Ramsey Cardinals and the HNN Embedding Theorem

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The HNN Embedding Theorem

Theorem (Higman-Neumann-Neumann 1949)

Every countable group G is embeddable in a 2-generator group K_G.

Remark

In the standard proofs, the construction of the group K_G involves an enumeration of a set $\{g_n \mid n \in \mathbb{N}\}$ of generators of the group G; and it is clear that the isomorphism type of K_G depends upon both the generating set and the particular enumeration that is used.

Question

Does there exist a more uniform construction with the property that the isomorphism type of K_G only depends upon the isomorphism type of G?

The answer ...

Notation

- $\mathcal{G}$ denotes the Polish space of countably infinite groups.
- $\mathcal{G}_{fg}$ denotes the Polish space of finitely generated groups.

Main Theorem (*LC*)

- *Suppose that* $G \mapsto K_G$ *is any Borel map from* $\mathcal{G}$ *to* $\mathcal{G}_{fg}$ *such that* $G \hookrightarrow K_G$ *for all* $G \in \mathcal{G}$.
- *Then there exists an uncountable Borel family* $\mathcal{F} \subseteq \mathcal{G}$ *of pairwise isomorphic groups such that the groups* $\{ K_G \mid G \in \mathcal{F} \}$ *are pairwise incomparable with respect to relative constructibility; i.e., if* $G \neq H \in \mathcal{F}$*, then* $K_G \notin L[K_H]$ *and* $K_H \notin L[K_G]$.

Remark

(*LC*): There exists a Ramsey cardinal κ.

The answer ...

Futher Remarks

- (Philip Welch) Enough to assume that $\omega_1^{L[r]} < \omega_1$ for all $r \in 2^{\mathbb{N}}$.
- In *ZFC*, we can find an uncountable Borel family $\mathcal{F}$ such that the groups $\{ K_G \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to embeddability ... or any other countable Borel quasi-order.
- For example, $\{ \text{Word}(K_G) \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to Turing reducibility.
- (Philip Welch) Or even $\{ \text{Word}(K_G) \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to hyperarithmetic reducibility.

Towards a proof of the Main Theorem ...

Definition

- $\text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ *is the Polish space of all injective maps* $z : \mathbb{N} \to 2^{\mathbb{N}}$.
- E_{cntble} *is the Borel equivalence relation on* $\text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ *defined by*

$$z \, E_{cntble} \, z' \iff \{ z(n) \mid n \in \mathbb{N} \} = \{ z'(n) \mid n \in \mathbb{N} \}.$$

Theorem

If E is any countable Borel equivalence relation, then $E \leq_B E_{cntble}$.

Proof.

An easy consequence of the Feldman-Moore Theorem. □

The Main Lemma

Main Lemma

Suppose that X is a Polish space and that $\theta : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to X$ is any Borel map. Then at least one of the following must hold:

(a) *There exists $x \in X$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \mathrm{range}(z)$ such that $\theta(z) = x$.*

(b) *For each countable Borel quasi-order $\preccurlyeq$ on X, there exists a perfect subset $P \subseteq \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that*

   (i) *$y \; E_{cntble} \; z$ for all $y, z \in P$; and*

   (ii) *$\theta(y), \theta(z)$ are incomparable with respect to $\preccurlyeq$ for all $y \neq z \in P$.*

Moreover, if (LC) holds, then the conclusion also holds with respect to the quasi-order $\leq_c$ of relative constructibility.

