Determinacy and Large Cardinals

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Use PageDown or the down arrow to scroll through slides.
Press Esc when done.

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$G_\omega(A)$ is *determined* if one of the players has a winning strategy.

(A *strategy* is a complete recipe that instructs the player precisely how to play in each conceivable situation.)

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But determinacy for *definable* sets is: (1) true; and (2) useful.

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{Borel sets} $\subsetneq$ {analytic sets} $\subsetneq$ {projective sets}.

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$\{$projective sets$\} \subset \mathsf{L}_1(\mathbb{R})$.

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Theorems 1 and 2 are in ZFC, the basic system of axioms for set theory.

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Theorems 3, 4, and 5 require large cardinal axioms.

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Every $\mathbf{\Sigma}^1_n$ set A is obtained from an underlying open set using negations and existential quantifiers. A is *(lightface)* Σ^1_n if the underlying open set is recursive. Similarly with Π^1_n.

Γ has the *reduction property* if for any $A, B \in \Gamma$, there are $A' \subset A$, $B' \subset B$ in Γ, so that $A' \cup B' = A \cup B$ and $A' \cap B' = \emptyset$.

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In fact they did more. They obtained a fundamental property, the prewellordering property, which implies reduction.

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The pointclasses in Theorem 10 are therefore precisely the pointclasses $\Game^{(n)}\mathbf{\Pi}^1_1$, $n < \omega$.

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Theorem 12 (Steel–Van Wesep–Woodin) *Assume* $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$. *Then it is consistent (with* $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ *and* AC*) that* $(\omega_2)^{\mathsf{L}(\mathbb{R})} = \omega_2$*, and hence* $\delta^1_2 = \omega_2$*.*

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Similar arguments show κ must be inaccessible, and in fact cannot be described from below in any absolute manner.

So the existence of non-trivial $\pi\colon \mathsf{V} \to M \subset \mathsf{V}$ cannot be proved in ZFC, and the first ordinal moved by π must be very large.

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(Using an ultrapower construction, the measurability of κ is equivalent to the existence of a total, non-principal, countably complete, 2-valued measure on κ.)

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κ is $<\!\delta$–*strong* if it is the critical point of a λ–strong embedding for each $\lambda < \delta$.

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Note: if κ is the first measurable cardinal, then κ is only κ^+–strong.

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δ is a *Woodin cardinal* if for every $D \subset \delta$ there is $\kappa < \delta$ which is $<\delta$–strong wrt D.

Let $\pi\colon \mathrm{V} \to M$. Let $\kappa = \mathrm{crit}(\pi)$ and $\lambda \leq \pi(\kappa)$. The (κ, λ)–*extender* induced by π is the function $E\colon \mathcal{P}(\kappa) \to \mathcal{P}(\lambda)$ defined by $E(X) = \pi(X) \cap \lambda$.

