The Structure of Tukey Types of Ultrafilters

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Let $\mathcal{U}$ and $\mathcal{V}$ be ultrafilters on ω. $(\mathcal{U}, \supseteq)$ is a directed partial ordering.

Def. $\mathcal{V} \leq_T \mathcal{U}$ iff there is a *Tukey* map $g : \mathcal{V} \to \mathcal{U}$ taking unbounded subsets of $\mathcal{V}$ to unbounded subsets of $\mathcal{U}$.

Equivalently, $\mathcal{U} \geq_T \mathcal{V}$ iff there is a *cofinal* map $f : \mathcal{U} \to \mathcal{V}$ taking cofinal subsets of $\mathcal{U}$ to cofinal subsets of $\mathcal{V}$.

Fact. $\mathcal{U} \geq_T \mathcal{V} \Rightarrow$ there is a *monotone* cofinal map witnessing this; i.e. $X \supseteq Y \Rightarrow f(X) \supseteq f(Y)$.

Fact. $\mathcal{U} \geq_T \mathcal{V} \Rightarrow \mathrm{cof}(\mathcal{U}) \geq \mathrm{cof}(\mathcal{V})$ and $\mathrm{add}(\mathcal{U}) \leq \mathrm{add}(\mathcal{V})$.

$\mathcal{U} \equiv_T \mathcal{V}$ iff $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$.

Fact. $\equiv_T$ is an equivalence relation. $\leq_T$ is a partial ordering on the equivalence classes.

Fact. $\mathcal{U} \equiv_T \mathcal{V}$ iff $\mathcal{U}$ and $\mathcal{V}$ are cofinally equivalent.

The collection of all ultrafilters cofinally equivalent to $\mathcal{U}$ is called the *Tukey type* or *cofinal type* of $\mathcal{U}$.

Motivations

1. A special class of directed systems of size $\mathfrak{c}$.
(In contrast to non-classification theorems of Todorcevic for directed posets of size $\mathfrak{c}$)

2. $\mathcal{U} \geq_{RK} \mathcal{V}$ implies $\mathcal{U} \geq_T \mathcal{V}$.

$\mathcal{U} \geq_{RK} \mathcal{V}$ iff there is a function $h : \omega \to \omega$ such that
$\mathcal{V} = h(\mathcal{U}) := \{X \subseteq \omega : h^{-1}(X) \in \mathcal{U}\}$.

3. Isbell's Problem.

Fact. There is an ultrafilter $(\mathcal{U}_{\mathsf{top}}, \supseteq) \equiv_T ([\mathfrak{c}]^{<\omega}, \subseteq)$.

Note: $\mathcal{V} \equiv_T [\mathfrak{c}]^{<\omega}$ iff $\exists S \in [\mathcal{V}]^{\mathfrak{c}}\ \forall T \in [S]^{\omega}\ (\bigcap T \notin \mathcal{V})$.

Fact. $\mathcal{U}_{\mathsf{top}}$ has Tukey type of size $2^{\mathfrak{c}}$.

Isbell's Problem. [Isbell 65] Is there always (in ZFC) an ultrafilter $\mathcal{U}$ such that $\mathcal{U} <_T \mathcal{U}_{\mathsf{top}}$?

Isbell's Problem

$\mathcal{U}_{\mathsf{top}}$

???

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Def. (follows from [Solecki/Todorcevic 04]) An ultrafilter $\mathcal{V}$ is *basic* if each convergent sequence has a bounded subsequence.

Fact. If $\mathcal{U}$ is basic, then $\mathcal{U} <_T [\mathfrak{c}]^{<\omega}$.

Thm. (follows from [Solecki/Todorcevic 04]) If $\mathcal{U}$ is basic, then whenever $\mathcal{U} \geq_T \mathcal{V}$, this is witnessed by a definable map. Thus, the Tukey type of $\mathcal{U}$ has size $\mathfrak{c}$.

Thm. An ultrafilter is basic iff it is a p-point. Hence, every p-point is not Tukey top.

