Determinacy and Large Cardinals

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University of California Los Angeles
Los Angeles, CA 90095-1555

25 August 2006
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\[
\begin{array}{c|c}
    I & II \\
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\end{array}
\]

If \( z \in A \) then player I wins. If \( z \notin A \) then player II wins.
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\[
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I & a_0 & a_2 & a_4 \\
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Players $I$ and $II$ alternate playing numbers $a_n \in \omega$, forming together an infinite sequence $z = \langle a_0, a_1, a_2, \cdots \cdots \rangle \in \omega^\omega$. 
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If $z$ belongs to $A$ then player $I$ wins.
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If \(z\) belongs to \(A\) then player \(I\) wins.
If \(z\) does not belong to \(A\) then player \(II\) wins.
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If \( z \) belongs to \( A \) then player I wins.
If \( z \) does not belong to \( A \) then player II wins.

\( G_\omega(A) \) is determined if one of the players has a winning strategy.

(A strategy is a complete recipe that instructs the player precisely how to play in each conceivable situation.)
For $\Gamma \subseteq \mathcal{P}(\omega^\omega)$, $\text{det}(\Gamma)$ is the statement that all sets in $\Gamma$ are determined.
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But determinacy for *definable* sets is: (1) true; and (2) useful.
\( \omega^\omega \) is the set of finite sequences of natural numbers.
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For $s \in \omega^\omega$ let $N_s = \{ x \in \omega^\omega \mid x \text{ extends } s \}$.
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$N_s$ ($s \in \omega^<\omega$) are the *basic open sets*. 
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$N_s$ ($s \in \omega^{<\omega}$) are the \textit{basic open sets}.

$A \subseteq \omega^\omega$ is \textit{open} if it is a union of basic open sets.
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Following standard abuse of notation identify it with \( \mathbb{R} \).
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The Borel sets are those that can be obtained from open sets using complementations and countable unions.
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The projection of $B \subset \mathbb{R} \times \mathbb{R}$ is the set $\{ x \mid (\exists y) \langle x, y \rangle \in B \}$. 
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\( \{ \text{Borel sets} \} \subset \{ \text{analytic sets} \} \subset \{ \text{projective sets} \} \).
Theorem 1 (Gale–Stewart 1953) *All open sets are determined.*
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$L(\mathbb{R})$ is the smallest model of set theory which contains all the reals and all the ordinals.
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$L(\mathbb{R})$ is the smallest model of set theory which contains all the reals and all the ordinals. It is obtained as the union $\bigcup_{\alpha \in \text{ON}} L_\alpha(\mathbb{R})$ where:
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$\{\text{projective sets}\} \subset L_1(\mathbb{R})$. 
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Theorem 5 (Woodin 1985) *All sets of reals in $L(\mathbb{R})$ are determined.*
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Theorems 1 and 2 are in ZFC, the basic system of axioms for set theory.
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Theorems 3, 4, and 5 require large cardinal axioms.
Theorem 6 (Banach, Oxtoby 1957) Assume $\det(\Gamma)$. Then all sets in $\Gamma$ have the Baire property.
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Theorem 7 (Mycielski–Swierczkowski 1964) Assume $\det(\Gamma)$. Then all sets in $\Gamma$ are Lebesgue measurable.

Theorem 8 (Davis 1964) Assume $\det(\Gamma)$. Let $A \in \Gamma$. Then either $A$ is countable or else it contains a perfect set.
Theorem 6 (Banach, Oxtoby 1957) Assume $\text{det}(\Gamma)$. Then all sets in $\Gamma$ have the Baire property.

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Every \( \Sigma^1_n \) set \( A \) is obtained from an underlying open set using negations and existential quantifiers. \( A \) is (lightface) \( \Sigma^1_n \) if the underlying open set is recursive. Similarly with \( \Pi^1_n \).
Γ has the \textit{reduction property} if for any $A, B \in \Gamma$, there are $A' \subset A$, $B' \subset B$ in $\Gamma$, so that $A' \cup B' = A \cup B$ and $A' \cap B' = \emptyset$. 
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In fact they did more. They obtained a fundamental property, the prewellordering property, which implies reduction.
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A pwo $\preceq$ induces an equivalence relation: $x \sim y$ iff $x \preceq y \land y \preceq x$. The pwo gives rise to a wellorder of the equivalence classes.
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$\preceq$ belongs to $\Gamma$ if there are $P, N$ in $\Gamma$, $\neg \Gamma$ respectively, so that for every $y \in A$, $\{ x \mid x \preceq y \} = \{ x \mid \langle x, y \rangle \in P \} = \{ x \mid \langle x, y \rangle \in N \}$. 
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**Theorem 10 (Martin, Addison–Moschovakis 1968)** Assume $\text{det}($projective$)$. Then the projective pointclasses with the pwo property, and similarly reduction, are $\Pi^1_1$, $\Sigma^1_2$, $\Pi^1_3$, $\Sigma^1_4$, $\ldots$.
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For \( B \subset \mathbb{R} \times \mathbb{R} \) set \( B_x = \{ y \mid \langle x, y \rangle \in B \} \).
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Easy to check $\partial \Pi^1_n = \Sigma^1_{n+1}$, and (using determinacy) $\partial \Sigma^1_n = \Pi^1_{n+1}$.

