Fragments of Martin's Maximum in the $\mathbb{P}_{\mathrm{max}}$ extension

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Martin's Maximum$^{++}(\kappa)$: if P is a partial order of cardinality at most κ which preserves stationary subsets of ω_1, D_α $(\alpha < \omega_1)$ are dense subsets of P and τ_α $(\alpha < \omega_1)$ are P-names for stationary subsets of ω_1, then there is a filter $G \subseteq P$ intersecting each D_α such that

$$\{\beta < \omega_1 \mid \exists p \in G\, p \Vdash \beta \in \tau_\alpha\}$$

is stationary for all $\alpha < \omega_1$.

$\mathbb{P}_{\text{max}}$ is a partial order whose conditions consist of pairs $\langle (M, I), a \rangle$ such that

- M is a countable transitive model of ZFC + $\text{MA}_{\aleph_1}$
- I is a normal precipitous ideal on ω_1^M and (M, I) is iterable
- $a \in \mathcal{P}(\omega_1)^M$ is such that for some $x \in \mathcal{P}(\omega)^M$,

$$\omega_1^{L[x,a]} = \omega_1^M$$

The order on $\mathbb{P}_{\mathsf{max}}$ is given by

$$\langle (M, I), a \rangle < \langle (N, J), b \rangle$$

if

$$\langle (N, J), b \rangle \in H(\aleph_1)^M$$

and there is an iteration

$$j\colon (N, J) \to (N^*, J^*)$$

in M such that $I \cap N^* = J^*$.

$\mathbb{P}_{\mathrm{max}}$ is a partial order in $L(\mathbb{R})$.

Conditions are countable transitive models of ZFC with some additional structure.

The order is given by iterations of generic embeddings via $\mathcal{P}(\omega_1)/I$, for some normal precipitious ideal I on ω_1.

Given a set of reals A, a pair (M, I) as above is said to be A-iterable if

$$A \cap M \in M$$

and

$$j(A \cap M) = A \cap M^*$$

for all iterations

$$j\colon (M, I) \to (M^*, I^*).$$

AD^+ is an ostensible strengthening of AD which is an attempt to to describe the properties of determinacy models whose sets of reals are all Suslin in some larger determinacy model with the same reals.

$\mathbb{P}_{max}$ can be applied to any model of AD^+.

What fragments of MM do the $\mathbb{P}_{\mathrm{max}}$ extensions of models of various forms of AD^+ satisfy?

If $L(\mathbb{R}) \models$ AD, the $\mathbb{P}_{\text{max}}$ extension of $L(\mathbb{R})$ satisfies $\text{MM}^{++}(\aleph_1)$, ψ_{AC}, $\mathfrak{c} = \aleph_2$, the saturation of NS_{ω_1}, plus all forceable Π_2 sentences for

$$\langle H(\aleph_2), \in, NS_{\omega_1}, A; A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})\rangle.$$

The $H(\aleph_2)$ of the $\mathbb{P}_{\max}$ extension of any model of AD^+ is the $H(\aleph_2)$ of the corresponding extension of $L(\mathbb{R})$.

$\mathbb{P}_{\mathrm{max}}$ names for subsets of $H(\aleph_2)$ are coded by sets of reals.

It follows that the $H(\aleph_3)$ of the $\mathbb{P}_{\mathrm{max}}$ extension is determined by the $L(\mathcal{P}(\mathbb{R}))$ of the ground model.

Theorem 1 (Woodin). *If* $\Gamma \subseteq \mathcal{P}(\mathbb{R})$,

$$L(\Gamma, \mathbb{R}) \models \mathsf{AD}_{\mathbb{R}} + \text{“}\Theta \textit{ is regular”},$$

$G \subseteq \mathbb{P}_{\mathsf{max}}$ *is a* $L(\Gamma, \mathbb{R})$*-generic filter and*

$$H \subseteq Coll(\omega_3, \mathcal{P}(\mathbb{R}))$$

is a $L(\Gamma, \mathbb{R})[G]$*-generic filter, then*

$$\mathsf{MM}^{++}(\mathfrak{c})$$

holds in $L(\Gamma, \mathbb{R})[G][H]$.