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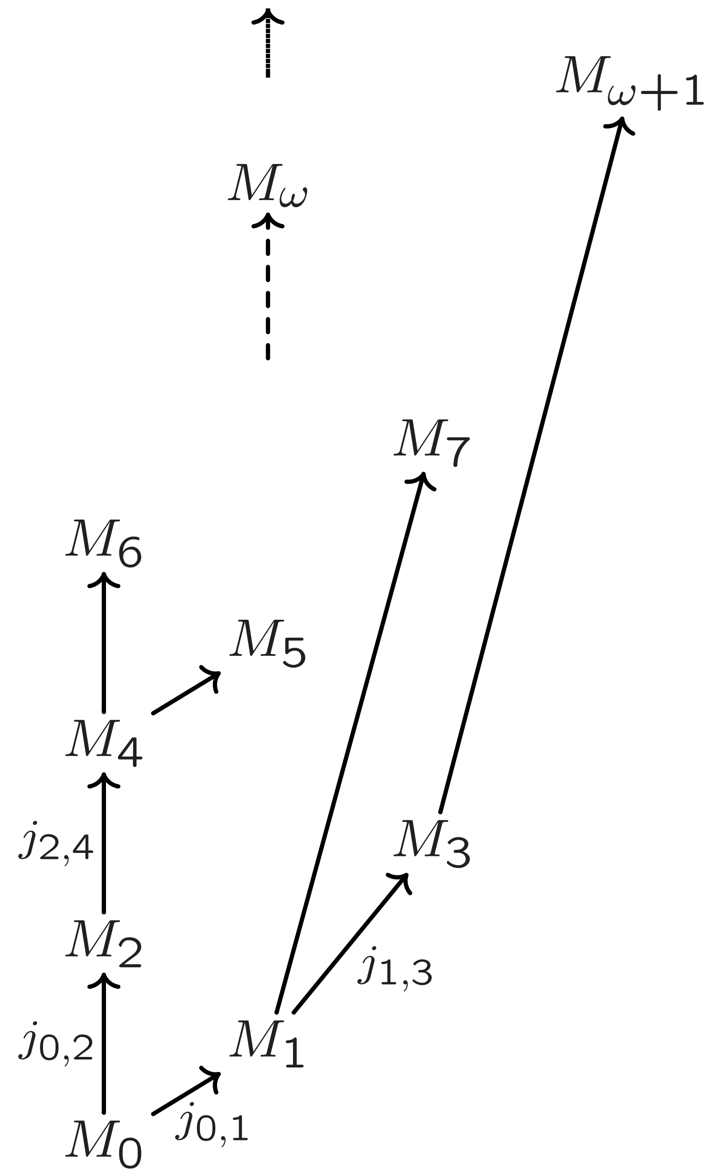
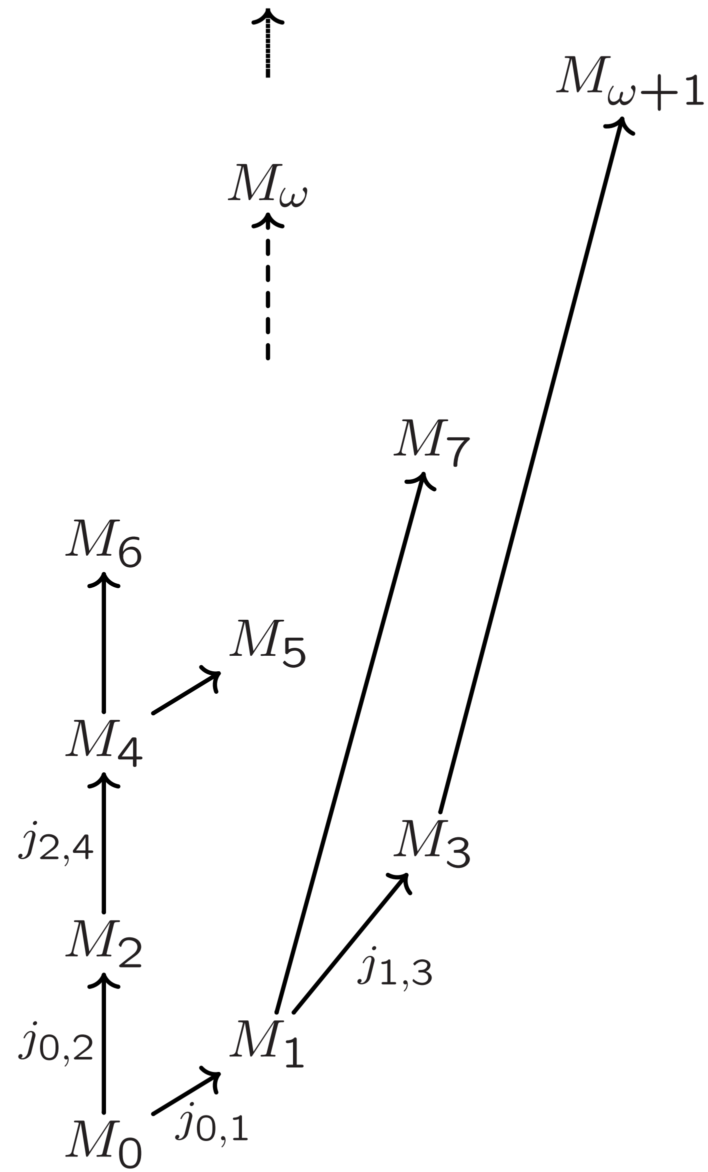
This allows constructing iterated ultrapowers with non-linear base orders.


$M_{\omega+1}$
M_ω
M_7
M_6
M_5
M_4
M_3
$j_{2,4}$
M_2
$j_{1,3}$
M_1
$j_{0,2}$
M_0
$j_{0,1}$

Constructed in stages, starting from a base model $M = M_0$.



