of p and q.

$\mathbb{P}$ is not even ω closed, but for each stem h, the collection of conditions with stem equal to h is $<\kappa_0$ closed, since the measures $\mathcal{U}_n$ are $<\kappa_0$ complete.

Let G be generic for $\mathbb{P}$ over $V[E]$. G is completely determined by $g = \bigcup_{p\in G} g_p$. A condition p belongs to G iff $g_p \subseteq g$ and $g(n) \in A_p(n)$ for each $n \geq \text{lh}(g_p)$.

The following properties of the generic extension $V[E][G]$ are clear:

1. κ_0 is a strong limit in $V[E][G]$, $(2^{\kappa_0})^{V[E][G]} = (\nu^{++})^V$ and the GCH holds in $V[E][G]$ from ν^+ upward.
2. $\langle g(n) \cap \kappa_0 \mid n < \omega\rangle$ is cofinal in κ_0. In particular κ_0 is singular, of cofinality ω, in $V[E][G]$.
3. ν is collapsed to κ_0 in the extension. Indeed, it is equal to $\bigcup_{n<\omega} g(n)$, a union of ω sets each of cardinality $< \kappa_0$.

4. No cardinals are collapsed below κ_0 (because of the closure of $\mathbb{A}$, the Prikry property for $\mathbb{P}$, and the closure for conditions with a fixed stem in $\mathbb{P}$), and no cardinals are collapsed above ν (because of the chain condition).
5. In fact cofinalities smaller than κ_0 and greater than ν are preserved. Cofinalities in the interval $[\kappa_0, \nu)$ are changed to cofinality ω.

It follows in particular that the extension $V[E][G]$ satisfies $\neg\mathsf{SCH}_{\kappa_0}$. In the extension, κ_0 is a strong limit of cofinality ω, and $2^{\kappa_0} = \kappa^{++}$.

Gitik–Sharon [9] introduced their poset so as to produce an extension with a singular strong limit κ so that SCH_κ fails, there is a very good scale at κ, and yet the approachability property fails at κ, and in particular so does the weak square property. Cummings–Foreman [3] showed that there is also a bad scale on κ in the Gitik–Sharon extension, and this implies the failure of the approachability property. All these results adapt with little change to our extension, yielding:

Fact 2.3. In $V[E][G]$, there is a very good scale on κ_0, and there is a bad scale on κ_0.

Let $\tau^n = g(n) \cap \kappa_0$, so that $\langle \tau^n \mid n < \omega \rangle$ is increasing and cofinal in κ_0. Using the property of π given by Lemma 2.1, fix for each $\alpha < \nu^+$ a function $f_\alpha \colon \kappa \to \kappa$ in $V[E]$ so that $\pi(f)(\kappa) = \alpha$. Let $\tau^n_i = f_{\kappa_i}(\tau^n)$, and let $\mu^n = f_\nu(\tau^n)$. By assuming that the generic G contains a condition $p_0 = \langle \emptyset, A_0 \rangle$ with an appropriately restricted sequence of measure one sets $A_0(n)$, we can reflect the fact that $\langle \kappa_i \mid i < \omega \rangle$ is increasing, and arrange that $\langle \tau^n_i \mid i < \omega \rangle$ is increasing for each n. Reflecting the fact that $\sup\{\kappa_i \mid i < \omega\} = \nu$ we may arrange that $\sup\{\tau^n_i \mid i < \omega\} = \mu^n$ for each n. Making sure that each of the elements of the measure one set $A_0(n+1)$ is sufficiently closed, we can arrange that $\tau^{n+1} > \mu^n$. Finally, by taking f_{κ_0} to be the identity, and adjusting each f_α on a measure zero set, we may assume that $\tau^n_0 = \tau^n$, and that $f_\alpha(\tau^n) < (\mu^n)^+$ for each $\alpha < \nu^+$ and each n.

The very good scale mentioned in Fact 2.3 is the sequence $\langle \varphi_\alpha \mid \alpha < \nu^+ \rangle$ defined by $\varphi_\alpha(n) = f_\alpha(\tau_n)$. It is a scale on $\Pi_{n<\omega}(\mu^n)^+$ in $V[E][G]$.

Fix, in $V[E]$, a scale $\langle \psi^*_\alpha \mid \alpha < \nu^+ \rangle$ on $\Pi_{n<\omega}{\kappa_n}^+$. Since ν is above a supercompact, $\langle \psi^*_\alpha \mid \alpha < \nu^+ \rangle$ is a bad scale on ν in $V[E]$. The bad scale mentioned in Fact 2.3 is the sequence $\langle \psi_\alpha \mid \alpha < \nu^+ \rangle$ defined by $\psi_\alpha(n) = f_{\psi^*_\alpha(n)}(\tau_n)$. It is a scale on $\Pi_{n<\omega}(\tau^n_n)^+$ in $V[E][G]$.

The proofs that $\vec{\varphi}$ and $\vec{\psi}$ are very good, and bad, scales respectively are direct adaptations of the corresponding proofs in [9] and [3], and we do not include them here. We proceed now to prove the extra property for which we created the extension: that in $V[E][G]$, ${\kappa_0}^+$ has the tree property.

§3. The tree property. Fix, in $V[E]$, a $\mathbb{P}$-name $\dot{T}$ which is forced by the empty condition to be a tree on $(\nu^+)^V = ({\kappa_0}^+)^{V[E][G]}$, with levels of size at most κ_0. We work to prove that in $V[E][G]$ there is a branch through $T = \dot{T}[G]$. Our argument is inspired by the proof in Magidor–Shelah [12] that the tree property holds at successors of limits of ω supercompact cardinals. In our context the Magidor–Shelah proof shows that the tree property at ν^+ holds in V. We shall have to do additional work to account for the move to an extension by $\mathbb{A} \times \mathbb{P}$.

Much of the difficulty is in dealing with $\mathbb{A}$, but of course we could not have defined $\mathbb{P}$ without first forcing with $\mathbb{A}$, since the definition of $\mathbb{P}$ uses the measures given by Lemma 2.1, and the lemma relies on the addition of subsets of κ_0.

Recall that $\dot{T} \in V[E]$ is a $\mathbb{P}$-name for T. Without loss of generality suppose that the nodes on level α of T are the elements of $\{\alpha\} \times \kappa_0$, and that this is forced by the empty condition in $\mathbb{P}$.