Very rough sketch of proof: By $\mathsf{AD}_{\mathbb{R}}$, the club filter $\mathcal{F}$ on $\mathcal{P}_{\aleph_1}(\mathbb{R})$ is an ultrafilter. Every name for P, D_α, τ_α ($\alpha < \omega_1$) as in the statement of $\mathsf{MM}^{++}(\mathfrak{c})$ is coded by a set of reals A. Applying the fact that all sets of reals are Suslin, for $\mathcal{F}$-many σ there exists an A-iterable $\mathbb{P}_{\mathsf{max}}$ condition $\langle (M, I), a \rangle$ such that

- $M \cap \mathbb{R} = \sigma$,
- M has the form $M'[g]$, where M' is a $\mathbb{P}_{\mathsf{max}}$ extension of an inner model of AD^+ and g is M'-generic for the realization of the corresponding fragment of the name for P, which preserves stationary subsets of ω_1 in M'.

$\mathsf{WRP}_n(\kappa)$ is the statement that whenever

$$A_1, \ldots, A_n$$

are stationary subsets of $[\kappa]^{\aleph_0}$ there exists an

$$X \in [\kappa]^{\aleph_1}$$

containing ω_1 such that

$$A_i \cap X$$

is stationary in $[X]^{\aleph_0}$ for all i.

Theorem 2 (Woodin). *If*

- $\Gamma \subseteq \mathcal{P}(\mathbb{R})$,
- $L(\Gamma, \mathbb{R})$ *satisfies* AD^+,
- *the* $\mathbb{P}_{\mathrm{max}}$ *extension of* $L(\Gamma, \mathbb{R})$ *satisfies* $WRP(\omega_2)$,

then $L(\Gamma, \mathbb{R})$ *satisfies* $AD_{\mathbb{R}}$.

Since $\mathsf{AD}_{\mathbb{R}}$ and "WRP(ω_2) holds in the $\mathbb{P}_{\max}$ extension" both depend only on $\mathcal{P}(\mathbb{R})$, it follows that for any model N of AD^+, N satisfies $\mathsf{AD}_{\mathbb{R}}$ if and only if the $\mathbb{P}_{\max}$ extension of N satisfies WRP(ω_2).

Trying to do with less than $\mathsf{AD}_{\mathbb{R}}$:

Theorem 3 (Woodin)**.** *If* $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ *and* $L(\Gamma, \mathbb{R})$ *satisfies* AD^+ *plus the existence of a normal fine measure on* $\mathcal{P}_{\aleph_1}(\mathbb{R})$*, then* $L(\Gamma, \mathbb{R})$ *satisfies* $AD_{\mathbb{R}}$*.*

The hypothesis that AD^+ holds and

$$L(\mathcal{F}, \mathbb{R}) \models \mathsf{AD}^+ + \text{“}\mathcal{F} \text{ is an ultrafilter,”}$$

where $\mathcal{F}$ is the club filter on $\mathcal{P}_{\aleph_1}(\mathbb{R})$, is equiconsistent with the existence of ω^2 Woodin cardinals.

In $L(\mathcal{F}, \mathbb{R})$, the Suslin-co-Suslin sets are $\Sigma_1(\mathcal{F})$ elementary in $\mathcal{P}(\mathbb{R})$, which means that for every set of reals A there are F-many countable sets $\sigma \subseteq \mathbb{R}$ for which there exist A-iterable $\mathbb{P}_{\text{max}}$ conditions $\langle (M', I), a \rangle$ such that

- $\sigma = \mathbb{R} \cap M'$,
- $\langle H(\aleph_1)^{M'}, \in, A \cap M' \rangle \prec \langle H(\aleph_1), \in, A \rangle$,
- M' is a $\mathbb{P}_{\text{max}}$ extension of an inner model of AD^+.