The pointclasses in Theorem 10 are therefore precisely the pointclasses $\partial^{(n)} \Pi^1_1$, $n < \omega$. 

8
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**Theorem 12 (Steel–Van Wesep–Woodin)** Assume $AD^{L(\mathbb{R})}$. Then it is consistent (with $AD^{L(\mathbb{R})}$ and AC) that $(\omega_2)^{L(\mathbb{R})} = \omega_2$, and hence $\delta_2^1 = \omega_2$. 
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$\kappa$ must be a limit cardinal. Otherwise have $\tau < \kappa$ so that $\kappa = \tau^+$. But then by elementarity $\pi(\kappa) = (\pi(\tau)^+)^M$. Yet $\pi(\tau) = \tau$, so $\pi(\kappa) = (\tau^+)^M = \kappa$, contradiction.
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The critical point of $\pi$ is the first ordinal $\kappa$ so that $\pi(\kappa) \neq \kappa$.

$\kappa$ must be a cardinal. Otherwise have $\tau < \kappa$ and a surjection $f : \tau \to \kappa$. But then by elementarity $\pi(f)$ is onto $\pi(\kappa)$. Since $f \subset \tau \times \kappa \subset \text{crit}(\pi)^2$, $\pi(f) = f$. So $\pi(\kappa) = \kappa$, contradiction.

$\kappa$ must be a limit cardinal. Otherwise have $\tau < \kappa$ so that $\kappa = \tau^+$. But then by elementarity $\pi(\kappa) = (\pi(\tau)^+)^M$. Yet $\pi(\tau) = \tau$, so $\pi(\kappa) = (\tau^+)^M = \kappa$, contradiction.

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Similar arguments show $\kappa$ must be inaccessible, and in fact cannot be described from below in any absolute manner.

So the existence of non-trivial $\pi : V \rightarrow M \subset V$ cannot be proved in ZFC, and the first ordinal moved by $\pi$ must be very large.
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(Using an ultrapower construction, the measurability of $\kappa$ is equivalent to the existence of a total, non-principal, countably complete, 2-valued measure on $\kappa$.)
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\( \kappa \) is \( <\delta \text{-strong} \) if it is the critical point of a \( \lambda \text{-strong} \) embedding for each \( \lambda < \delta \).
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Note: if \( \kappa \) is the first measurable cardinal, then \( \kappa \) is only \( \kappa^+ \)-\textit{strong}.
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\( \delta \) is a *Woodin cardinal* if for every \( D \subset \delta \) there is \( \kappa < \delta \) which is \( <\delta \text{-}strong \) wrt \( D \).
Let $\pi: V \to M$. Let $\kappa = \text{crit}(\pi)$ and $\lambda \leq \pi(\kappa)$. The $(\kappa, \lambda)-extender$ induced by $\pi$ is the function $E: \mathcal{P}(\kappa) \to \mathcal{P}(\lambda)$ defined by $E(X) = \pi(X) \cap \lambda$. 
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From $E$ one can construct (using ultrapowers) an embedding $\sigma: V \to \text{Ult}(V, E)$ which agrees with $\pi$ to $\lambda$, meaning that $\sigma(X) \cap \lambda = \pi(X) \cap \lambda$. 
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This allows constructing iterated ultrapowers with non-linear base orders.
Constructed in stages, starting from a base model $M_0$. E.g., having constructed $M_1, \ldots, M_6$: pick an extender $E_6 \in M_6$, apply it to $M_1$, setting $M_7 = \text{Ult}(M_1, E_6)$ and letting $j_{1,7} : M_1 \rightarrow M_7$ be the ultrapower embedding. At limit $\lambda$: pick a branch through the tree, cofinal in $\lambda$. Set $M_\lambda$ equal to the direct limit of models and embeddings along this branch. The result is an iteration tree on $M_\omega$. 

\[ \begin{align*} M_0 &\rightarrow M_1 & M_1 &\rightarrow M_2 & M_2 &\rightarrow M_3 \ldots \rightarrow M_\omega \rightarrow M_{\omega+1} \\ j_{0,1} &\rightarrow j_{1,2} & j_{1,3} &\rightarrow j_{2,4} & \vdots & \rightarrow \vdots \end{align*} \]
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M_3 & \rightarrow M_4 \\
M_4 & \rightarrow M_5 \\
M_5 & \rightarrow M_6 \\
M_6 & \rightarrow M_7 \\
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The result is an iteration tree on $M$. 
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The creation of iteration trees with several cofinal branches requires many strong extenders.