For every $\alpha < \beta < \nu^+$ there are $\xi, \zeta < \kappa_0$, and $k < \omega$, so that $\langle \alpha, \xi \rangle \, T \, \langle \beta, \zeta \rangle$, and so that this is forced by a condition with stem of length k. We begin by finding a cofinal set $I \subseteq \nu^+$ on which k can be fixed.

Lemma 3.1. *There is $\bar{k} < \omega$, and a cofinal $I \subseteq \nu^+$ in $V[E]$, so that for all $\alpha, \beta \in I$, there exists $\xi, \zeta < \kappa_0$ and $p \in \mathbb{P}$, so that $\mathrm{lh}(g_p) = \bar{k}$ and $p \Vdash \langle \check{\alpha}, \check{\xi} \rangle \, \dot{T} \, \langle \check{\beta}, \check{\zeta} \rangle$.*

Proof. We work in $V[E]$. Recall that $\pi \colon V[E] \to M$ is a ν^+ supercompactness embedding. Let G^* be generic for $\pi(\mathbb{P})$ over M. Using the fact that ν^+ is a discontinuity point of π (which follows from the closure of M under ν^+ sequences) fix γ between $\sup(\pi''\nu^+)$ and $\pi(\nu^+)$. Fix a node u of $\pi(\dot{T})[G^*]$ on level γ of the tree, and a name $\dot{u}$ for this node.

For every $\alpha < \nu^+$, there is ξ_α so that $\langle \pi(\alpha), \xi_\alpha \rangle$ is a node of $\pi(\dot{T})[G^*]$ on level $\pi(\alpha)$, and is below u in the tree order. Let $p_\alpha \in G^*$ force that $\langle \pi(\check{\alpha}), \check{\xi}_\alpha \rangle \, \pi(\dot{T}) \, \dot{u}$. Let $k_\alpha = \mathrm{lh}(g_{p_\alpha})$. Since $\pi''\nu^+$ belongs to M, all this can be done inside $M[G^*]$. Since ν^+ is a regular cardinal in $M[G^*]$, there is a fixed $\bar{k}$, so that $k_\alpha = \bar{k}$ for cofinally many $\alpha < \nu^+$.

Let h^* be the restriction to $\bar{k}$ of the stem of some (equivalently all) condition in G^* with stem of length $> \bar{k}$. Let $I \subseteq \nu^+$ be the set of all α so that there is a condition $r \in \pi(\mathbb{P})$ with stem h^*, and an ordinal $\zeta < \pi(\kappa_0)$, so that $r \Vdash \langle \pi(\check{\alpha}), \check{\zeta} \rangle \, \pi(\dot{T}) \, \dot{u}$. The definition of I is made with reference to π but without reference to G^*. So I belongs to $V[E]$. By the way we fixed $\bar{k}$ in the last paragraph, I is cofinal in ν^+.

Suppose now that $\alpha < \beta$ both belong to I. By the definition of I there are conditions $r_\alpha, r_\beta \in \pi(\mathbb{P})$, both with stem h^*, and ordinals $\zeta_\alpha, \zeta_\beta$, so that $r_\alpha \Vdash \langle \pi(\check{\alpha}), \check{\zeta}_\alpha \rangle \, \pi(\dot{T}) \, \dot{u}$ and $r_\beta \Vdash \langle \pi(\check{\beta}), \check{\zeta}_\beta \rangle \, \pi(\dot{T}) \, \dot{u}$

Since r_α and r_β have the same stem h^*, $r_\alpha \wedge r_\beta$ is defined and is a common extension of the conditions, again with stem h^*. It forces that both $\langle \pi(\check{\alpha}), \check{\zeta}_\alpha \rangle$ and $\langle \pi(\check{\beta}), \check{\zeta}_\beta \rangle$ are below $\dot{u}$ in $\pi(\dot{T})$. Since $\pi(\dot{T})$ is forced to be a tree, it follows that the condition forces the two nodes to be compatible, i.e., $r_\alpha \wedge r_\beta \Vdash \langle \pi(\check{\alpha}), \check{\zeta}_\alpha \rangle \pi(\dot{T}) \langle \pi(\check{\beta}), \check{\zeta}_\beta \rangle$. By elementarity of π then, there exists $p \in \mathbb{P}$ with stem of length $\bar{k}$, and $\zeta, \zeta' < \kappa_0$, so that $p \Vdash \langle \check{\alpha}, \check{\zeta} \rangle \, \dot{T} \, \langle \check{\beta}, \check{\zeta}' \rangle$. ⊣

Having fixed length, we now proceed to fix the stem itself, and also the nodes of $\dot{T}$ involved.

Lemma 3.2. *There is in $V[E]$ a cofinal $J \subseteq \nu^+$, a map $\alpha \mapsto \xi_\alpha$ ($\alpha \in J$) and a stem $\bar{h}$ of length $\bar{k}$, so that for every $\alpha < \beta$ both in J, there is a condition p with stem $\bar{h}$, that forces $\langle \check{\alpha}, \check{\xi}_\alpha \rangle \, \dot{T} \, \langle \check{\beta}, \check{\xi}_\beta \rangle$.*

Proof. Let $\sigma \colon V \to N$ be a ν^+ supercompactness embedding with critical point $\kappa_{\bar{k}+1}$. Let $\mathbb{B}$ be the poset for adding $\sigma(\nu^{++})$ subsets of κ_0 with conditions of size $< \kappa_0$, and let F be generic for $\mathbb{B}$ over $V[E]$. Since $\sigma(\mathbb{A})$ is itself the poset for adding $\sigma(\nu^{++})$ subsets of κ_0, we can in $V[E][F]$ combine $\sigma''E$ and F to find $E^* \supseteq \sigma''E$ which is generic for $\sigma(\mathbb{A})$ over N. The embedding σ then extends to an embedding $\sigma^*\colon V[E] \to N[E^*]$. We have $\sigma^* \in V[E][F]$.

ν^+ is a discontinuity point of σ, and I is cofinal in ν^+, so we can find $\gamma > \sigma''\nu^+$ with $\gamma \in \sigma^*(I)$.

