Introduction

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(Magidor - Shelah, 1996) Suppose there is a model with a huge cardinal and $\omega$ many supercompact cardinals above it. Then there is a model with the tree property at $\aleph_{\omega+1}$. 

We reduce the large cardinal hypothesis to $\omega$ many supercompact cardinals. Our construction is motivated by the Prikry type forcing in Gitik-Sharon (2008) and arguments in Neeman (2009).
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Our construction is motivated by the Prikry type forcing in Gitik-Sharon (2008) and arguments in Neeman (2009).
The Main Theorem

Theorem
(S) Suppose that in $V$, $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals and GCH holds. Then there is a generic extension in which:

1. $\kappa_0 = \aleph_\omega$,
2. the tree property holds at $\aleph_{\omega + 1}$.

Furthermore, there is a bad scale at $\kappa_0$. 
In $V$, $\langle \kappa_n \mid n < \omega \rangle$ are increasing supercompact cardinals, $\kappa_0 = \kappa$ indestructably supercompact.
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Force with $C$ to make each $\kappa_n$ be the $n$-th successor of $\kappa$. Let $H$ be $C$-generic over $V$. 
In $V$, $\langle \kappa_n \mid n < \omega \rangle$ are increasing supercompact cardinals, $\kappa_0 = \kappa$ indestructably supercompact.

Force with $\mathbb{C}$ to make each $\kappa_n$ be the $n$-th successor of $\kappa$. Let $H$ be $\mathbb{C}$-generic over $V$.

In $V[H]$, we have:

$\langle U_n \mid n < \omega \rangle$ are supercompactness measures on $\mathcal{P}_\kappa(\kappa^{+n})$.
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- $\langle K_n \mid n < \omega \rangle$, such that $K_0$ is $Ult_{U_0}$-generic for $Col(\kappa^{+\omega+2}, < j_{U_0}(\kappa))$ and for $n > 0$, $K_n$ is $Ult_{U_n}$-generic for $Col(\kappa^{+n+2}, < j_{U_n}(\kappa))$. 

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The Main Forcing

Conditions in $\mathbb{P}$ are of the form $p = \langle d, \langle p_n \mid n < \omega \rangle \rangle$, where setting $l = lh(p)$, we have:

1. For $0 \leq n < l$, $p_n = \langle x_n, c_n \rangle$ such that:
   - $x_n \in \mathcal{P}\kappa(\kappa + n)$ and for $i < n$, $x_i \prec x_n$,
   - $c_0 \in \text{Col}(\kappa + \omega + 2 x_0, <\kappa x_1)$ if $1 < l$,
   - if $1 < l$, for $0 < n < l - 1$, $c_n \in \text{Col}(\kappa + n + 2 x_n, <\kappa x_{n+1})$, and $c_{l-1} \in \text{Col}(\kappa + l + 1 x_{l-1}, <\kappa)$.

2. For $n \geq l$, $p_n = \langle A_n, C_n \rangle$ such that:
   - $A_n \in U_n$, $A_n \subset X_n$, and $x_{l-1} \prec y$ for all $y \in A_n$.
   - $[C_n] U_n \in K_n$.

3. If $l > 0$, then $d \in \text{Col}(\omega, \kappa + \omega x_0)$, otherwise $d \in \text{Col}(\omega, \kappa)$.
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   - if $1 < l$, for $0 < n < l - 1$, $c_n \in Col(\kappa^{x_n + n + 2}, < \kappa_{x_n+1})$, and
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3. if $l > 0$, then $d \in Col(\omega, \kappa_{\chi_0}^{+\omega})$, otherwise $d \in Col(\omega, \kappa)$. 
Properties of the forcing

Let $G$ be $\mathbb{P}$-generic over $V[H]$

1. $G$ determines a generic sequence $\langle x_n \mid n < \omega \rangle$, such that $\bigcup_n x_n = (\kappa + \omega)^{V[H]}$. 

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4. $\mathbb{P}$ has the Prikry property.