Woodin cardinals give precisely the extenders needed.
In fact using the extenders given by Woodin cardinals one can construct iteration trees very flexibly, reducing quantifiers over real numbers to quantifiers over iteration trees and branches through them.
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**Theorem (Woodin)** Suppose there are $\omega$ Woodin cardinals and a measurable cardinal above them. Then all sets in $L(\mathbb{R})$ are determined.
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**Theorem (Woodin)** Suppose there are $\omega$ Woodin cardinals and a measurable cardinal above them. Then all sets in $L(\mathbb{R})$ are determined.

In both cases Woodin cardinals in iterable inner models (rather than the actual universe $V$) are enough, and moreover necessary.
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But some make direct use of inner models for Woodin cardinals.
A set $A$ is $\alpha$–$\Pi_1^1$ if there is a sequence $\langle A_\xi \mid \xi < \alpha \rangle$ of $\Pi_1^1$ sets so that $x \in A$ iff the least $\xi$ so that $x \notin A_\xi \lor \xi = \alpha$ is odd.
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(The hierarchy generated by this definition is the difference hierarchy on $\Pi^1_1$ sets. If $\alpha = 2$ for example, then the condition states simply that $A = A_0 - A_1$.)
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The lightface notion is defined similarly, requiring a recursive code for the sequence.
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**Theorem 13 (Neeman–Woodin)** $\det(\Pi^1_{n+1})$ implies determinacy for all sets in the larger pointclass $\mathcal{C}^{(n)}(\omega^2–\Pi^1_1)$.
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**Theorem 13 (Neeman–Woodin)** \( \text{det}(\Pi^1_{n+1}) \) implies determinacy for all sets in the larger pointclass $\mathcal{C}(n)(<\omega^2–\Pi^1_1)$.

For $n = 0$: $\Pi^1_1$ determinacy gives a non-trivial $\pi: L \rightarrow L$ (Harrington), which in turn gives $<\omega^2–\Pi^1_1$ determinacy (Martin).
A set $A$ is $\alpha$–$\Pi^1_1$ if there is a sequence $\langle A_\xi \mid \xi < \alpha \rangle$ of $\Pi^1_1$ sets so that $x \in A$ iff the least $\xi$ so that $x \notin A_\xi \lor \xi = \alpha$ is odd.

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Generally: $\Pi^1_{n+1}$ determinacy gives non-trivial $\pi: M \to M$ where $M$ is an iterable class model with $n$ Woodin cardinals (Woodin), which in turn gives $\deltac(n)(<\omega^2–\Pi^1_1)$ determinacy (Neeman).
A set $A$ is $\alpha-\Pi^1_1$ if there is a sequence $\langle A_\xi \mid \xi < \alpha \rangle$ of $\Pi^1_1$ sets so that $x \in A$ iff the least $\xi$ so that $x \notin A_\xi \lor \xi = \alpha$ is odd.

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Theorem known previously for odd $n$, not using large cardinals (Kechris–Woodin).
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**Theorem 13 (Neeman–Woodin)** \( \det(\Pi^1_{n+1}) \) implies determinacy for all sets in the larger pointclass $\mathcal{O}^{(n)}(<\omega^2-\Pi^1_1)$. 
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**Theorem 13 (Neeman–Woodin)** $\text{det}(\Pi^1_{n+1})$ implies determinacy for all sets in the larger pointclass $\varnothing^{(n)}(<\omega^2–\Pi^1_1)$.

**Theorem 14 (Hjorth)** Work in $L(\mathbb{R})$ assuming $\text{AD}$. Let $\preceq$ be a $\varnothing(\alpha–\Pi^1_1)$ prewellorder with $\alpha < \omega \cdot k$. Then the ordertype of $\preceq$ is smaller than $\omega_{k+1}$. 

17
A set $A$ is $\alpha$–$\Pi^1_1$ if there is a sequence $\langle A_\xi | \xi < \alpha \rangle$ of $\Pi^1_1$ sets so that $x \in A$ iff the least $\xi$ so that $x \notin A_\xi \lor \xi = \alpha$ is odd.

