

On the strength of (failure of) square

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The main theorem

Theorem (S.)

Suppose that for some singular strong limit cardinal κ , \square_{κ} fails. Then there is a nontame mouse. In particular, there is an inner model with a Woodin cardinal δ and a strong cardinal $\lambda < \delta$.

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A stronger lower bound

Remark

- 1 *One can show that in fact “ $AD_{\mathbb{R}} + \Theta$ is regular” is a lower bound but that is a different story for some other time.*
- 2 *The proof is via core model induction and builds on Steel’s proof that “ $\neg \square_{\kappa} \implies AD^{L(\mathbb{R})}$ ”. (Recall that $AD^{L(\mathbb{R})}$ is equiconsistent with ω Woodins.)*

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- 1 Brief sketch of how things are set up.
- 2 A brief outline of the proof.

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What is the core model induction?

It is a technique for calibrating lower bounds of consistency strengths of set theoretic statements.

Typical applications of the core model induction

- 1 Forcing axioms: *PFA* and etc.
- 2 Combinatorial statements: $\neg \square_\kappa$ where κ is a singular strong limit cardinal and etc.
- 3 Generic embeddings: generic embeddings given by precipitous ideals, dense ideals and etc.

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How does the core model induction work?

- 1 It can be viewed as a way of constructing models of determinacy while working in extensions of ZFC .
- 2 There is a collection of companion theorems that link the determinacy theories with large cardinal theories.
- 3 Both together give large cardinal lower bounds.

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Solovay sequence

First, recall that assuming AD,

$$\Theta = \sup\{\alpha : \text{there is a surjection } f : \mathbb{R} \rightarrow \alpha\}.$$

Then, assuming AD, the Solovay sequence is a closed sequence of ordinals $\langle \theta_\alpha : \alpha \leq \Omega \rangle$ defined by:

- 1 $\theta_0 = \sup\{\alpha : \text{there is an ordinal definable surjection } f : \mathbb{R} \rightarrow \alpha\},$
- 2 If $\theta_\alpha < \Theta$ then fixing $A \subseteq \mathbb{R}$ of Wadge rank θ_α ,
 $\theta_{\alpha+1} = \sup\{\alpha : \text{there is a surjection } f : \mathbb{R} \rightarrow \alpha \text{ such that } f \text{ is ordinal definable from } A\},$
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The hierarchy: Solovay hierarchy

$$AD^+ + \Theta = \theta_0 <_{con} AD^+ + \Theta = \theta_1 <_{con} \dots AD^+ + \Theta = \theta_\omega <_{con} \\ \dots AD^+ + \Theta = \theta_{\omega_1} <_{con} AD^+ + \Theta = \theta_{\omega_1+1} <_{con} \dots$$

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Connections to large cardinals

- 1 (Woodin, AD^+) $AD_{\mathbb{R}} \Leftrightarrow AD^+ + “\Theta = \theta_\alpha \text{ for some limit } \alpha”$.
- 2 (Steel) $AD_{\mathbb{R}} \rightarrow$ there is a proper class model M of ZFC such that in M there is λ which is a limit of Woodin cardinals and $< \lambda$ -strong cardinals.
- 3 (Woodin) If λ is a limit of Woodin cardinals and $< \lambda$ -strong cardinals then the derived model at λ satisfies $AD_{\mathbb{R}}$.

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Theorem (Woodin)

$AD^+ + \Theta = \theta_1$ implies the existence of a nontame mouse. In particular, there is an inner model with a Woodin cardinal δ and a strong cardinal $\lambda < \delta$.

Remark

Hence, to prove the theorem it is enough to construct a model of $AD^+ + \Theta = \theta_1$ from $\neg \square_\kappa$ where κ is a singular strong limit cardinal.

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How do we construct models for these axioms?

- 1 Recall that the goal of CMI is to construct models for the axioms from the Solovay hierarchy. In our case, we want a model of $AD^+ + \Theta = \theta_1$.
- 2 With an apology to the experts, the model is essentially $L(\Gamma_{max}, \mathbb{R})$ where

$$\Gamma_{max} = \{A \subseteq \mathbb{R} : L(A, \mathbb{R}) \models AD^+\}.$$

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- 1** Core model induction is used to show that Γ_{max} has various closure properties.
- 2 Here we will concentrate on the following closure property.
- 3 Given a theory S from the Solovay hierarchy, is there $\Gamma \subseteq \Gamma_{max}$ such that $L(\Gamma, \mathbb{R}) \models S$ and $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$.
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- 1** Let κ be a singular strong limit cardinal such that $\neg \square_\kappa$ and let $\mu > \text{cf}(\kappa)$ be a regular cardinal such that $\mu^\omega = \mu$.
- 2** Fix $\lambda \gg \kappa$ and let X be a submodel of V_λ such that letting \mathcal{N} be the transitive collapse of X
 - 1** $\mu + 1 \subseteq \mathcal{N}$,
 - 2** $\mathcal{N}^\omega \subseteq \mathcal{N}$,
 - 3** $|\mathcal{N}| = \mu$.
- 3** Let $\pi : \mathcal{N} \rightarrow V_\lambda$.
- 4** We start working in $V[g]$ (backed up by V) where $g \subseteq \text{Coll}(\omega, \mu)$.
- 5** We can extend π to $\pi : \mathcal{N}[g] \rightarrow V_\lambda[g]$.