The Proof of the Main Theorem

- Suppose that $\varphi : \mathcal{G} \to \mathcal{G}_{fg}$ is a Borel map such that $G \hookrightarrow \varphi(G)$ for all $G \in \mathcal{G}$.
- Let $\{ H_r \mid r \in 2^{\mathbb{N}} \} \subseteq \mathcal{G}$ be a Borel family of pairwise nonisomorphic 2-generator groups.
- Let $\psi : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to \mathcal{G}$ be the injective Borel map defined by

$$\psi(z) = H_{z(0)} \times H_{z(1)} \times \cdots \times H_{z(n)} \times \cdots$$

  and consider $\theta = \varphi \circ \psi : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to \mathcal{G}_{fg}$.
- First suppose that there exists a group $G \in \mathcal{G}_{fg}$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that $r \in \mathrm{range}(z)$ and $\theta(z) = G$.
- Then H_r embeds into G for all $r \in 2^{\mathbb{N}}$, which is impossible since G has only countably many 2-generator subgroups!

The Proof of the Main Theorem

- Let $\preceq$ be either a countable Borel quasi-order or the relative constructibility relation on $\mathcal{G}_{fg}$.
- Then there exists a perfect subset $P \subseteq \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that
  - (i) $y \; E_{cntble} \; z$ for all $y, z \in P$; and
  - (ii) $\theta(y), \theta(z)$ are incomparable with respect to $\preccurlyeq$ for all $y \neq z \in P$.
- Hence $\mathcal{F} = \psi(P) \subseteq \mathcal{G}$ is an uncountable Borel family of pairwise isomorphic groups such that the groups $\{ \varphi(G) \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to $\preceq$.

Towards a proof of the Main Lemma ...

Notation

From now on, we work within a fixed set-theoretic universe V.

Definition

Suppose that R is a projective relation and $\mathbb{P}$ is a forcing notion.

- *$R^{V^{\mathbb{P}}}$ denotes the relation obtained by applying the definition of R within the generic extension $V^{\mathbb{P}}$.*
- *R is absolute for $V^{\mathbb{P}}$ iff $R^{V^{\mathbb{P}}} \cap V = R$.*

The Main Ingredients

- The Shoenfield and Martin-Solovay Absoluteness Theorems.
- Kanovei's notion of a virtual equivalence class.

Absoluteness

Theorem (Shoenfield)

If $R \in V$ is a $\mathbf{\Sigma}^1_2$ relation, then R is absolute for every generic extension $V^{\mathbb{P}}$.

An Application

If $\preceq$ is a countable Borel quasi-order on the Polish space X, then $\preceq^{V^{\mathbb{P}}}$ is a countable Borel quasi-order on $X^{V^{\mathbb{P}}}$.

Theorem (Martin-Solovay)

Suppose that κ is a Ramsey cardinal. If $R \in V$ is a $\mathbf{\Sigma}^1_3$ relation and $|\mathbb{P}| < \kappa$, then R is absolute for $V^{\mathbb{P}}$.

An Application (LC)

$\leq_c$ is a countable $\mathbf{\Sigma}^1_2$ quasi-order on $2^{\mathbb{N}}$.

Virtual equivalence classes

Definition (Kanovei après Hjorth)

Let E be a Borel equivalence relation on X and let $\mathbb{P}$ be a forcing notion. Then a $\mathbb{P}$-name τ is a virtual E-class if:

- $\Vdash_{\mathbb{P}} \quad \tau \in X^{V^{\mathbb{P}}}$
- $\Vdash_{\mathbb{P}\times\mathbb{P}} \quad \tau_{left} \; E^{V^{\mathbb{P}\times\mathbb{P}}} \; \tau_{right}$

Here τ_{left}, τ_{right} are the $(\mathbb{P}\times\mathbb{P})$-names such that if $G \times H$ is $(\mathbb{P}\times\mathbb{P})$-generic, then $\tau_{left}[G \times H] = \tau[G]$ and $\tau_{right}[G \times H] = \tau[H]$.

Virtual equivalence classes

Example

- Let $E = E_{cntble}$ and let $\mathbb{P}$ consist of all finite injective partial functions $p : \mathbb{N} \to 2^{\mathbb{N}}$.
- If G is $\mathbb{P}$-generic, then $g = \bigcup G$ is a bijection between $\mathbb{N}$ and $2^{\mathbb{N}} \cap V$.
- Hence if τ is the canonical $\mathbb{P}$-name such that $\tau[G] = g$, then τ is a virtual E_{cntble}-class.