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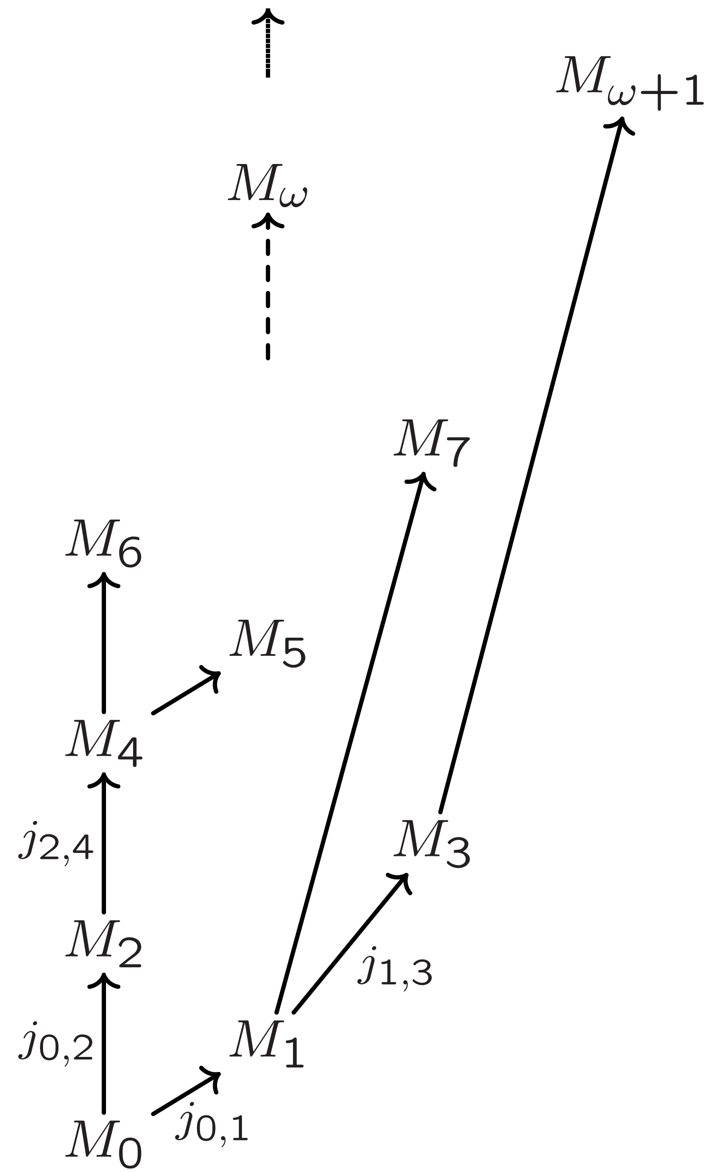

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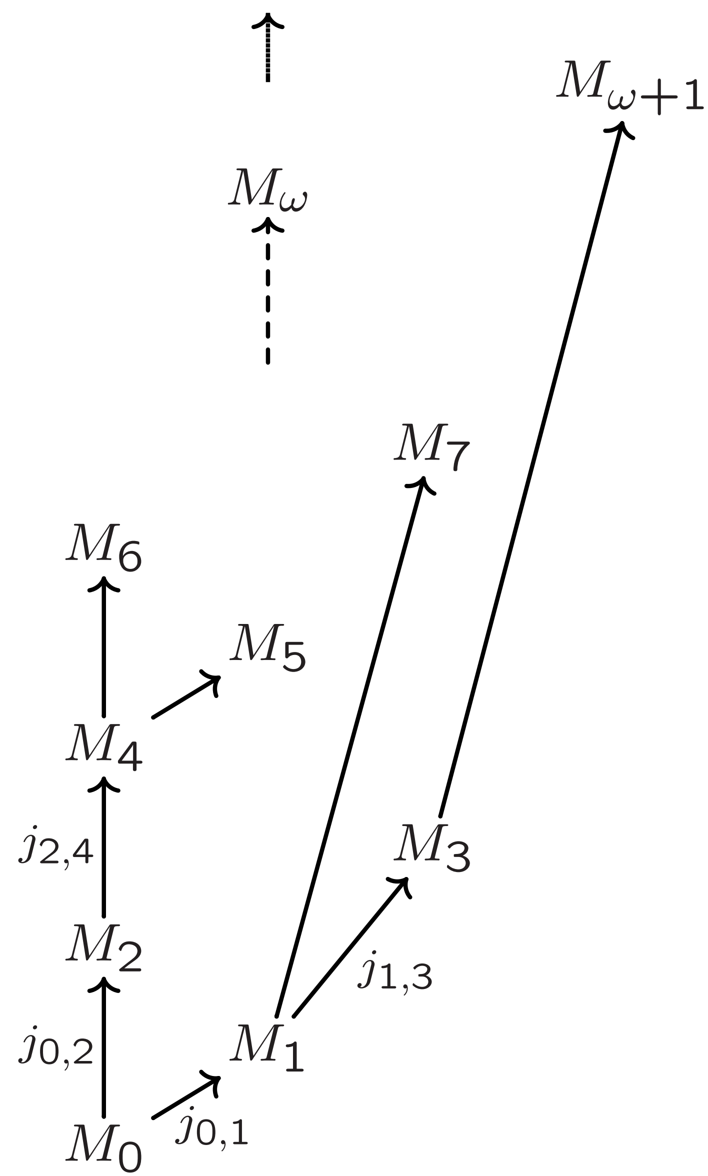

