Special Session on Large Cardinals and the Continuum

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Cardinal invariants of monotone and porous sets

Michael Hrušák
IMUNAM-Morelia
Universidad Nacional Autónoma de México
michael@matmor.unam.mx
(joint work with Ondřej Zindulka)

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Content

1. Monotone sets
2. Cardinal invariants
3. Mon and its cardinal invariants
   - Additivity and cofinality
   - Porous sets
   - Covering and uniformity
4. Open problems

Monotone sets

Definition (Ondřej Zindulka)

Let (X, d) be a metric space.
• (X, d) is called *monotone* if there is $c > 0$ and a linear order $<$ on X such that $d(x, y) \leqslant c\, d(x, z)$ for all $x < y < z$ in X.
• (X, d) is called *σ-monotone* if it is a countable union of monotone subspaces (with possibly different witnessing constants).

Zindulka used it to prove: • the existence of universal measure zero sets of large Hausdorff dimension, and
• that a Borel set in $\mathbb{R}^n$ of Hausdorff dimension greater than m maps onto the m-dimensional cube by a Lipschitz map.

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Basic properties of monotone sets

- Any monotone space is suborderable, i.p. any monotone set in the plane is homeomorphic to a subset of the line and any monotone connected set in the plane is homeomorphic to an interval.

- The closure of any monotone subspace of a metric space is monotone.

- (Nekvinda-Zindulka) Every discrete metric space is σ-monotone.

- The graph $\sin(1/x)$ is not monotone but it is σ-monotone. (Hint: Many "bad" triangles.)

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Functions with σ-monotone graphs

• (Zindulka) Every continuous function $f : [0,1] \to [0,1]$ with a σ-monotone graph has a dense set of points of differentiability,

on the other hand,

• (Mátrai-Vlasák) There is a continuous function $f : [0,1] \to [0,1]$ with a σ-monotone graph such that the set of points of differentiability has Lebesgue measure zero.

• There is an absolutely continuous function $f : [0,1] \to [0,1]$ which does not have a σ-monotone graph.

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Small non-σ-monotone spaces

Question

(Zindulka) Is there a (separable) metric space of size $\aleph_1$ which is not σ-monotone?

Proposition

($\mathrm{MA}_{\sigma\text{-linked}}$) *Every separable metric space of size $\aleph_1$ is σ-monotone.*

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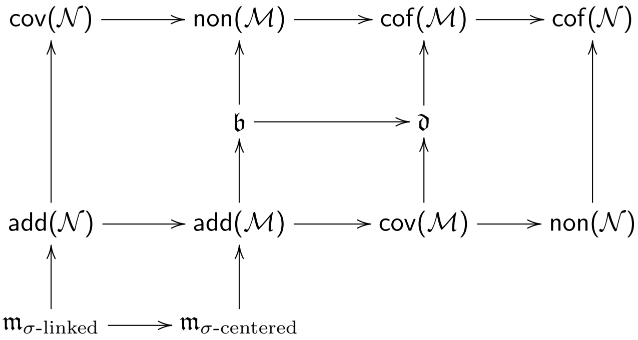
Cardinal invariants of σ-ideals

Definition

Given an ideal $\mathcal{I}$ on a set X, the following are the usual cardinal invariants of $\mathcal{I}$:

$$\begin{aligned}
\mathrm{add}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup\mathcal{A} \notin \mathcal{I}\},\\
\mathrm{cov}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup\mathcal{A} = X\},\\
\mathrm{cof}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall I \in \mathcal{I})(\exists A \in \mathcal{A})(I \subseteq A)\},\\
\mathrm{non}(\mathcal{I}) &= \min\{|Y| : Y \subseteq X \wedge Y \notin \mathcal{I}\}.
\end{aligned}$$

Cichoń's diagram



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Additivity and cofinality of **Mon**

Definition

The ideal of all σ-monotone sets in the plane is denoted **Mon**.

Theorem

(i) $\text{add}(\mathbf{Mon}) = \omega_1$,

(ii) $\text{cof}(\mathbf{Mon}) = \mathfrak{c}$.