Def. $\mathcal{U}$ is a *p-point* if for each sequence $X_0 \supseteq X_1 \supseteq \ldots$ in $\mathcal{U}$, there is a $Y \in \mathcal{U}$ such that for each $n < \omega$, $Y \subseteq^* X_n$ (i.e. $|Y \setminus X_n| < \omega$).

Note. p-points exist under CH, MA, or just $\mathfrak{d} = \mathfrak{u} = \mathfrak{c}$. It is consistent with ZFC that there are no p-points [Shelah].

Note. Fubini products of p-points and more generally, *basically generated* ultrafilters are strictly below the top.

Continuous Cofinal Maps

Thm. Suppose $\mathcal{U}$ is a p-point and $f : \mathcal{U} \rightarrow \mathcal{V}$ is a monotone cofinal map. Then there is an $\tilde{X} \in \mathcal{U}$ such that $f \restriction (\mathcal{U} \restriction \tilde{X})$ is continuous.

Moreover, $f \restriction (\mathcal{U} \restriction \tilde{X})$ can be extended to a continuous monotone map $\tilde{f} : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$ such that $\tilde{f} : \mathcal{P}(k) \rightarrow \mathcal{P}(k)$ for each $k < \omega$.

(This is useful in forcing constructions where the forcing is σ-closed, and in the next theorems.)

Thm. [Raghavan/Todorcevic] If $\mathcal{V}$ is selective and $\mathcal{U} \geq_T \mathcal{V}$ is witnessed by a continuous cofinal map, then $\mathcal{U} \geq_{RK} \mathcal{V}$.

Thm. [Todorcevic] If $\mathcal{V}$ is selective and $\mathcal{W} \leq_T \mathcal{V}$, then $\mathcal{W} \equiv_{RK} \mathcal{V}^\alpha$ for some $\alpha < \omega_1$.

Questions.

1. Are there ultrafilters besides p-points which carry continuous cofinal maps?
2. Does the existence of continuous cofinal maps get inherited downwards in the Tukey ordering?

Extending continuous maps on cofinal subsets of $\mathcal{U}$ to all of $\mathcal{P}(\omega)$

Thm 1. Suppose $f : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal map, and there is a cofinal subset $\mathcal{X} \subseteq \mathcal{U}$ such that

1. $f \restriction \mathcal{X}$ is continuous;
2. $f \restriction \mathcal{X}$ is given by a map $\hat{f}$ (defined on a subset of $2^{<\omega}$ which is level and initial segment preserving).

Then there is a continuous, monotone $\tilde{f} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $\tilde{f} \restriction \mathcal{X} = f \restriction \mathcal{X}$, $\tilde{f} \restriction \mathcal{U} : \mathcal{U} \to \mathcal{V}$ is a cofinal map, and $\tilde{f}$ is given by a monotone, level and initial segment preserving map.

Rk. Every p-point satisfies these conditions.

A guarantee of Tukey-downward preservation of existence of continuous cofinal maps

Thm 2. Let $\mathcal{U}$ be an ultrafilter such that whenever $f : \mathcal{U} \to \mathcal{V}$ is a monotone cofinal function, then there exists an $\mathcal{X} \subseteq \mathcal{U}$ such that

1. $\mathcal{X}$ is cofinal in $\mathcal{U}$;
2. for each $Z \in \overline{\mathcal{X}}$, there is a decreasing sequence $X_0 \supseteq X_1 \supseteq \ldots \supseteq Z$ such that each $X_n \in \mathcal{X}$ and $\bigcap_{n<\omega} X_n = Z$;
3. $f \restriction \mathcal{X}$ is continuous and given by a level and initial segment preserving monotone map $\hat{f}$.

Then for each $\mathcal{W} \leq_T \mathcal{U}$, if $h : \mathcal{W} \to \mathcal{V}$ is a monotone cofinal map, then there is a cofinal $\mathcal{Y} \subseteq \mathcal{W}$ such that $h \restriction \mathcal{Y}$ is continuous and given by a monotone, level and initial segment preserving map.