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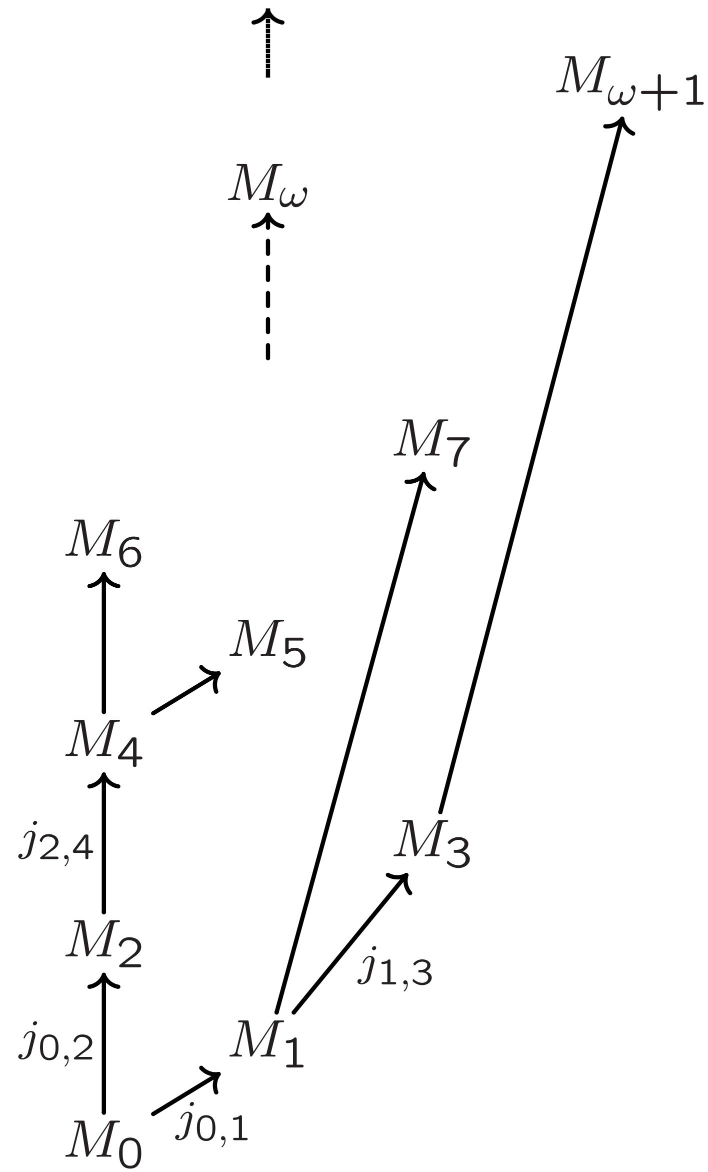

