Complete metric groups acting on trees

Main motivation:

Provide a uniform approach to several "rigidity" type theorems from the literature. E.g.

Theorem (R.M. Dudley)

Suppose G is a complete metric or loc. comp. Hausdorff group and $\pi: G \longrightarrow H$ is a homomorphism into a free or free-abelian group. Then $\ker \pi \leq G$ is an open subgroup.

Theorem (R. Alperin)

Suppose G is complete metric or loc. comp. Hausdorff. If $G = A *_C B$ is a non-trivial decomposition as a free product with amalgamation, then A, B, C are open subgps of G.

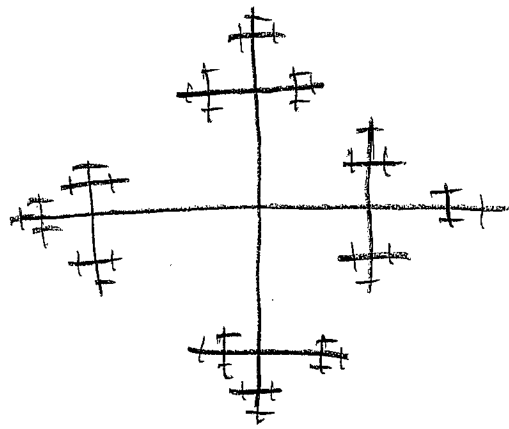
(Other results due to Bass, Koppelberg-Tits, Shelah, Macpherson-Thomas, ...)

Basics of Bass–Serre theory

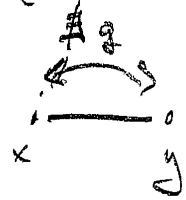
A **tree** is a connected, acyclic graph $\Gamma = (V(\Gamma), E(\Gamma))$, which we equip with the path distance.

E.g., the Cayley graph of a free group:

Cayley$(\mathbb{F}_2, \{a, b\})$:



An isometry (i.e., automorphism) g of Γ is w/o **inversion** if $\nexists \{x, y\} \in E(\Gamma)$ $(gx = y \ \& \ gy = x)$



Any isometry g is either

elliptic, i.e., fixes some vertex $x \in V(\Gamma)$

or

hyperbolic, i.e., acts by translation $g x_n = x_{n+m}$ on some biinfinite line $(\dots, x_{-1}, x_0, x_1, x_2, \dots)$ in Γ.

If G is a group of isometries of Γ w/o inversion we have the following options:

- G fixes some vertex $x \in V$
- each elt of G is elliptic, but there is no common fixed vertex, in which case $G = \bigcup_{n \in \mathbb{N}} G_n$ for some chain $G_0 < G_1 < \dots$ of proper subgps.

- the subgroup $R < G$ generated by elliptic isometries is proper and $G/R \cong F$ is a free group surjecting onto $\mathbb{Z}$.
- $G = R$ admits a non-trivial decomp. $G = A *_C B$.

<u>Main equivalence</u> Suppose G is complete metric and acts w/o inversions on a tree Γ. TFAE

(i) the amplitude $\|\cdot\| : G \to \mathbb{N}$, $\|g\| = \min_{x \in V(\Gamma)} d(x, g(x))$, is continuous,

(ii) there is a somewhere comeagre set of elliptic isometries,

(iii) there is an open subgroup of elliptic isometries

(iv) G either fixes an end or there is an open subgroup K fixing a vertex.

Moreover, the above 4 conditions hold for any action w/o inversions of G if and only if whenever $G = A *_C B$ is non-trivial, then A, B, C are open in G.

<u>Corollary</u>

If G is complete metric and the set

$\{g \in G \mid \langle g \rangle \text{ is finite or non-discrete}\}$

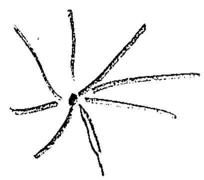
is somewhere dense, then G satisfies the equiv. conditions of the theorem.

<u>Lemma</u> Any Polish group satisfies these cond.

<u>Other results</u>:

Discontinuous actions

Not fixing an end, vertex stab. are not open



star of G

$V(T) = \{*\} \cup G$

No open vertex stabilizer

$G_0 < G_1 < \dots < G = \bigcup_n G_n$

$V(T) = G/G_0 \sqcup G/G_1 \sqcup \dots$

$\{gG_n, gG_{n+1}\} \in E(T) \qquad g \in G,\ n \in \mathbb{N}$

$\mathbb{Z}^{\mathbb{N}}$ has no open subgroup with property (FA').

<u>Thm</u> If G is loc. comp Hausdorff, then G has an open subgroup with property (FA).