In particular, in $V[H][G]$, $\mu$ is the successor of $\kappa$, and $\mu = \kappa^+ \omega_1$. 
The preservation theorem

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- instead of trees, here we work with narrow systems

\[ S = \langle I, \mathcal{R} \rangle \] is a **narrow system** of height \( \nu^+ \) and levels of size \( \kappa < \nu \) if:

\begin{itemize}
  
  \item \( I \subset \nu^+ \) is unbounded; for \( \alpha \in I \), \( S_\alpha = \{ \alpha \} \times \kappa \) is the \( \alpha \)-level of \( S \),
  
  \item \( \mathcal{R} \) is a set of transitive binary relations on \( S \), \( |\mathcal{R}| < \nu \),
  
  \item for \( \alpha < \beta \) in \( I \), there are \( u \in S_\alpha, v \in S_\beta, R \in \mathcal{R} \), s.t. \( \langle u, v \rangle \in R \),
  
  \item for \( R \in \mathcal{R} \), if \( u_1, u_2 \) are distinct, \( \langle u_1, v \rangle \in R, \langle u_2, v \rangle \in R \), then \( \langle u_1, u_2 \rangle \in R \) or \( \langle u_2, u_1 \rangle \in R \).
\end{itemize}

A branch of \( S \) is a set \( b \subset \bigcup \alpha \in I S_\alpha \) s.t. for every \( \alpha \), \( |b \cap S_\alpha| \leq 1 \), and for some \( R \in \mathcal{R} \), for all \( u, v \in b \), \( \langle u, v \rangle \in R \) or \( \langle v, u \rangle \in R \); \( b \) is unbounded if for unboundedly many \( \alpha \in I \), \( b \cap S_\alpha \neq \emptyset \).
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Theorem

(S) Suppose that $\text{cof}(\nu) = \omega$ and $S = \langle I, \mathcal{R} \rangle$ is a narrow system in $V$ of height $\nu^+$, levels of size $\kappa$, $|\mathcal{R}| = \tau$, where $\kappa, \tau < \nu$. Suppose also that $\mathbb{R}$ is a $<\chi$ closed notion of forcing where $\chi > \max(\kappa, \tau)^{+}$, and let $F$ be $\mathbb{R}$-generic over $V$. Suppose that in $V[F]$ there are (not necessarily all unbounded) branches $\langle b_{R,\delta} \mid R \in \mathcal{R}, \delta < \kappa \rangle$, such that:

1. every $b_{R,\delta}$ is a branch through $R$,
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Finally suppose that for some $\langle R, \delta \rangle \in \mathcal{R} \times \kappa$, $b_{R,\delta}$ is unbounded. Then $S$ has an unbounded branch in $V$. 

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Theorem

(S) Suppose that $\text{cof}(\nu) = \omega$ and $S = \langle I, \mathcal{R} \rangle$ is a narrow system in $V$ of height $\nu^+$, levels of size $\kappa$, $|\mathcal{R}| = \tau$, where $\kappa, \tau < \nu$. Suppose also that $\mathcal{R}$ is a $< \chi$ closed notion of forcing where $\chi > \max(\kappa, \tau)^+$, and let $F$ be $\mathcal{R}$-generic over $V$. Suppose that in $V[F]$ there are (not necessarily all unbounded) branches $\langle b_{R,\delta} | R \in \mathcal{R}, \delta < \kappa \rangle$, such that:

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In $V[H][G]$, the tree property holds at $\aleph_{\omega+1}$. The proof is motivated by Neeman. The main difference is that we have to deal with the poset $C$ and rely on the Preservation Theorem.
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Connection with the SCH

**Definition**
The Singular Cardinal Hypothesis (SCH) states that if $\kappa$ is singular and $2^{\text{cf}(\kappa)} < \kappa$, then $\kappa^{\text{cf}(\kappa)} = \kappa^+$. 

**Theorem** (Magidor) If there exists a supercompact cardinal, then there is a forcing extension in which $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_\omega + 2$.

Gitik and Woodin significantly reduced the large cardinal hypothesis to a measurable cardinal $\kappa$ of Mitchell order $\kappa^{++}$. This hypothesis was shown to be optimal by Gitik and Mitchell using core model theory.
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**Theorem**

*(Neeman, 2009)* The tree property at $\kappa^+$ is consistent with the failure of SCH at $\kappa$. 
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**Question**
Can Neeman’s result be obtained for $\kappa = \aleph_\omega$, or even $\aleph_{\omega_2}$?
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The strategy in the proof our theorem suggests some hope of answering the above question in the positive.