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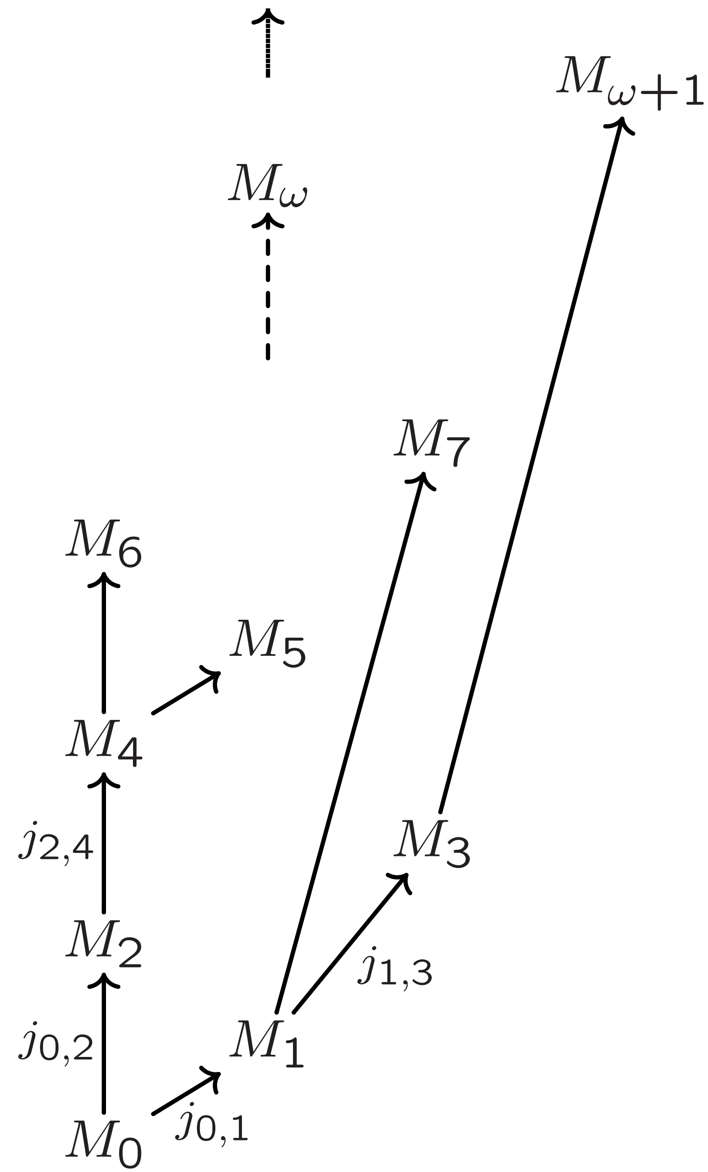

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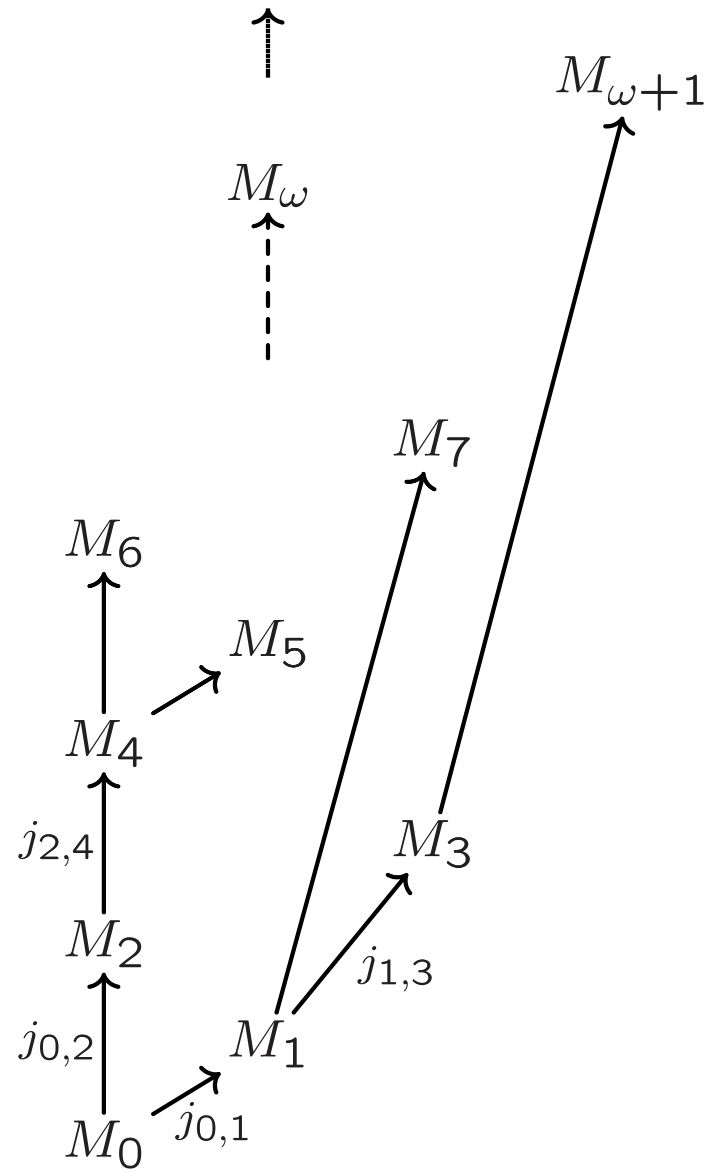