Rk. This is the only property known to be inherited under Tukey reducibility.

Ultrafilters on FIN

FIN $= [\omega]^{<\omega} \setminus \{\emptyset\}$.

Def. An (idempotent) ultrafilter $\mathcal{U}$ on FIN is *ordered union* if it is generated by sets of the form $[X]$ where X is an infinite block-sequence. ($[X]$ is the collection of all finite unions of members of X.)

Thm. [Blass 87] If $\mathcal{U}$ is an ordered union ultrafilter on FIN, then both $\mathcal{U}_{\text{min}}$ and $\mathcal{U}_{\text{max}}$ are selective.

Fact. If $\mathcal{U}$ is a ordered union ultrafilter on FIN, then $\mathcal{U}_{\text{min,max}}$ is rapid but neither a p-point nor a q-point.

Def. An ordered union ultrafilter $\mathcal{U}$ is *block-basic* if whenever we are given a sequence (X_n) of infinite block sequences in FIN such that each $[X_n] \in \mathcal{U}$ and (X_n) converges to some infinite block sequence X such that $[X] \in \mathcal{U}$, then there is an infinite subsequence (X_{n_k}) such that $\bigcap_{k<\omega}[X_{n_k}] \in \mathcal{U}$.

Thm. [Blass 87 and D/T] The following are equivalent for an ordered union ultrafilter $\mathcal{U}$ on FIN.

1. $\mathcal{U}$ is block-basic.

2. $\mathcal{U}$ is stable ordered-union. (For every sequence (X_n) of infinite block sequences of FIN such that $[X_n] \in \mathcal{U}$ and $X_{n+1} \leq^* X_n$ for each n, there is an infinite block sequence X such that $[X] \in \mathcal{U}$ and $X \leq^* X_n$ for each n.)

3. $\mathcal{U}$ has the 2-dimensional Ramsey property.

4. $\mathcal{U}$ has the Ramsey property.

Lots of Continuous Cofinal Maps

Main Theorem Suppose $\mathcal{W}$ is Tukey below some countable iteration of Fubini products of p-points and/or block-basic ultrafilters, or any ultrafilter with enough p-point-like structure.

Then whenever $\mathcal{V} \leq_T \mathcal{W}$, this is witnessed by a continuous monotone cofinal map $f : \mathcal{W} \to \mathcal{V}$.

Def. The *Fubini product* of $\mathcal{U}$ and $\mathcal{V}$ is
$\mathcal{U} \cdot \mathcal{V} = \{A \subseteq \omega \times \omega : \mathcal{U}n \ \mathcal{V}j \ ((n, j) \in A)\}$.

$\lim_{n \to \mathcal{U}} \mathcal{V}_n = \{A \subseteq \omega \times \omega : \mathcal{U}n \ \mathcal{V}_n j \ ((n, j) \in A)\}$.

Some Structure Theorems

(some using continuous cofinal maps)

Antichains

Thm. 1. If $\text{cov}(\mathcal{M}) = \mathfrak{c}$, and $2^{<\kappa} = \mathfrak{c}$, then there are 2^κ pairwise incomparable selective ultrafilters.

2. If $\mathfrak{d} = \mathfrak{u} = \mathfrak{c}$ and $2^{<\kappa} = \mathfrak{c}$, then there are 2^κ pairwise incomparable p-points.

Thm. Every family of p-points of cardinality $> \mathfrak{c}^+$ contains a subfamily of equal size of pairwise Tukey incomparable p-points.