A reminder ...

Main Lemma

Suppose that X is a Polish space and that $\theta : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to X$ is any Borel map. Then at least one of the following must hold:

(a) *There exists $x \in X$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \mathrm{range}(z)$ such that $\theta(z) = x$.*

(b) *For each countable Borel quasi-order $\preccurlyeq$ on X, there exists a perfect subset $P \subseteq \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that*
   (i) *$y \; E_{cntble} \; z$ for all $y, z \in P$; and*
   (ii) *$\theta(y), \theta(z)$ are incomparable with respect to $\preccurlyeq$ for all $y \neq z \in P$.*

Moreover, if (LC) holds, then the conclusion also holds with respect to the quasi-order $\leq_c$ of relative constructibility.

Towards a proof of the Main Lemma ...

- Let $\theta : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to X$ be any Borel map.
- Let $\preceq$ be either a countable Borel quasi-order on X or else the relative constructibility relation $\leq_c$.

Notation

- $x \perp y \iff x, y$ are $\preceq$-incomparable.
- $x \,||\, y \iff x, y$ are $\preceq$-comparable.

- Let $\mathbb{P}$ consist of all finite injective partial functions $p : \mathbb{N} \to 2^{\mathbb{N}}$ and let τ be the corresponding virtual E_{cntble}-class.

The Fundamental Dichotomy

Are $\theta(\tau_{left}), \theta(\tau_{right})$ comparable with respect to $\preceq^{V^{\mathbb{P}\times\mathbb{P}}}$?

Case 1: $(\exists\, p_0 \in \mathbb{P})\ \langle\, p_0, p_0\, \rangle \Vdash \theta(\tau_{\text{left}}) \,||\, \theta(\tau_{\text{right}})$.

Claim

There exists $p_1 \leq p_0$ such that $\langle\, p_1, p_1\, \rangle \Vdash \theta(\tau_{\text{left}}) = \theta(\tau_{\text{right}})$.

Proof.

- Suppose not and let $\mathbb{Q}$ collapse $\mathcal{P}(\mathbb{P} \times \mathbb{P})$ to a countable set.
- Working in $V^{\mathbb{Q}}$, there exists a perfect subset $P \subseteq \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that $\theta(P)$ is an uncountable Borel set of pairwise $\preceq$-comparable elements.
- Let $Z \subseteq \theta(P)$ be a perfect subset.
- By Kuratowski-Ulam, both $A = \{\, (x, y) \in Z \times Z \mid x \preceq y\, \}$ and $B = \{\, (x, y) \in Z \times Z \mid y \preceq x\, \}$ are meager subsets of $Z \times Z$.
- Since $Z \times Z = A \cup B$, this contradicts the Baire Category Theorem.

$\square$

Case 1: $(\exists p_0 \in \mathbb{P}) \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \mid\mid \theta(\tau_{\text{right}})$.

Working in V and assuming that $X = [0, 1]$, we can inductively define conditions

$$p_1 \geq p_2 \geq p_3 \geq \cdots \geq p_n \geq \cdots$$

and closed intervals $I_n \subseteq [0, 1]$ with rational endpoints

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

such that the following conditions hold:

- $|I_n| = 2^{-(n-1)}$
- $p_n \Vdash \theta(\tau) \in I_n$.

Still working in V, let

$$\bigcap_{n \geq 1} I_n = \{x\}.$$

Case 1: $(\exists p_0 \in \mathbb{P}) \; \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \,||\, \theta(\tau_{\text{right}})$.

Claim

$p_1 \Vdash \theta(\tau) = x$.

Proof.