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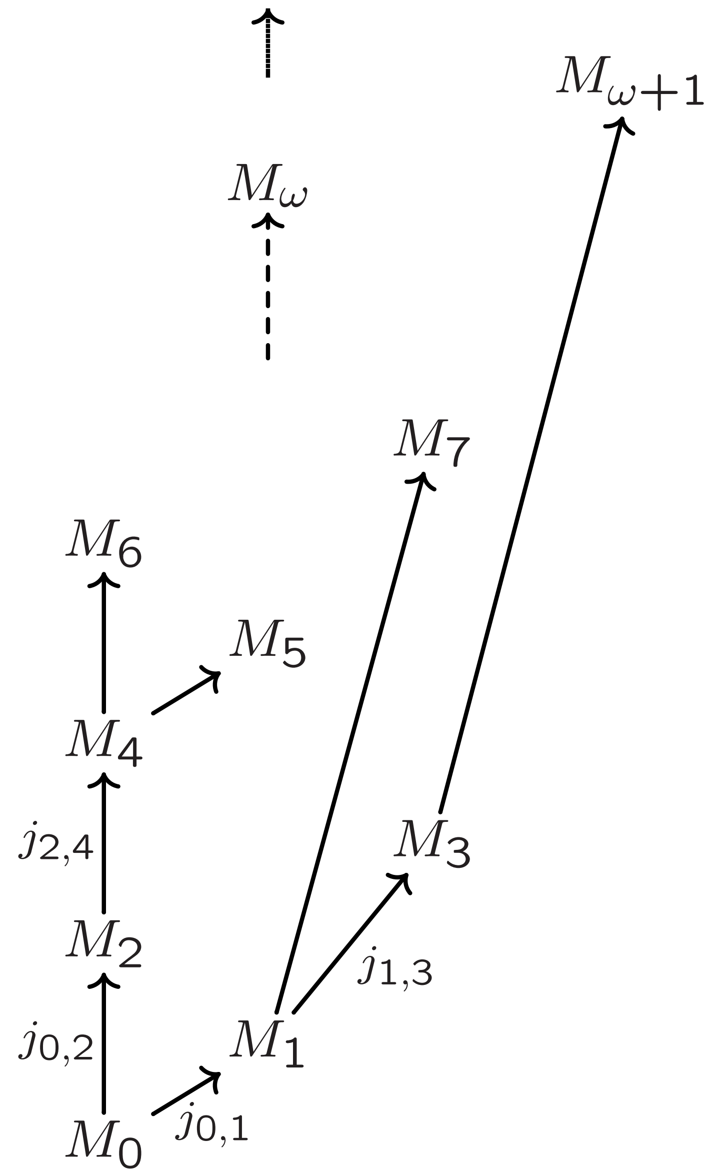

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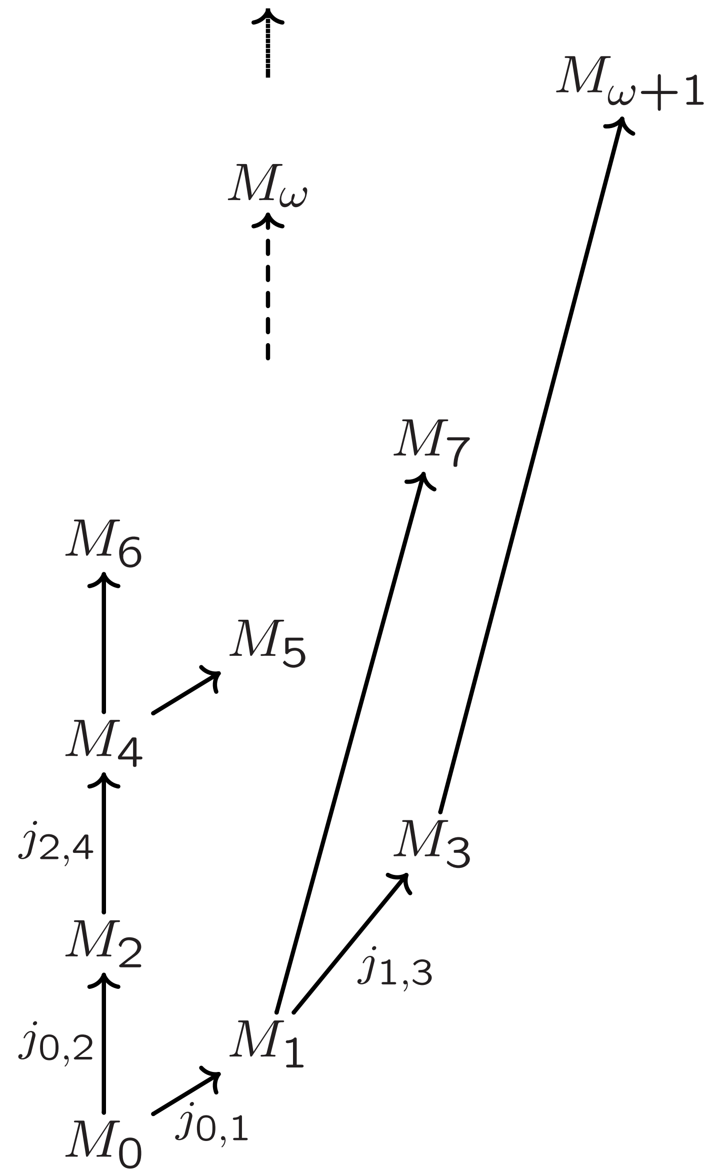