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- 2 In an earlier work, Steel has shown that $L(\Gamma_{max}, \mathbb{R}) \models AD^+$. Therefore, $L(\Gamma_{max}, \mathbb{R}) \models AD^+ + \Theta = \theta_0$.
- 3 To get a contradiction, we try to construct $A \subseteq \mathbb{R}$ such that $A \notin \Gamma_{max}$ yet $L(A, \mathbb{R}) \models AD^+$.

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The nature of A

- 1** The general idea is to produce a countable mouse \mathcal{M} such that \mathcal{M} has a “nice” iteration strategy Σ such that $Code(\Sigma) \notin \Gamma_{max}$.
- 2** Then use *CMI* to show that $L(Code(\Sigma), \mathbb{R}) \models AD^+$.
- 3** So A is really a code set of an iteration strategy for some countable mouse \mathcal{M} .

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The nature of \mathcal{M}

- 1** Let $\mathcal{H}^* = (V_\Theta)^{\text{HOD}^{L(\Gamma_{max}, \mathbb{R})}}$ and let $\mathcal{P}^* = \pi^{-1}(\mathcal{H}^*)$. Let $\mathcal{H} = Lp_\omega(\mathcal{H}^*)$ and $\mathcal{P} = Lp_\omega(\mathcal{P}^*)$.
- 2 Under $AD^+ + MSC$, \mathcal{H} is a mouse.
- 3 Hence, \mathcal{P} is a mouse.
- 4 We then try to construct a strategy for \mathcal{P} . Let Σ be this strategy.
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The final plan.

Construct a strategy Σ for \mathcal{P} such that

- 1 HOD is a Σ -iterate of \mathcal{P} .
- 2 $L(\text{Code}(\Sigma), \mathbb{R}) \models AD^+$.

The construction of Σ

Diagram on the board.

Showing that $L(\text{Code}(\Sigma), \mathbb{R}) \models AD^+$

- 1 To show that $L(\text{Code}(\Sigma), \mathbb{R}) \models AD^+$ one needs to show that Σ has *branch condensation*.

Definition (Branch condensation)

An iteration strategy Σ has *branch condensation* if for any two stacks \vec{T} and \vec{U} on M_Σ and branch c of \vec{U} if

- 1 \vec{T} and \vec{U} are according to Σ , $lh(\vec{U}) = \gamma + 1$ and $lh(\mathcal{U}_\gamma)$ is limit,
- 2 if $b = \Sigma(\vec{T})$ then $i_b^{\vec{T}}$ exists,
- 3 $i_c^{\vec{U}}$ -exists and for some $\pi : \mathcal{M}_c^{\vec{U}} \rightarrow_{\Sigma_1} \mathcal{M}_b^{\vec{T}}$,
$$i^{\vec{T}} = \pi \circ i_c^{\vec{U}}$$

then $c = \Sigma(\vec{U})$.

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What is needed to get more?

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Mouse Capturing

Definition

The **Mouse Capturing** is the statement that for any two reals x and y , x is $OD(y)$ iff there is a mouse \mathcal{M} over y such that $x \in \mathcal{M}$.

The Mouse Set Conjecture

Conjecture (Steel and Woodin)

Assume AD^+ and that there is no inner model with a superstrong cardinal. Then Mouse Capturing holds.

Instances of the Mouse Capturing

Theorem

- 1** (Kleene) $x \in \Delta_1^1(y) \leftrightarrow x \in L_{\omega_1^{ck}(y)}[y]$.
- 2** (Shoenfield) x is $\Delta_2^1(y)$ in a countable ordinal iff $x \in L[y]$.
- 3** etc.

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A partial result

Theorem (S.)

Assume AD^+ and there is no inner model containing the reals and satisfying $AD_{\mathbb{R}} + \text{“}\Theta \text{ is regular”}$. Then Mouse Capturing holds.

How are hods computed?

Assume Mouse Capturing and work under AD^+ . As a first step, notice that if $x \in \text{HOD}$ then x is in a mouse. So \mathbb{R}^{HOD} is a set of reals of a mouse. We just generalize this but it is much harder. HOD is shown to be a hod premouse.

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The hod theorems

Theorem (S.)

HOD of the minimal model of $AD_{\mathbb{R}}$ + “ Θ is regular” is a hod premouse.

This much is enough to carry out the general outline and get
“ $AD_{\mathbb{R}} + \Theta$ is regular” as a lower bound for $\neg \square_{\kappa}$.

The definition of a hod mouse is motivated by the following theorem of Woodin.

Theorem (Woodin)

Assume AD^+ . For every α , if $\theta_{\alpha+1}$ exists then it is a Woodin cardinal in HOD.