Using the conclusion of the previous lemma, shifted to $N[E^*]$ via the elementary embedding σ^*, we can find for each $\alpha \in I$ some $\xi_\alpha, \zeta_\alpha < \kappa_0 = \sigma(\kappa_0)$ and $p_\alpha \in \sigma^*(\mathbb{P})$ so that $p_\alpha \Vdash \langle \sigma(\check{\alpha}), \check{\xi}_\alpha \rangle \, \sigma^*(\dot{T}) \, \langle \check{\gamma}, \check{\zeta}_\alpha \rangle$, and so that the length of g_{p_α} is $\bar{k}$. Since $\sigma''\nu^+$ belongs to N, all this can be done inside $N[E^*]$. Continuing to work inside $N[E^*]$, we can find some fixed stem $\bar{h}$, some fixed ζ, and a cofinal $J \subseteq I$, so that for every $\alpha \in J$, $g_{p_\alpha} = \bar{h}$, and $\zeta_\alpha = \zeta$.

Suppose now that $\alpha < \beta$ both belong to J. Then $p_\alpha \wedge p_\beta$ is a condition with stem $\bar{h}$, which forces both $\langle \sigma(\check{\alpha}), \check{\xi}_\alpha \rangle$ and $\langle \sigma(\check{\beta}), \check{\xi}_\beta \rangle$ to be below $\langle \check{\gamma}, \zeta \rangle$ in the tree order $\sigma^*(\dot{T})$, and therefore forces them to be comparable in the tree order. Pulling back to $V[E]$ using the elementarity of σ^*, it follows that there is a condition $p \in \mathbb{P}$, with stem equal to $\bar{h}$, forcing over $V[E]$ that $\langle \check{\alpha}, \check{\xi}_\alpha \rangle \, \dot{T} \, \langle \check{\beta}, \check{\xi}_\beta \rangle$. Note that $\bar{h}$, ξ_α, and ξ_β are not affected by the pull back to $V[E]$, as they are all below the critical point of σ^*. ξ_α and ξ_β are smaller than κ_0, and $\bar{h}$ is a finite sequence from $\mathcal{P}_{\kappa_0}(\kappa_{\bar{k}})$, while the critical point of σ^* is $\kappa_{\bar{k}+1}$.

So far we found J, $\bar{h}$, and a map $\alpha \mapsto \xi_\alpha$ satisfying the condition in the claim, except that J and the map $\alpha \mapsto \xi_\alpha$ belong to $V[E][F]$, not to $V[E]$. It remains to see that we can find similar objects in $V[E]$.

Let $Z \in V[E]$ be the set of tuples $\langle \bar{h}, \alpha, \xi, \alpha', \xi' \rangle$ so that there exists a condition $p \in \mathbb{P}$ with $g_p = \bar{h}$ forcing $\langle \check{\alpha}, \check{\xi} \rangle \, \dot{T} \, \langle \check{\alpha}', \check{\xi}' \rangle$ over $V[E]$.

Let $\theta(Z, \bar{h}, J, f, \nu^+)$ be the statement that J is cofinal in ν^+, $\bar{h}$ has length $\bar{k}$, and for every $\alpha < \beta$ both in J, $\langle \bar{h}, \alpha, f(\alpha), \beta, f(\beta) \rangle \in Z$. We proved that $V[E][F] \models (\exists \bar{h}, J, f)\, \theta(Z, \bar{h}, J, f, \nu^+)$. (Take f to be the function $\alpha \mapsto \xi_\alpha$ obtained above, with the objects I' and $\bar{h}$ obtained above.) To obtain $\bar{h}$, J, and f inside $V[E]$, we simply use the fact that Z can be coded by a subset of ν^+, and θ is absolute.

Precisely, let H be a $<\!\kappa_0$ closed elementary model of a sufficiently large rank initial segment of $V[E]$, with $\mathrm{card}(H)^{V[E]} = \nu^+$ and $\nu^+ \cup \{\nu^+, Z, \mathbb{B}\} \subseteq H$. Let Q be the transitive collapse of H, and let $c\colon H \to Q$ be the collapse embedding. Let $\delta = H \cap \nu^{++}$. Then $\delta < \nu^{++}$, and Q has the form $R[E \restriction \delta]$, where by $E \restriction \delta$ we mean the part of E adding the sets E_ξ, $\xi < \delta$. Since Z can be coded by a subset of ν^+, it is not moved by c. Nor is ν^+ itself moved. The poset $c(\mathbb{B})$ is $\mathrm{Add}(\kappa_0, c(\sigma(\nu^{++})))$, and $c(\sigma(\nu^{++}))$ is smaller than ν^{++}, since $\mathrm{card}(H) < \nu^{++}$. So $E \restriction [\delta, \nu^{++})$ supplies more than enough subsets of κ_0 that are generic over $Q = R[E \restriction \delta]$, to construct $\bar{F} \in V[E]$ which is generic for $c(\mathbb{B})$ over Q.

By the elementarity of the anticollapse embedding, $Q[\bar{F}]$ satisfies $(\exists \bar{h}, J, f)$ $\theta(Z, \bar{h}, J, f, \nu^+)$. Since $Q[\bar{F}]$ belongs to $V[E]$, we can find $\bar{h}$, J, and f in $V[E]$ so that $Q[\bar{F}] \models \theta(Z, \bar{h}, J, f, \nu^+)$. Now since θ is absolute, $V[E] \models \theta(Z, \bar{h}, J, f, \nu^+)$. $\dashv$

Remark 3.3. For any condition $q \in \mathbb{P}$, Lemmas 3.1 and 3.2 can be strengthened to give $\bar{h} = g_r$ for some $r \leq q$, simply by restricting attention to conditions stronger than q. It follows that the set of conditions r so that J, $\alpha \mapsto \xi_\alpha$, and $\bar{h}$

as in Lemma 3.2 can be found, with $\bar{h} = g_r$, is dense in $\mathbb{P}$. The generic G meets every dense set below every $q_0 \in G$. So we may assume, for an arbitrary $q_0 \in G$, that $\bar{h} = g_r$ for some $r \in G$ stronger than q_0.

We continue to work with J and the map $\alpha \mapsto \xi_\alpha$ for the rest of the section. We use ξ to denote the map. We know that for $\alpha < \beta$ both in J, there is a condition with stem $\bar{h}$ forcing $\langle \check{\alpha}, \check{\xi}_\alpha \rangle \, \dot{T} \, \langle \check{\beta}, \check{\xi}_\beta \rangle$. Our next goal is to find a map $\alpha \mapsto p_\alpha$, so that for all $\alpha < \beta$ both in J, the condition $p_\alpha \wedge p_\beta$ forces this compatibility. We do this in stages. We shall set $p_\alpha = \langle \bar{h}, A^\alpha \rangle$, and we work recursively on $n \geq \bar{k}$ to define $A^\alpha(n)$.