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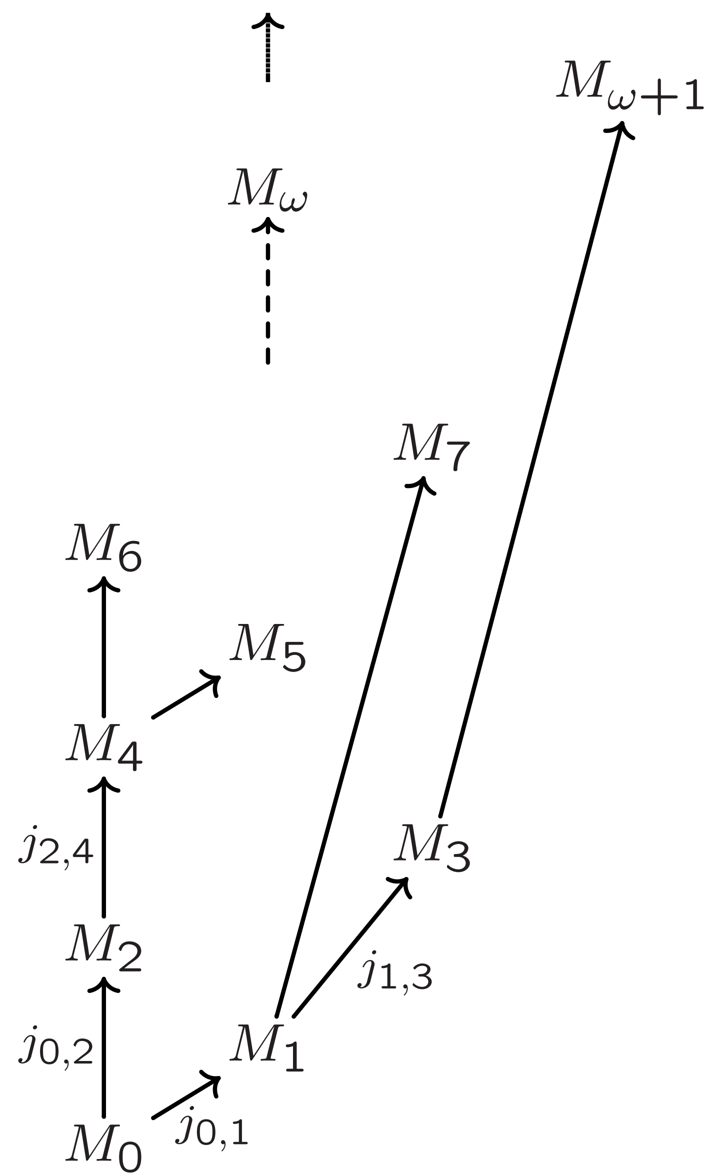
The result is an *iteration tree* on M.



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In both cases Woodin cardinals in iterable inner models (rather than the actual universe V) are enough, and moreover *necessary*.

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But some make direct use of inner models for Woodin cardinals.

A set A is α–$\mathbf{\Pi}^1_1$ if there is a sequence $\langle A_\xi \mid \xi < \alpha\rangle$ of $\mathbf{\Pi}^1_1$ sets so that $x \in A$ iff the least ξ so that $x \notin A_\xi \vee \xi = \alpha$ is odd.

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(The hierarchy generated by this definition is the *difference hierarchy* on $\mathbf{\Pi}^1_1$ sets. If $\alpha = 2$ for example, then the condition states simply that $A = A_0 - A_1$.)

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Theorem known previously for odd n, not using large cardinals (Kechris–Woodin).