You don't get that in M' the realization P of the name corresponding to A preserves stationary subsets of ω_1, or that the P-names coded are names for stationary subsets of ω_1 in M'.

In the corresponding $\mathbb{P}_{\mathsf{max}}$ extension, we get $\mathsf{MM}^{++}(\mathfrak{c})$ for those P for which the fact that they preserve stationary subsets of ω_1 is absolute to all outer models with the same $\mathcal{P}(\omega_1)$, and names for stationary subsets of ω_1 which similarly persist as such names (possibly witnessed by a subset of $H(\aleph_2)$).

Woodin calls this fragment Martin's $\text{Maximum}^{++}_{\mathsf{ZF}}(\mathfrak{c})$, and he notes that it holds in the $\mathbb{P}_{\mathsf{max}}$ extension of a weaker fragment of determinacy.

This includes c.c.c. forcing, σ-closed forcing, and maximal antichain sealing forcing for $\mathcal{P}(\omega_1)/NS_{\omega_1}$. It's enough to show the failure of $\square_{\omega_1}$, but I don't know if it includes the Tree Property at ω_2.

We have seen that it does not include WRP(ω_2). The failure of Todorcevic's $\square(\omega_2)$ is also not included.

$\mathsf{WRP}^*_n(\kappa)$ is the statement that there exists a normal, fine, proper ideal I on $[\kappa]^{\aleph_0}$, not containing any set of the form

$$\{b \in [\kappa]^{\aleph_0} \mid b \cap \omega_1 \in T\}$$

for T a stationary subset of ω_1, such that whenever

$$A_1 \ldots, A_n$$

are I-positive there exist an $X \in [\kappa]^{\aleph_1}$ containing ω_1 such that $A_i \cap [X]^{\aleph_0}$ is stationary for each i.

In the $\mathbb{P}_{\text{max}}$ extension of $L(\mathcal{F}, \mathbb{R})$, let I be the collection of subsets A of $[\omega]^{\aleph_0}$ such that there do not exist a set of reals B coding a name giving rise to and name τ for A and $\mathcal{F}$-many σ such that there exist models M as above in which the corresponding realization of τ is stationary.

Then I witnesses $\mathsf{WRP}^*_n(\omega_2)$ for all n.

In fact, $\mathrm{WRP}^*(\kappa)$ is a theorem of ZFC for all regular $\kappa \geq \omega_2$: let I be the set of $A \subseteq [\kappa]^{\aleph_0}$ such that A reflects to nonstationarily many $X \in [\kappa]^{\aleph_1}$.

Theorem 4 (Steel-Zoble). *If* $\mathfrak{c} = \aleph_2$, NS_{ω_1} *is saturated and* $\mathit{WRP}^*_2(\omega_2)$ *holds then AD holds in* $L(\mathbb{R})$.

They conjecture that $\mathrm{WRP}^*_2(\omega_2)$ and $\mathrm{SRP}^*(\omega_2)$ are each equiconsistent with the existence of ω^2 Woodin cardinals.

$\mathrm{WRP}_2^*(\omega_2)$ implies that if every subset of ω_1 has a sharp then every subset of ω_2 has a sharp.

Do there exist other interesting “idealized” fragments of $\mathrm{MM}^{++}(\mathfrak{c})$?

Woodin has conjectured that the following holds: If

- $\Gamma \subseteq \mathcal{P}(\mathbb{R})$,
- $L(\Gamma, \mathbb{R})$ satisfies AD^+,
- the $\mathbb{P}_{\mathsf{max}}$ extension of $L(\Gamma, \mathbb{R})$ satisfies $\mathsf{SRP}^*(\omega_2)$,

then $L(\Gamma, \mathbb{R})$ satisfies $\mathsf{AD}_{\mathbb{R}}$.