Cardinal invariants of monotone and porous sets

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(joint work with Ondřej Zindulka)

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Content

1 Monotone sets

2 Cardinal invariants

3 Mon and its cardinal invariants
   - Additivity and cofinality
   - Porous sets
   - Covering and uniformity

4 Open problems
Monotone sets

Definition (Ondřej Zindulka)

Let $(X, d)$ be a metric space.

- $(X, d)$ is called monotone if there is $c > 0$ and a linear order $<$ on $X$ such that $d(x, y) \leq c \, d(x, z)$ for all $x < y < z$ in $X$.
- $(X, d)$ is called $\sigma$-monotone if it is a countable union of monotone subspaces (with possibly different witnessing constants).

Zindulka used it to prove: • the existence of universal measure zero sets of large Hausdorff dimension, and
• that a Borel set in $\mathbb{R}^n$ of Hausdorff dimension greater than $m$ maps onto the $m$-dimensional cube by a Lipschitz map.
Monotone sets

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Let \((X, d)\) be a metric space.
- \((X, d)\) is called *monotone* if there is \(c > 0\) and a linear order \(<\) on \(X\) such that \(d(x, y) \leq c \cdot d(x, z)\) for all \(x < y < z\) in \(X\).
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Definition (Ondřej Zindulka)

Let \((X, d)\) be a metric space.
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- the existence of universal measure zero sets of large Hausdorff dimension, and  
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Basic properties of monotone sets

- Any monotone space is suborderable, i.e., any monotone set in the plane is homeomorphic to a subset of the line and any monotone connected set in the plane is homeomorphic to an interval.

- The closure of any monotone subspace of a metric space is monotone.

- (Nekvinda-Zindulka) Every discrete metric space is $\sigma$-monotone.

- The graph $\sin\left(\frac{1}{x}\right)$ is not monotone but it is $\sigma$-monotone. (Hint: Many "bad" triangles.)
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Functions with $\sigma$-monotone graphs

- (Zindulka) Every continuous function $f : [0, 1] \to [0, 1]$ with a $\sigma$-monotone graph has a dense set of points of differentiability,

on the other hand,

- (Mátrai-Vlasák) There is a continuous function $f : [0, 1] \to [0, 1]$ with a $\sigma$-monotone graph such that the set of points of differentiability has Lebesgue measure zero.

- There is an absolutely continuous function $f : [0, 1] \to [0, 1]$ which does not have a $\sigma$-monotone graph.
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Question

(Zindulka) Is there a (separable) metric space of size $\aleph_1$ which is not $\sigma$-monotone?

Proposition

($\text{MA}_{\sigma\text{-linked}}$) Every separable metric space of size $\aleph_1$ is $\sigma$-monotone.
Question

(Zindulka) Is there a (separable) metric space of size $\mathfrak{c}$ which is not $\sigma$-monotone?

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($\text{MA}_{\sigma}$-linked) Every separable metric space of size $\mathfrak{c}$ is $\sigma$-monotone.
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1. Monotone sets

2. Cardinal invariants

3. Mon and its cardinal invariants
   - Additivity and cofinality
   - Porous sets
   - Covering and uniformity

4. Open problems
Definition

Given an ideal $I$ on a set $X$, the following are the usual cardinal invariants of $I$:

- $\text{add}(I) = \min\{|A| : A \subseteq I \land \bigcup A \notin I\}$,
- $\text{cov}(I) = \min\{|A| : A \subseteq I \land \bigcup A = X\}$,
- $\text{cof}(I) = \min\{|A| : A \subseteq I \land (\forall I \in I)(\exists A \in A)(I \subseteq A)\}$,
- $\text{non}(I) = \min\{|Y| : Y \subseteq X \land Y \notin I\}$.

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Cichoń’s diagram

\[
\begin{align*}
\text{cov}(\mathcal{N}) & \to \text{non}(\mathcal{M}) \to \text{cof}(\mathcal{M}) \to \text{cof}(\mathcal{N}) \\
\text{add}(\mathcal{N}) & \to \text{add}(\mathcal{M}) \to \text{cov}(\mathcal{M}) \to \text{non}(\mathcal{N}) \\
\text{m}_\sigma\text{-linked} & \to \text{m}_\sigma\text{-centered}
\end{align*}
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Definition

The ideal of all $\sigma$-monotone sets in the plane is denoted $\text{Mon}$. 

Theorem

(i) $\text{add}(\text{Mon}) = \omega_1$, 
(ii) $\text{cof}(\text{Mon}) = c$. 