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Theorem 15 (Neeman, Woodin) *Assume* $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$. *Then it is consistent (with* $\mathsf{AD}^{\mathsf{L}(\mathbb{R})}$ *and the axiom of choice) that* $\boldsymbol{\delta}^1_3 = \omega_2$.

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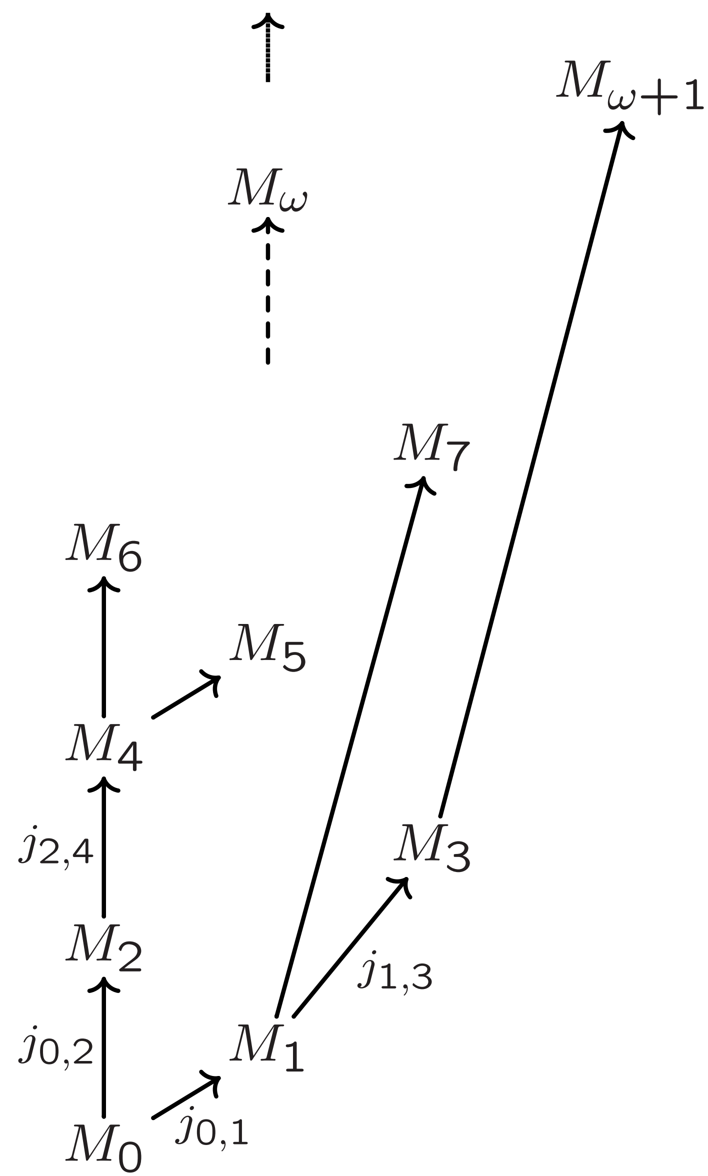

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M is *iterable* if these choices can be made in a way that secures the wellfoundedness of all the models created.

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Comparisons through iterated ultrapowers show that any two ways to witness θ are compatible.

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Theorem 17 (Neeman) *Let B_i be a recursive enumeration of the $\Game^{(n)}(<\omega^2$–$\Pi^1_1)$ sets. Suppose a sharp for n Woodin cardinals exists. Then all $\Game^{(n)}(<\omega^2$–$\Pi^1_1)$ games are determined, and $\{i \mid I$ has a w.s. in $G_\omega(B_i)\}$ is recursively isomorphic to the theory of the sharp for n Woodin cardinals.*

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The connection (with analogues for ω Woodin cardinals) is crucial for Theorems 13–15.

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Let $[\vec{S}]$ denote the set

$$\{\langle \alpha_0, \ldots, \alpha_{k-1} \rangle \in [\omega_1]^{<\omega} \mid (\forall i < k)\ \alpha_i \in S_{\langle \alpha_0, \ldots, \alpha_{i-1} \rangle}\}.$$

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If neither condition holds then both players lose.

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The theorem establishes a precise analogue of Theorems 16 and 17, but for embeddings concentrating on Woodin cardinals and for games of length ω_1.

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Games motivated by Theorem 18 were used by Woodin in results on Σ^2_2 absoluteness. Other games similar to those in the theorem are enough to capture the theory of superstrong cardinals. But there are no determinacy proofs for these games from large cardinals, and indeed there are some negative results (Larson).

The End

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