The lightface notion is defined similarly, requiring a recursive code for the sequence.

**Theorem 13 (Neeman–Woodin)** $\det(\Pi^1_{n+1})$ implies determinacy for all sets in the larger pointclass $\mathcal{D}(n)(<\omega^2 - \Pi^1_1)$.

**Theorem 14 (Hjorth)** Work in $L(\mathbb{R})$ assuming $\text{AD}$. Let $\preceq$ be a $\mathcal{D}(\alpha - \Pi^1_1)$ prewellorder with $\alpha < \omega \cdot k$. Then the ordertype of $\preceq$ is smaller than $\omega_{k+1}$.

**Theorem 15 (Neeman, Woodin)** Assume $\text{AD}^{L(\mathbb{R})}$. Then it is consistent (with $\text{AD}^{L(\mathbb{R})}$ and the axiom of choice) that $\delta^1_3 = \omega_2$. 

17
Let $\theta(\nu)$ be a formula. A *sharp* for $\theta$ is a non-trivial embedding $\pi: M \to M$ where $M$ is the minimal iterable class model admitting a non-trivial embedding $\pi$ and satisfying $\theta[\text{crit}(\pi)]$. 
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**Iterable:**

The creation of iteration trees requires some choice at limits.
Constructed in stages, starting from a base model $M = M_0$.

E.g., having constructed $M_1, \ldots, M_6$: pick an extender $E_6 \in M_6$, apply it to $M_1$, setting $M_7 = \text{Ult}(M_1, E_6)$ and letting $j_{1,7}: M_1 \to M_7$ be the ultrapower embedding.

At limit $\lambda$: pick a branch through the tree, cofinal in $\lambda$. Set $M_\lambda$ equal to the direct limit of models and embeddings along this branch.

The result is an iteration tree on $M$. 
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$M$ is *iterable* if these choices can be made in a way that secures the wellfoundedness of all the models created.
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Comparisons through iterated ultrapowers show that any two ways to witness $\theta$ are compatible.
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\textbf{Theorem 17 (Neeman)} Let $B_i$ be a recursive enumeration of the $\mathcal{D}^{(n)}(<\omega^2-\Pi^1_1)$ sets. Suppose a sharp for $n$ Woodin cardinals exists. Then all $\mathcal{D}^{(n)}(<\omega^2-\Pi^1_1)$ games are determined, and $\{i \mid I \text{ has a w.s. in } G_{\omega}(B_i)\}$ is recursively isomorphic to the theory of the sharp for $n$ Woodin cardinals.
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These theorems give tight connection between the theory of embeddings acting on models for large cardinals, and determinacy.

The connection (with analogues for $\omega$ Woodin cardinals) is crucial for Theorems [13–15].
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Let $[\vec{S}]$ denote the set

$$\{ \langle \alpha_0, \ldots, \alpha_{k-1} \rangle \in [\omega_1]^{<\omega} \mid (\forall i < k) \alpha_i \in S_{\langle \alpha_0, \ldots, \alpha_{i-1} \rangle} \}.$$
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If there is a club $C \subset \omega_1$ so that $(L_{\omega_1}[r]; r) \models \varphi[\alpha_0, \ldots, \alpha_{k-1}]$ for all $\langle \alpha_0, \ldots, \alpha_{k-1} \rangle \in [\vec{S}] \cap [C]^k$ then player I wins the run $r$. 
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If neither condition holds then both players lose.
Theorem 18 (Neeman) Let $\varphi_i$ be a recursive enumeration of formulae. Suppose that there is a sharp $\pi: M \rightarrow M$ for the statement “crit($\pi$) is a Woodin cardinal.” Then:
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3. The set \( \{i \mid I \text{ has a w.s. in } G_{\omega_1}(\vec{S}, \varphi_i) \} \) is recursively isomorphic to the theory of the sharp for “$\text{crit}(\pi)$ is a Woodin cardinal.”
Theorem 18 (Neeman) Let $\phi_i$ be a recursive enumeration of formulae. Suppose that there is a sharp $\pi: M \rightarrow M$ for the statement “crit($\pi$) is a Woodin cardinal.” Then:

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The theorem establishes a precise analogue of Theorems 16 and 17, but for embeddings concentrating on Woodin cardinals and for games of length $\omega_1$. 
**Question** How high in the large cardinal hierarchy can such tight connections between games and the theories of embeddings be found?
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Games motivated by Theorem 18 were used by Woodin in results on $\Sigma_2^2$ absoluteness. Other games similar to those in the theorem are enough to capture the theory of superstrong cardinals. But there are no determinacy proofs for these games from large cardinals, and indeed there are some negative results (Larson).
The End

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