Lemma

Let $\mathscr{L}$ be a family of lines in $\mathbb{R}^2$. Then $\bigcup \mathscr{L}$ is σ-monotone if and only if $\mathscr{L}$ is countable.

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Strongly porous sets

Definition

Let (X, d) be a metric space. A set $A \subseteq X$ is

- *porous at a point* $x \in X$ if there is $p > 0$ and $r_0 > 0$ such that for any $r \leqslant r_0$ there is $y \in X$ such that $B(y, pr) \subseteq B(x, r) \setminus A$,
- *porous* if it is porous at each point $x \in A$, and
- *σ-porous* if it is a countable union of porous sets.

Definition

Let X be a metric space. The ideal of all σ-porous sets in X is denoted $\mathbf{SP}(X)$.

Proposition

$\mathrm{cov}(\mathbf{SP}(\mathbb{R})) = \mathrm{cov}(\mathbf{SP}(\mathbb{R}^2)) = \mathrm{cov}(\mathbf{SP}(2^\omega))$ *and likewise for* non.

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Monotone vs. porous

Proposition

Every monotone set $X \subseteq \mathbb{R}^2$ is porous. Consequently $\mathbf{Mon} \subseteq \mathbf{SP}(\mathbb{R}^2)$.

Proposition

If $A, B \subseteq \mathbb{R}$ are porous, then $A \times B \subseteq \mathbb{R}^2$ is monotone.

Corollary

$\mathrm{cov}(\mathbf{Mon}) = \mathrm{cov}(\mathbf{SP})$ *and* $\mathrm{non}(\mathbf{Mon}) = \mathrm{non}(\mathbf{SP})$ *(from now on* $\mathbf{SP}$ *denotes* $\mathbf{SP}(2^\omega)$*).*

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Bounds on cov and non

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Every monotone set $X \subseteq \mathbb{R}^2$ is contained in a closed set of measure zero.

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$\mathrm{non}(\mathbf{Mon}) = \mathrm{non}(\mathbf{SP}) \leqslant \min\{\mathrm{non}(\mathcal{N}), \mathrm{non}(\mathcal{M})\}$ *and* $\max\{\mathrm{cov}(\mathcal{N}), \mathrm{cov}(\mathcal{M})\} \leqslant \mathrm{cov}(\mathbf{Mon}) = \mathrm{cov}(\mathbf{SP})$.

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Uniformity

Theorem

$\mathfrak{m}_{\sigma\text{-linked}} \leqslant \mathrm{non}(\mathbf{SP}) = \mathrm{non}(\mathbf{Mon})$.

Theorem

It is relatively consistent with ZFC *that*
$\mathrm{add}(\mathcal{N}) = \mathfrak{m}_{\sigma\text{-centered}} = \mathfrak{c} > \mathrm{non}(\mathbf{SP}) = \mathrm{non}(\mathbf{Mon}) = \omega_1$.

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It is relatively consistent with ZFC *that* $\mathrm{cov}(\mathbf{Mon}) = \mathrm{cov}(\mathbf{SP}) < \mathfrak{c}$.

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Hyper-perfect tree forcing

Definition

A tree $T \subseteq 2^{<\omega}$ is *hyper-perfect* if

$$\forall s \in T\ \forall n\ \exists t \supseteq s\ \forall r \in 2^n\ t^\frown r \in T.$$

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Open problems

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Is it true that

(i) $\mathsf{add}(\mathbf{SP}) = \omega_1$,

(ii) $\mathsf{cof}(\mathbf{SP}) = \mathfrak{c}$,

Question

What can one say about the cardinal invariants of $\mathbf{Mon}(X)$ *when* X *is*

(i) *the non-σ-monotone graph of an absolutely continuous function* $f : [0,1] \to [0,1]$,

(ii) *the Hilbert cube,*

(iii) *the Urysohn space?*

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Is there a metric space of cardinality $\aleph_1$ *that is not σ-monotone?*

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