For a stem h, we write that $h \Vdash \varphi$ iff there is a condition $p \in \mathbb{P}$ with $g_p = h$ so that $p \Vdash \varphi$. Note that if $h \Vdash \varphi$, then any condition q with $g_q = h$ either forces φ, or does not decide φ. This is because any two conditions p, q with the same stem are compatible. Note further that, by the Prikry property, for every φ and any stem h, either $h \Vdash \varphi$ or $h \Vdash \neg\varphi$. It is important to emphasize though, that even if $h \Vdash \varphi$, there may well be stems h' extending h so that $h' \Vdash \neg\varphi$. The fact that $h \Vdash \varphi$ merely implies that there are not very many such h'.

Lemma 3.4. *Let h be a stem of length k extending $\bar{h}$. Let $J^h \subseteq J$ be unbounded in ν^+, and suppose that for all $\alpha < \beta$ both in J^h, $h \Vdash \langle \check{\alpha}, \check{\xi}_\alpha \rangle \, \dot{T} \, \langle \check{\beta}, \check{\xi}_\beta \rangle$. Then there is $\rho^h < \nu^+$, and a map $\alpha \mapsto A^h_\alpha$ ($\alpha \in J^h - \rho^h$) in $V[E]$, so that:*

1. *$\mathcal{U}_k(A^h_\alpha) = 1$ for each α.*
2. *For every $\alpha < \beta$ both in J^h and greater than ρ^h, and for every $x \in A^h_\alpha \cap A^h_\beta$, $h^\frown\langle x \rangle \Vdash \langle \check{\alpha}, \check{\xi}_\alpha \rangle \, \dot{T} \, \langle \check{\beta}, \check{\xi}_\beta \rangle$.*

Proof. Let $\sigma \colon V \to N$ be a ν^+ supercompactness embedding with critical point κ_{k+1}. Let $\hat{\mathbb{B}} = \mathrm{Add}(\kappa_0, \nu^{++})$, and let $\hat{F}$ be generic for $\mathbb{B}$ over $V[E]$. In $V[E][F]$ we can combine $\sigma''E$ and F to find $E^* \supseteq \sigma''E$ which is generic for $\sigma(\mathbb{A})$ over N. The embedding σ then extends to an embedding $\sigma^* \colon V[E] \to N[E^*]$. We have $\sigma^* \in V[E][F]$.

ν^+ is a discontinuity point of σ, and J^h is cofinal in ν^+, so we can find $\gamma > \sigma''\nu^+$ with $\gamma \in \sigma^*(J^h)$. Let $\zeta = \sigma^*(\xi)_\gamma$.

Claim 3.5. *There is, in $V[E][F]$, a map $\alpha \mapsto A^*_\alpha$ ($\alpha \in J^h$) so that:*

- *A^*_α has $\sigma^*(\mathcal{U}_k)$ measure one.*
- *For each $\alpha \in J^h$ and $x \in A^*_\alpha$, $h^\frown\langle x \rangle \Vdash \langle \sigma(\check{\alpha}), \sigma(\check{\xi}_\alpha) \rangle \, \sigma^*(\dot{T}) \, \langle \check{\gamma}, \check{\zeta} \rangle$.*

Proof. By assumption of the lemma and the elementarity of σ^*, $h = \sigma^*(h)$ forces in $\sigma^*(\mathbb{P})$ over $N[E^*]$ that $\langle \sigma(\check{\alpha}), \sigma(\check{\xi}_\alpha) \rangle \, \sigma^*(\dot{T}) \, \langle \check{\gamma}, \check{\zeta} \rangle$. Fix a condition r_α with stem h forcing this, and let $A^*_\alpha = A_{r_\alpha}(k)$. ⊣

It is tempting to think that we can set $A^h_\alpha = A^*_\alpha$, and use a trick similar to that in the proof of the previous lemma to pull the existence of the resulting map back to $V[E]$. Unfortunately the sets A^*_α are given measure one not by $\mathcal{U}_k$ but by $\sigma^*(\mathcal{U}_k)$. Both measures are on $\mathcal{P}_{\kappa_0}(\kappa_k)$, and this domain is not moved by σ^* whose critical point is κ_{k+1}. Under GCH σ^* would not affect the measures either. But we do not have the GCH, and since $2^{\kappa_0} = \nu^{++} > \mathrm{crit}(\sigma^*)$, there are more subsets of $\mathcal{P}_{\kappa_0}(\kappa_k)$ in $N[E^*]$ than in $V[E]$. The measures $\mathcal{U}_k$ and $\sigma^*(\mathcal{U}_k)$ are different, and the sets A^*_α we obtained above need not even belong to the domain of $\mathcal{U}_k$, let alone have $\mathcal{U}_k$ measure one.

Our biggest problem in proving the lemma is to produce sets A^h_α which belong to $V[E]$, so that they are measured by $\mathcal{U}_k$. Claim 3.7 below, which is a consequence of the next claim, 3.6, provides our initial tool in pulling existence of sets from $V[E][\hat{F}]$, back to $V[E]$. Unfortunately it handles the wrong sets—"vertical" subsets of ν^+ rather than "horizontal" subsets of $\mathcal{P}_{\kappa_0}(\kappa_k)$—but we shall deal with that problem later.

Claim 3.6. *Let S be a tree of height θ in a model M of* ZFC*. Let $\mathbb{B} \in M$ be a poset, and suppose that, in M, $\mathbb{B} \times \mathbb{B}$ has the* $\operatorname{cof}(\theta)$ *chain condition. Suppose further that a power $\mathbb{B}^{|S|^+}$ does not collapse $|S|^+$. (Which power is used, meaning which support is used to form the power, is irrelevant to the claim, so long as the resulting power does not collapse $|S|^+$.) Then $\mathbb{B}$ does not add new branches to S. Precisely, if F is generic for $\mathbb{B}$ over M, and $b \in M[F]$ is a branch of S, then $b \in M$.*

Proof. We work over M. Replacing θ by $\operatorname{cof}(\theta)$, and replacing S by a restriction of S to nodes on levels in a set of order type $\operatorname{cof}(\theta)$ cofinal in θ, we may assume that θ is regular. Let $\dot{b}$ name a branch of S, and supposed that it is forced that $\dot{b}$ does not belong to M. Let $\mathbb{B}^*$ denote the power $\mathbb{B}^{|S|^+}$, and let $F^* = \Pi_{\delta<|S|^+} F_\delta$ be generic for $\mathbb{B}^*$ over M. Let $b_\delta = \dot{b}[F_\delta]$.

