The Tree property for small cardinals

Dima Sinapova
UCI

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- The tree property at κ^+ states that every κ^+-tree has an unbounded branch.
- (Magidor - Shelah, 1996) Suppose there is a model with a huge cardinal and ω many supercompact cardinals above it. Then there is a model with the tree property at $\aleph_{\omega+1}$.
- We reduce the large cardinal hypothesis to ω many supercompact cardinals.
- Our construction is motivated by the Prikry type forcing in Gitik-Sharon (2008) and arguments in Neeman (2009).

The Main Theorem

Theorem

(S) Suppose that in V, $\langle \kappa_n \mid n < \omega \rangle$ is an increasing sequence of supercompact cardinals and GCH holds. Then there is a generic extension in which:

1. $\kappa_0 = \aleph_\omega$,
2. *the tree property holds at* $\aleph_{\omega+1}$.

Furthermore, there is a bad scale at κ_0.

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- In $V[H]$, we have:
    - $\langle U_n \mid n < \omega \rangle$ are supercompactness measures on $\mathcal{P}_\kappa(\kappa^{+n})$
    - $\langle K_n \mid n < \omega \rangle$, such that K_0 is Ult_{U_0}-generic for $Col(\kappa^{+\omega+2}, < j_{U_0}(\kappa))$ and for $n > 0$, K_n is Ult_{U_n}-generic for $Col(\kappa^{+n+2}, < j_{U_n}(\kappa))$.

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   - $x_n \in \mathcal{P}_\kappa(\kappa^{+n})$ and for $i < n$, $x_i \prec x_n$,
   - $c_0 \in Col(\kappa_{x_0}^{+\omega+2}, < \kappa_{x_1})$ if $1 < l$, and if $l = 1$, $c_0 \in Col(\kappa_{x_0}^{+\omega+2}, < \kappa)$.
   - if $1 < l$, for $0 < n < l - 1$, $c_n \in Col(\kappa_{x_n}^{+n+2}, < \kappa_{x_{n+1}})$, and $c_{l-1} \in Col(\kappa_{x_{l-1}}^{+l+1}, < \kappa)$.

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2. For $n \geq l$, $p_n = \langle A_n, C_n \rangle$ such that:
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3. if $l > 0$, then $d \in Col(\omega, \kappa_{x_0}^{+\omega})$, otherwise $d \in Col(\omega, \kappa)$.

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In particular, in $V[H][G]$, μ is the successor of κ, and $\mu = \aleph_{\omega+1}$.

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- for $R \in \mathcal{R}$, if u_1, u_2 are distinct, $\langle u_1, v \rangle \in R$, $\langle u_2, v \rangle \in R$, then $\langle u_1, u_2 \rangle \in R$ or $\langle u_2, u_1 \rangle \in R$.

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A **branch** of S is a set $b \subset \bigcup_{\alpha \in I} S_\alpha$ s. t. for every α, $|b \cap S_\alpha| \leq 1$, and for some $R \in \mathcal{R}$, for all $u, v \in b$, $\langle u, v \rangle \in R$ or $\langle v, u \rangle \in R$; b is unbounded if for unboundedly many $\alpha \in I$, $b \cap S_\alpha \neq \emptyset$.

The preservation theorem

Theorem

(S) Suppose that cof$(\nu) = \omega$ and $S = \langle I, \mathcal{R} \rangle$ is a narrow system in V of height ν^+, levels of size κ, $|\mathcal{R}| = \tau$, where $\kappa, \tau < \nu$. Suppose also that $\mathbb{R}$ is a $<\chi$ closed notion of forcing where $\chi > \max(\kappa, \tau)^+$, and let F be $\mathbb{R}$-generic over V. Suppose that in $V[F]$ there are (not necessarily all unbounded) branches $\langle b_{R,\delta} \mid R \in \mathcal{R},\ \delta < \kappa \rangle$, such that:

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1. *every* $b_{R,\delta}$ *is a branch through* R,
2. *for all* $\alpha \in I$, *there is* $\langle R, \delta \rangle \in \mathcal{R} \times \kappa$, *such that* $S_\alpha \cap b_{R,\delta} \neq \emptyset$.

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(S) Suppose that $cof(\nu) = \omega$ *and* $S = \langle I, \mathcal{R} \rangle$ *is a narrow system in* V *of height* ν^+*, levels of size* κ*,* $|\mathcal{R}| = \tau$*, where* $\kappa, \tau < \nu$*. Suppose also that* $\mathbb{R}$ *is a* $<\chi$ *closed notion of forcing where* $\chi > \max(\kappa, \tau)^+$*, and let* F *be* $\mathbb{R}$*-generic over* V*. Suppose that in* $V[F]$ *there are (not necessarily all unbounded) branches* $\langle b_{R,\delta} \mid R \in \mathcal{R},\ \delta < \kappa \rangle$*, such that:*

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Finally suppose that for some $\langle R, \delta \rangle \in \mathcal{R} \times \kappa$*,* $b_{R,\delta}$ *is unbounded.*

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Finally suppose that for some $\langle R, \delta \rangle \in \mathcal{R} \times \kappa$, $b_{R,\delta}$ *is unbounded. Then* S *has an unbounded branch in* V.

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- The proof is motivated by Neeman
- The main difference is that we have to deal with the poset $\mathbb{C}$ and rely on the Preservation Theorem.

Connection with the SCH

Definition

The Singular Cardinal Hypothesis (SCH) states that if κ is singular and $2^{\mathrm{cf}(\kappa)} < \kappa$, then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.

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Gitik and Woodin significantly reduced the large cardinal hypothesis to a measurable cardinal κ of Mitchell order κ^{++}. This hypothesis was shown to be optimal by Gitik and Mitchell using core model theory.

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Can Neeman's result be obtained for $\kappa = \aleph_\omega$, or even $\aleph_{\omega^2}$?

The strategy in the proof our theorem suggests some hope of answering the above question in the positive.