- Otherwise, there exists $q \leq p_1$ and $n \geq 1$ such that $q \Vdash \theta(\tau) \notin I_n$.
- But then $\langle q, p_n \rangle \leq \langle p_1, p_1 \rangle$ satisfies

$$\langle q, p_n \rangle \Vdash \theta(\tau_{\text{left}}) \notin I_n \quad \text{and} \quad \theta(\tau_{\text{right}}) \in I_n,$$

which is a contradiction.

□

Case 1: $(\exists p_0 \in \mathbb{P}) \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \,||\, \theta(\tau_{\text{right}})$.

- Let $G \subseteq \mathbb{P}$ be V-generic with $p_1 \in G$.
- Then $V[G] \vDash \theta(\tau[G]) = x$.
- Hence for each $r \in 2^{\mathbb{N}} \cap V$,

$$V[G] \vDash (\exists z \in \text{Inj}(\mathbb{N}, 2^{\mathbb{N}}))(\exists n \in \mathbb{N})\, [z(n) = r \text{ and } \theta(z) = x].$$

- By Shoenfield Absoluteness, this $\mathbf{\Sigma}^1_1$ property of the reals $r, x \in 2^{\mathbb{N}} \cap V$ must also hold in V.
- Thus, in V, for all $r \in 2^{\mathbb{N}}$, there exists $z \in \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \text{range}(z)$ such that $\theta(z) = x$.

Case 2: $(\forall p \in \mathbb{P})\ \langle p, p \rangle \not\Vdash \theta(\tau_{\text{left}}) \mid\mid \theta(\tau_{\text{right}})$.

- Once again, let $\mathbb{Q}$ collapse $\mathcal{P}(\mathbb{P} \times \mathbb{P})$ to a countable set.
- Then $V^{\mathbb{Q}}$ satisfies the following statement:

$$(\exists P \in \text{Perf}(\text{Inj}(\mathbb{N}, 2^{\mathbb{N}})))\,(\forall x)\,(\forall y)$$
$$[\,(x, y \in P \wedge x \neq y) \implies (x\, E_{cntble}\, y \wedge \theta(x) \perp \theta(y))\,].$$

- Applying either Shoenfield or Martin-Solovay Absoluteness, this statement also holds in V.
- This completes the proof of the Main Lemma.

The word problem for finitely generated groups

Theorem (Folklore)

For each subset $A \subseteq \mathbb{N}$, there exists a finitely generated group G_A such that $\mathrm{Word}(G_A) \equiv_T A$.

Theorem

- *Suppose that $A \mapsto G_A$ is a Borel map from $2^{\mathbb{N}}$ to $\mathcal{G}_{fg}$ such that $\mathrm{Word}(G_A) \equiv_T A$ for all $A \in 2^{\mathbb{N}}$.*
- *Then there exists a Turing degree $\mathbf{d}_0$ such that for all $\mathbf{d} \geq_T \mathbf{d}_0$, there exists an infinite subset $\{ A_n \mid n \in \mathbb{N} \} \subseteq \mathbf{d}$ such that the groups $\{ G_{A_n} \mid n \in \mathbb{N} \}$ are pairwise incomparable with respect to embeddability.*

Sketch Proof.

A very easy consequence of Borel Determinacy. □

Some Open Questions

Theorem

There does not exist a Borel choice of generators for each f.g. group which has the property that isomorphic groups are assigned isomorphic Cayley graphs.

Problem

Formulate and prove a corresponding "gregification".

Theorem (Folklore)

Every finitely generated group G has a just infinite quotient Q_G.

Conjecture

There does not exist a Borel choice such that the isomorphism type of Q_G only depends on the isomorphism type of G.

Some Open Questions

Theorem

There does not exist a Borel choice of generators for each f.g. group which has the property that isomorphic groups are assigned isomorphic Cayley graphs.

Problem

Formulate and prove a corresponding "gregification".

Theorem (Folklore)

Every finitely generated group G has a just infinite quotient Q_G.

Remark

It is enough to show that the isomorphism relation on simple finitely generated groups isn't smooth.