We shall work with the product $\mathbb{B} \times \mathbb{B}$, and with generics $F_{\delta_1} \times F_{\delta_2}$, $\delta_1 \neq \delta_2$, for this product. We use b_{left} and b_{right} to refer to the branches $\dot{b}[F_{\delta_1}]$ and $\dot{b}[F_{\delta_2}]$ in the extension $M[F_{\delta_1} \times F_{\delta_2}]$.

Let H be an elementary submodel of a sufficiently large rank initial segment of M, with $\operatorname{card}(H) < \theta$, $H \cap \theta$ an ordinal, and $\{\theta, S, \dot{b}, \mathbb{B}, \mathbb{B}^*\} \subseteq H$. Let $\eta = H \cap \theta < \theta$.

Because H is elementary, $H \cap \theta$ is an ordinal, and $\mathbb{B}$ has the θ chain condition, every antichain of $\mathbb{B}$ that belongs to H is contained in H. It follows that $H[F_\delta]$ is an elementary substructure of (a rank initial segment of) $M[F_\delta]$ for each δ, and $H[F_\delta] \cap M = H$. Similarly $H[F_{\delta_1} \times F_{\delta_2}]$ is an elementary substructure of (a rank initial segment of) $M[F_{\delta_1} \times F_{\delta_2}]$ for $\delta_1 \neq \delta_2$, and $H[F_{\delta_1} \times F_{\delta_2}] \cap M = H$.

For each $\delta < |S|^+$, let β_δ be the node of b_δ of height η. As $\beta_\delta \in S$ for each $\delta < |S|^+$, and $|S|^+$ is not collapsed in $M[F^*]$, there must be $\delta_1 \neq \delta_2$ so that $\beta_{\delta_1} = \beta_{\delta_2}$. By elementarity of $H[F_{\delta_1}]$, $b_{\delta_1} \cap H$ consists of all nodes of b_{δ_1} of height $< \eta$. Thus $b_{\delta_1} \cap H$ is equal to the set of nodes of S below β_{δ_1}. Similar reasoning applies to $b_{\delta_2} \cap H$, and since $\beta_{\delta_1} = \beta_{\delta_2}$ it follows that $b_{\delta_1} \cap H = b_{\delta_2} \cap H$.

Consider now the situation in $M[F_{\delta_1} \times F_{\delta_2}]$. From the conclusion of the previous paragraph we get $b_{\delta_1} \cap H[F_{\delta_1} \times F_{\delta_2}] = b_{\delta_2} \cap H[F_{\delta_1} \times F_{\delta_2}]$, in other words $(b_{\text{left}} = b_{\text{right}})^{H[F_{\delta_1} \times F_{\delta_2}]}$. By elementarity of $H[F_{\delta_1} \times F_{\delta_2}]$ in $M[F_{\delta_1} \times F_{\delta_2}]$ it follows that $\dot{b}_{\text{left}}[F_{\delta_1} \times F_{\delta_2}] = \dot{b}_{\text{right}}[F_{\delta_1} \times F_{\delta_2}]$. From this, standard arguments produce a condition in $\mathbb{B}$ forcing $\dot{b}$ to belong to M. ⊣

Recall that we are working with an embedding σ^*, which we obtained in an extension $V[E][\hat{F}]$, where $\hat{F}$ is generic for $\hat{\mathbb{B}} = \operatorname{Add}(\kappa_0, \nu^{++})$. h and J^h satisfy the assumptions in Lemma 3.4.

Claim 3.7. *Let h^* be a stem of length $k+1$ extending h. Suppose that $J^* \in V[E][\hat{F}]$ is a subset of J^h so that:*

1. *J^* is unbounded in ν^+.*

2. *For* $\alpha < \beta$ *both in* J^h, *with* $\beta \in J^*$, *we have* $\alpha \in J^*$ *iff* $h^* \Vdash \langle \check{\alpha}, \check{\xi}_\alpha \rangle \dot{T} \langle \check{\beta}, \check{\xi}_\beta \rangle$. *Then* J^* *belongs to* $V[E]$.

By condition (2), knowledge that $\beta \in J^*$ is sufficient to completely determine $J^* \cap \beta$, in $V[E]$. We shall use this in the proof.

Proof of Claim 3.7. Let $M = V[E]$, let $\theta = \nu^+$, and let $S \in M$ be the tree of attempts to construct an increasing function $b\colon \nu^+ \to J^h$, so that condition (2) in the claim holds with J^* replaced by range(b). (A node in S is an initial segment of b.)

By condition (2), every strict initial segment of J^* belongs to M. So the function enumerating J^* in increasing order is a branch of S. By Claim 3.6, the function belongs to M, and therefore so does J^*. The claim is applied with the poset $\hat{\mathbb{B}}$, whose κ_0 support powers have the κ_0 chain condition. In particular they have the $\nu^+ = \theta$ chain condition, and do not collapse $|S|^+ = \nu^{++}$. ⊣

For every $x \in \mathcal{P}_{\kappa_0}(\kappa_k)$, let h_x be the stem $h^\frown\langle x \rangle$ of length $k+1$. Let J_x be the set of $\alpha \in J^h$ so that $h_x \Vdash \langle \sigma(\check{\alpha}), \sigma(\check{\xi}_\alpha) \rangle\, \sigma^*(\dot{T})\, \langle \check{\gamma}, \check{\zeta} \rangle$. Let $\dot{J}_x \in V[E]$ be a $\hat{\mathbb{B}}$ name for the set defined this way. J_x is defined in $V[E][\hat{F}]$, where we have access to σ^*. Still, using the previous claims, we get:

Claim 3.8. *If* J_x *is unbounded in* ν^+, *then it belongs to* $V[E]$.