Lemma

Let $\mathcal{L}$ be a family of lines in $\mathbb{R}^2$. Then $\bigcup \mathcal{L}$ is $\sigma$-monotone if and only if $\mathcal{L}$ is countable.
Additivity and cofinality of $\text{Mon}$

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Additivity and cofinality of \( \text{Mon} \)

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Strongly porous sets

Definition

Let \((X, d)\) be a metric space. A set \(A \subseteq X\) is

- **porous at a point** \(x \in X\) if there is \(p > 0\) and \(r_0 > 0\) such that for any \(r \leq r_0\) there is \(y \in X\) such that \(B(y, pr) \subseteq B(x, r) \setminus A\),
- **porous** if it is porous at each point \(x \in A\), and
- **\(\sigma\)-porous** if it is a countable union of porous sets.

Definition

Let \(X\) be a metric space. The ideal of all \(\sigma\)-porous sets in \(X\) is denoted \(\text{SP}(X)\).

Proposition

\[
\text{cov}(\text{SP}(\mathbb{R})) = \text{cov}(\text{SP}(\mathbb{R}^2)) = \text{cov}(\text{SP}(2^\omega)) \text{ and likewise for non.}
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\text{cov(\text{SP}(\mathbb{R})) = cov(\text{SP}(\mathbb{R}^2)) = cov(\text{SP}(2^\omega)) and likewise for non.}
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Monotone vs. porous

**Proposition**

*Every monotone set $X \subseteq \mathbb{R}^2$ is porous. Consequently $\text{Mon} \subseteq \text{SP}(\mathbb{R}^2)$.***

**Proposition**

*If $A, B \subseteq \mathbb{R}$ are porous, then $A \times B \subseteq \mathbb{R}^2$ is monotone.*

**Corollary**

$cov(\text{Mon}) = cov(\text{SP})$ and $\text{non}(\text{Mon}) = \text{non}(\text{SP})$ (from now on $\text{SP}$ denotes $\text{SP}(2^\omega)$).

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Proposition

Every monotone set \( X \subseteq \mathbb{R}^2 \) is contained in a closed set of measure zero.

Corollary

\[
\text{non}(\text{Mon}) = \text{non}(\text{SP}) \leq \min\{\text{non}(\mathcal{N}), \text{non}(\mathcal{M})\} \quad \text{and} \\
\max\{\text{cov}(\mathcal{N}), \text{cov}(\mathcal{M})\} \leq \text{cov}(\text{Mon}) = \text{cov}(\text{SP}).
\]
Bounds on cov and non

**Proposition**

Every monotone set $X \subseteq \mathbb{R}^2$ is contained in a closed set of measure zero.

**Corollary**

$$\text{non}(\text{Mon}) = \text{non}(\text{SP}) \leq \min\{\text{non}(\mathcal{N}), \text{non}(\mathcal{M})\} \quad \text{and}$$

$$\max\{\text{cov}(\mathcal{N}), \text{cov}(\mathcal{M})\} \leq \text{cov}(\text{Mon}) = \text{cov}(\text{SP}).$$
Uniformity

Theorem

\[ m_{\sigma\text{-linked}} \leq \text{non}(\text{SP}) = \text{non}(\text{Mon}). \]

Theorem

It is relatively consistent with ZFC that
\[ \text{add}(\mathcal{N}) = m_{\sigma\text{-centered}} = c > \text{non}(\text{SP}) = \text{non}(\text{Mon}) = \omega_1. \]
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$m_{\sigma\text{-linked}} \leq \text{non}(\text{SP}) = \text{non}(\text{Mon})$. 

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It is relatively consistent with ZFC that $\text{cov}(\text{Mon}) = \text{cov}(\text{SP}) < \mathfrak{c}$.

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It is relatively consistent with ZFC that $\text{cof}(\mathcal{N}) = \omega_1$ and $\text{cov}(\text{SP}) = \omega_2$. 
It is relatively consistent with ZFC that \( \text{cov}(\text{Mon}) = \text{cov}(\text{SP}) < c \).

It is relatively consistent with ZFC that \( \text{cof}(\mathcal{N}) = \omega_1 \) and \( \text{cov}(\text{SP}) = \omega_2 \).
A tree $T \subseteq 2^{<\omega}$ is hyper-perfect if

$$\forall s \in T \ \forall n \exists t \supseteq s \ \forall r \in 2^n \ t \upharpoonright r \in T.$$
Hyper-perfect tree forcing

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$$\forall s \in T \\forall n \exists t \supseteq s \\forall r \in 2^n \ t \upharpoonright r \in T.$$  

Definition

$\mathsf{HP} = \{ T \subseteq 2^{<\omega} : T \text{ is hyper-perfect} \}$
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Open problems

Question

*Is it true that*

(i) \( \text{add}(\text{SP}) = \omega_1, \)

(ii) \( \text{cof}(\text{SP}) = c, \)

Question

*What can one say about the cardinal invariants of \( \text{Mon}(X) \) when \( X \) is*

(i) the non-\( \sigma \)-monotone graph of an absolutely continuous function \( f : [0, 1] \to [0, 1] \),

(ii) the Hilbert cube,

(iii) the Urysohn space?

Question

*Is there a metric space of cardinality \( \aleph_1 \) that is not \( \sigma \)-monotone?*
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