Proof. We check that h_x and J_x satisfy the conditions in Claim 3.7, and then appeal to the claim. Condition (1) is clear as we explicitly assume that J_x is unbounded. As for condition (2), if $h_x \Vdash \langle \sigma(\check{\beta}), \sigma(\check{\xi}_\beta) \rangle\, \sigma^*(\dot{T})\, \langle \check{\gamma}, \check{\zeta} \rangle$, then for $\alpha < \beta$ (in J^h, so that ξ_α is defined),

$$\begin{aligned} h_x \Vdash \langle \alpha, \xi_\alpha \rangle\, \dot{T}\, \langle \beta, \xi_\beta \rangle &\Leftrightarrow h_x \Vdash \langle \sigma(\check{\alpha}), \sigma(\check{\xi}_\alpha) \rangle\, \sigma^*(\dot{T})\, \langle \sigma(\check{\beta}), \sigma(\check{\xi}_\beta) \rangle \\ &\Leftrightarrow h_x \Vdash \langle \sigma(\check{\alpha}), \sigma(\check{\xi}_\alpha) \rangle\, \sigma^*(\dot{T})\, \langle \check{\gamma}, \check{\zeta} \rangle \\ &\Leftrightarrow \alpha \in J_x. \end{aligned}$$

The first equivalence uses the elementarity of σ^*, the second uses the fact that $\sigma^*(\dot{T})$ is forced by the empty condition to be a tree order, and the third is by definition.

Now Claim 3.7 yields $J_x \in V[E]$. ⊣

Let K_x be the set of $C \subseteq J^h$ in $V[E]$ so that C is unbounded in ν^+ and there is $b \in \hat{\mathbb{B}}$ forcing $\dot{J}_x = \check{C}$. K_x and the map $x \mapsto K_x$ belong to $V[E]$. Since $\hat{\mathbb{B}}$ has the κ_0 chain condition, card$(K_x) < \kappa_0$ in $V[E]$. By the last claim, J_x, if unbounded, belongs to K_x. But we cannot in $V[E]$ tell which element of K_x it is.

Claim 3.9. *Suppose* $C \in K_x$. *Then for* $\alpha < \beta$ *both in* J^h, *with* $\beta \in C$, *we have* $\alpha \in C$ *iff* $h_x \Vdash \langle \check{\alpha}, \check{\xi}_\alpha \rangle\, \dot{T}\, \langle \check{\beta}, \check{\xi}_\beta \rangle$.

Proof. Since C can be a value of $\dot{J}_x$, the calculation ending the proof of the previous claim applies (with C for J_x), yielding the current claim. ⊣

Claim 3.10. *Suppose that* C *and* C' *are two distinct elements of* K_x. *Then they are disjoint on a tail-end of* ν^+.

PROOF. If $\beta \in C \cap C'$, then by the previous claim $C \cap \beta = C' \cap \beta$. As $C \neq C'$ there is some $\alpha < \nu^+$ which belongs to one but not the other. Then for any $\beta > \alpha$, $\beta \notin C \cap C'$. ⊣

Fix for each $x \in \mathcal{P}_{\kappa_0}(\kappa_k)$ and each $C, C' \in K_x$ with $C \neq C'$ an ordinal $\rho_{x,C,C'} < \nu^+$ so that C and C' are disjoint above $\rho_{x,C,C'}$. Let ρ^h be the supremum of the ordinals $\rho_{x,C,C'}$. Since $\mathcal{P}_{\kappa_0}(\kappa_k)$ and K_x have cardinalities smaller than $\mathrm{cof}(\nu^+)$, $\rho^h < \nu^+$.

We have that for every x and $\alpha \in J^h - \rho^h$, α belongs to at most one $C \in K_x$. Define a function f on $\mathcal{P}_{\kappa_0}(\kappa_k) \times (J^h - \rho^h)$ letting $f(x, \alpha)$ be the unique $C \in K_x$ so that $\alpha \in C$ if there is such a C, and leaving $f(x, \alpha)$ undefined otherwise. The function is defined in $V[E]$.

CLAIM 3.11. *Let $\alpha \in J^h - \rho^h$. Then $\{x \in \mathcal{P}_{\kappa_0}(\kappa_k) \mid f(x, \alpha)$ is defined$\}$ is given measure one by $\mathcal{U}_k$.*

PROOF. Note that the set belongs to $V[E]$. Let Y be its complement, namely the set $\{x \in \mathcal{P}_{\kappa_0}(\kappa_k) \mid f(x, \alpha)$ is not defined$\}$. Suppose for contradiction that $\mathcal{U}_k(Y) = 1$.

We intend to find $x \in Y$ so that J_x is unbounded in ν^+ and $\alpha \in J_x$. Since $J_x \in K_x$ it follows then that $f(x, \alpha)$ is defined (and equal to J_x), contradicting the fact that $x \in Y$.

We work with the sets given by Claim 3.5. The set A^*_α is given measure one by $\sigma^*(\mathcal{U}_k)$, and so is each of the sets A^*_β for $\beta \in J^h$. As $Y \subseteq \mathcal{P}_{\kappa_0}(\kappa_k)$, Y is not moved by σ^*. So $\sigma^*(\mathcal{U}_k)(Y) = \sigma^*(\mathcal{U}_k)(\sigma^*(Y)) = \sigma^*(\mathcal{U}_k(Y))$, which again is one. The intersection $A^*_\alpha \cap A^*_\beta \cap Y$ of these three $\sigma^*(\mathcal{U}_k)$ measure one sets is non-empty. So we can fix for each $\beta \in J^h$ some $x_\beta \in A^*_\alpha \cap A^*_\beta \cap Y$.

Since J^h is unbounded in ν^+, and ν^+ has cofinality greater than ${\kappa_k}^+ = \mathrm{card}(\mathcal{P}_{\kappa_0}(\kappa_k))$, there is $x \in \mathcal{P}_{\kappa_0}(\kappa_k)$ and an unbounded $U \subseteq J^h$, so that $x_\beta = x$ for all $\beta \in U$.

By Claim 3.5 and since $x \in A^*_\alpha$, $h^\frown\langle x\rangle \Vdash \langle \sigma(\check{\alpha}), \sigma(\check{\xi}_\alpha)\rangle\, \sigma^*(\dot{T})\, \langle \check{\gamma}, \check{\zeta}\rangle$. It follows by the definition of J_x that $\alpha \in J_x$. Similarly, since $x = x_\beta \in A^*_\beta$ for each $\beta \in U$, $\beta \in J_x$ and therefore $U \subseteq J_x$. Thus $x \in Y$ is such that $\alpha \in J_x$ and J_x is unbounded in ν^+, as required. ⊣

CLAIM 3.12. *Let $\alpha, \alpha' \in J^h - \rho^h$. Then $\{x \in \mathcal{P}_{\kappa_0}(\kappa_k) \mid f(x, \alpha) = f(x, \alpha')\}$ is given measure one by $\mathcal{U}_k$.*

PROOF. Again the set belongs to $V[E]$. Let Y be its complement, namely the set $\{x \in \mathcal{P}_{\kappa_0}(\kappa_k) \mid f(x, \alpha) \neq f(x, \alpha')\}$ and suppose for contradiction that $\mathcal{U}_k(Y) = 1$.

An argument similar to that in the proof of the previous claim, adding $A^*_{\alpha'}$ to the list of sets in the intersection, produces $x \in Y$ so that J_x is unbounded, and both α and α' belong to J_x. Then both $f(x, \alpha) = J_x$ and $f(x, \alpha') = J_x$, so $f(x, \alpha) = f(x, \alpha')$, contradicting the fact that $x \in Y$. ⊣

REMARK 3.13. For each $\alpha \in J^h - \rho^h$, the function $x \mapsto f(x, \alpha)$ belongs to $V[E]$. The proofs of the last two claims show that it agrees with the function $x \mapsto J_x$ on a $\sigma^*(\mathcal{U}_k)$ measure one set. But of course $x \mapsto J_x$ does not belong to $V[E]$, nor does the measure $\sigma^*(\mathcal{U}_k)$.

We are ready now to complete the proof of Lemma 3.4. Let α_0 be the first element of J^h above ρ^h. Define A^h_α for $\alpha \in J^h - \rho^h$ to be the set of x so that $f(x,\alpha)$ is defined and equal to $f(x,\alpha_0)$. By the last two claim, $\mathcal{U}_k(A^h_\alpha) = 1$, so condition (1) in Lemma 3.4 holds. As for condition (2): Suppose $\alpha < \beta$ both belong to $J^h - \rho^h$, and $x \in A^h_\alpha \cap A^h_\beta$. Then $f(x,\alpha)$ and $f(x,\beta)$ are both defined and both are equal to $f(x,\alpha_0)$. Let $C = f(x,\alpha_0)$. Then $C \in K_x$ and since both $f(x,\alpha)$ and $f(x,\beta)$ are equal to C, both α and β belong to C. Using Claim 3.9 it follows that $h_x \Vdash \langle\check{\alpha},\check{\xi}_\alpha\rangle\, \dot{T}\, \langle\check{\beta},\check{\xi}_\beta\rangle$. This gives condition (2) of Lemma 3.4, completing its proof. ⊣

Lemma 3.14. *There is $\rho < \nu^+$, and a map $\alpha \mapsto p_\alpha$ ($\alpha \in J - \rho$) in $V[E]$, so that:*

1. *$p_\alpha \in \mathbb{P}$, with stem equal to $\bar{h}$.*
2. *For any $\alpha < \beta$ both in $J - \rho$, $p_\alpha \wedge p_\beta \Vdash \langle\check{\alpha},\check{\xi}_\alpha\rangle\, \dot{T}\, \langle\check{\beta},\check{\xi}_\beta\rangle$.*

Proof. We intend to set $p_\alpha = \langle \bar{h}, A_\alpha\rangle$, defining $A_\alpha(k)$ for $k \geq \bar{k}$ by recursion on k. We also define an increasing sequence of ordinals $\rho_k < \nu^+$. $A_\alpha(k)$ will be defined for all $\alpha \in J - \rho_k$. We work in $V[E]$ throughout. We shall define the sets $A_\alpha(k)$ by taking diagonal intersections of sets A^h_α given by the previous lemma.

A stem h of length k can be *prepended* to $x \in \mathcal{P}_{\kappa_0}(\kappa_k)$ if $h(n) \subseteq x$ and $\text{card}(h(n)) < x \cap \kappa_0$ for each $n < k$. Note that if $h^\frown\langle x\rangle$ is a stem, then h can be prepended to x. This follows from the third condition in the definition of $\mathbb{P}$.

For a set H of stems of length k and a mapping $h \mapsto Z^h$, the set $D = \{x \in \mathcal{P}_{\kappa_0}(\kappa_k) \mid x \in Z^h$ for every h which can be prepended to $x\}$ is the *diagonal intersection* of the sets Z^h, $h \in H$. If each of the sets Z^h has $\mathcal{U}_k$ measure one, then their diagonal intersection too has $\mathcal{U}_k$ measure one. The proof of this fact is standard. Let us just comment that it uses the restriction that $(\forall n < k)(h(n) \subseteq x \wedge \text{card}(h(n)) < x \cap \kappa_0)$ of the previous paragraph, to make sure that for each individual x, not too many $h \in H$ are involved in determining whether $x \in D$.

A stem $h \supseteq \bar{h}$ of length $k \geq \bar{k}$ is said to *fit* $\alpha \geq \rho_k$ if $h(n) \in A_\alpha(n)$ for $\bar{k} \leq n < k$. The concept assumes that ρ_k has already been defined, and that $A_\alpha(n)$ has already been defined for $\bar{k} \leq n < k$ and $\alpha \geq \rho_k$. Let J^h be the set of $\alpha \geq \rho_k$ so that h fits α.

During the recursive definition of ρ_k and $A_\alpha(k)$ we intend to make sure that:

(i) If $\alpha < \beta$ both belong to J^h (equivalently, h fits both ordinals), then $h \Vdash \langle\check{\alpha},\check{\xi}_\alpha\rangle\, \dot{T}\, \langle\check{\beta},\check{\xi}_\beta\rangle$.
(ii) $\bigcup_{h\supseteq\bar{h},\text{lh}(h)=k} J^h = J - \rho_k$.

The sets J^h need not be disjoint.

Set to start $\rho_{\bar{k}} = 0$, and $J^{\bar{h}} = J$. That condition (ii) holds is clear. Condition (i) holds as J and $\alpha \mapsto \xi_\alpha$ were given by Lemma 3.2.

Suppose ρ_k and $A_\alpha(n)$ for $\alpha \in J - \rho_k$ and $\bar{k} \leq n < k$ have been defined, and conditions (i) and (ii) hold for stems of length k.

For each stem $h \supseteq \bar{h}$ of length k so that J^h is bounded in ν^+, let $\rho^h < \nu^+$ be a bound for J^h.

For every other stem $h \supseteq \bar{h}$ of length k, let ρ^h and A^h_α ($\alpha \in J^h - \rho^h$) be given by Lemma 3.4. Note that the assumptions of the lemma are satisfied, because J^h is unbounded, and because of condition (i) above.

Let $\rho_{k+1} = \sup\{\rho^h \mid h \supseteq \bar{h}, \mathrm{lh}(h) = k\}$. The supremum is taken over a set of size $< \nu^+$, so $\rho_{k+1} < \nu^+$.

Let $H_\alpha(k)$ be the set of stems $h \supseteq \bar{h}$ of length k which fit α. $H_\alpha(k)$ is non-empty for $\alpha \in J - \rho_{k+1}$ (in fact even $\alpha \in J - \rho_k$), by condition (ii) for k. If J^h has elements above ρ_{k+1} then it is unbounded in ν^+, by definition of ρ_{k+1}. So $h \in H_\alpha(k)$ and $\alpha > \rho_{k+1}$ implies that A^h_α is defined using Lemma 3.4, and in particular it has $\mathcal{U}_k$ measure one. For each $\alpha \in J - \rho_{k+1}$ define $A_\alpha(k)$ to be the diagonal intersection of the sets A^h_α, $h \in H_\alpha(k)$.

This completes the recursive definition. Note that $A_\alpha(k)$ has $\mathcal{U}_k$ measure one, because it is a diagonal intersection of measure one sets. Condition (ii) above holds for $k+1$, since $J^{h^\frown\langle x\rangle} = \{\alpha \in J^h \mid x \in A_\alpha(k)\}$, and for every $\alpha \in J^h - \rho_{k+1}$, there are $x \in A_\alpha(k)$ which can be appended to h (measure one many in fact). Condition (i) for $k+1$ follows from condition (1) in Lemma 3.4. If $h^\frown\langle x\rangle$ fits both α and β, then h belongs to both $H_\alpha(k)$ and $H_\beta(k)$, and x belongs to both $A_\alpha(k)$ and $A_\beta(k)$. By the definition of $A_\alpha(k), A_\beta(k)$, and the definition of diagonal intersection, x belongs to both A^h_α and A^h_β. By the use of Lemma 3.4 then, $h^\frown\langle x\rangle \Vdash \langle\check{\alpha}, \check{\xi}_\alpha\rangle \,\dot{T}\, \langle\check{\beta}, \check{\xi}_\beta\rangle$.

We have now defined $\rho_k < \nu^+$, and $A_\alpha(k)$ for $k \geq \bar{k}$ and $\alpha \in J - \rho_k$. The definition is such that conditions (i) and (ii) above hold, and $\mathcal{U}_k(A_\alpha(k)) = 1$ for each α and each k.

Let $A_\alpha = \langle A_\alpha(k) \mid \bar{k} \leq k < \omega\rangle$, and let $p_\alpha = \langle \bar{h}, A_\alpha\rangle$. Let $\rho = \sup\{\rho_k \mid \bar{k} \leq k < \omega\}$. To complete the proof of the lemma, suppose $\alpha < \beta$ both belong to $J - \rho$. We have to show that $p_\alpha \wedge p_\beta \Vdash \langle\check{\alpha}, \check{\xi}_\alpha\rangle \,\dot{T}\, \langle\check{\beta}, \check{\xi}_\beta\rangle$.

Suppose $q \in \mathbb{P}$ is stronger than $p_\alpha \wedge p_\beta$. It is enough to show that q does not forces $\langle\check{\alpha}, \check{\xi}_\alpha\rangle$ and $\langle\check{\beta}, \check{\xi}_\beta\rangle$ to be incomparable in $\dot{T}$.

Let $h = g_q$. Since $q \leq p_\alpha \wedge p_\beta$, $h \supseteq \bar{h}$ and $h(n)$ belongs to both $A_\alpha(n)$ and $A_\beta(n)$ for $\bar{k} \leq n < \mathrm{lh}(h)$. In other words, h fits both α and β. By condition (i) above, h forces $\langle\check{\alpha}, \check{\xi}_\alpha\rangle \,\dot{T}\, \langle\check{\beta}, \check{\xi}_\beta\rangle$. By definition this means that there is a condition with stem h forcing the statement. Since q has stem h, it cannot force the negation of the statement, in other words it cannot force $\langle\check{\alpha}, \check{\xi}_\alpha\rangle$ and $\langle\check{\beta}, \check{\xi}_\beta\rangle$ to be incomparable in $\dot{T}$. ⊣

We finally have the tools necessary to show that the tree $T = \dot{T}[G]$ has a branch in $V[E][G]$. We just have to show that enough of the conditions p_α given by the previous lemma belong to G.

Claim 3.15. *If the set $\{\alpha \in J - \rho \mid p_\alpha \in G\}$ is unbounded in ν^+, then T has a branch.*

Proof. Let $B = \{\alpha \in J - \rho \mid p_\alpha \in G\}$. If $\alpha < \beta$ both belong to B, then $p_\alpha \wedge p_\beta \in G$ and therefore $\langle\alpha, \xi_\alpha\rangle \, T \, \langle\beta, \xi_\beta\rangle$ (the condition $p_\alpha \wedge p_\beta$ forces this, by the last lemma). So the set $\{\langle\alpha, \xi_\alpha\rangle \mid \alpha \in B\}$, if unbounded, generates a branch of T. ⊣

Lemma 3.16. *$T = \dot{T}[G]$ has a branch.*

Proof. Suppose not, and let $q_0 \in G$ force that there are no branches through $\dot{T}$. By Remark 3.3, and strengthening q_0 if needed, we may assume that $\bar{h} = g_{q_0}$.

By the last claim, $B = \{\alpha \in J - \rho \mid p_\alpha \in G\}$ must be bounded in ν^+, and this must be forced by q_0. Since $\mathbb{P}$ has the ν^+ chain condition, q_0 must in fact force a specific bound, $\delta < \nu^+$, for the set.

Let $\alpha \in J - \rho$ be greater than δ. The conditions q_0 and p_α have the same stem, $\bar{h}$. They are therefore compatible. Let r be a common extension of these conditions. Then r forces $\check{p}_\alpha \in \dot{G}$, since $r \leq p_\alpha$. On the other hand r forces $\check{p}_\alpha \notin \dot{G}$, since $r \leq q_0$ and q_0 forces α, and indeed all ordinals above δ, to not belong to B. This contradiction completes the proof. ⊣

Recall that in the extension $V[E][G]$, SCH fails at κ_0 ($2^{\kappa_0} = {\kappa_0}^{++}$ in the extension), there is a bad scale on κ_0, and there is a very good scale on κ_0. We proved that every $({\kappa_0}^+)^{V[E][G]}$-tree $T \in V[E][G]$ has a branch. Thus, in the extension, ${\kappa_0}^+$ has the tree property. This completes the proof of Theorem 1.1.

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