Chains

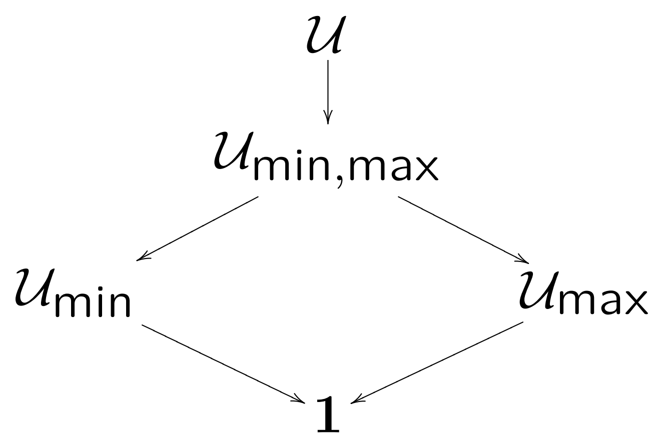
Thm. Every $\leq_T$ chain of p-points has cardinality $\leq \mathfrak{c}^+$.

Thm. (also independently by Raghavan) CH implies for each p-point $\mathcal{U}$ there is a Tukey strictly increasing chain of p-points of length ω_1.

Incomparable Predecessors

Thm. (MA) There is a p-point with 2 Tukey incomparable predecessors, each of which is also a p-point.

Thm. Assuming CH, there is a block-basic ultrafilter $\mathcal{U}$ on FIN such that $\mathcal{U}_{\mathrm{min,max}} <_T \mathcal{U}$ and $\mathcal{U}_{\mathrm{min}}$ and $\mathcal{U}_{\mathrm{max}}$ are Tukey incomparable.



Comparing with ω^ω.

Thm. The following are equivalent for a p-point $\mathcal{U}$

1. $\mathcal{U} \geq_T \omega^\omega$;
2. $\mathcal{U} \equiv_T \mathcal{U} \cdot \mathcal{U}$;
3. $\mathcal{U} \equiv_T \mathcal{U}^\omega$.

Cor. If $\mathcal{U}$ is a p-point of cofinality $< \mathfrak{d}$, then $\mathcal{U} \not\geq_T \omega^\omega$ and therefore $\mathcal{U} \cdot \mathcal{U} >_T \mathcal{U}$.

Thm. Assuming $\mathfrak{p} = \mathfrak{c}$, there is a p-point $\mathcal{U}$ such that $\mathcal{U} \not\geq_T \omega^\omega$ and therefore $\mathcal{U} <_T \mathcal{U} \cdot \mathcal{U} <_T \mathcal{U}_{\text{top}}$.

Conditions for $\mathcal{U} <_T \mathcal{U}_{\text{top}}$.

Prop. [Milovich 09] There is an ultrafilter $\mathcal{U}$ such that $(\mathcal{U}, \supseteq) <_T \mathcal{U}_{\text{top}}$ iff there is an ultrafilter $\mathcal{V}$ such that $(\mathcal{V}, \supseteq^*) <_T \mathcal{U}_{\text{top}}$.

Assume $\neg$CH. Then $\diamondsuit^-_{[[\omega]^\omega]^\omega}$ holds.

Def. [Todorcevic] $\diamondsuit^-_{[[\omega]^\omega]^\omega}$: There exist ordered pairs $(\mathcal{U}_A, \mathcal{X}_A)$, where $A \in [[\omega]^\omega]^\omega$ and $\mathcal{X}_A \subseteq \mathcal{U}_A \subseteq A$, such that for each pair $(\mathcal{U}, \mathcal{X})$ with $\mathcal{X} \subseteq \mathcal{U}$ and $\mathcal{X}, \mathcal{U} \in [[\omega]^\omega]^{\mathfrak{c}}$, $\{A \in [[\omega]^\omega]^\omega : \mathcal{U}_A = \mathcal{U} \cap A, \mathcal{X}_A = \mathcal{X} \cap A\}$ is stationary in $[[\omega]^\omega]^\omega$.

Let $P_A = \{W \in [\omega]^\omega : \exists X \in \mathcal{U}_A (W \cap X = \emptyset)\}$,
$Q_A = \{W \in [\omega]^\omega : \exists (B_n)_{n<\omega} \subseteq \mathcal{X}_A (\forall n < \omega,\ W \subseteq^* B_n)\}$, and
$D_A = P_A \cup Q_A$.
Then for each $A \in [\omega]^\omega$, D_A is dense open in $[\omega]^\omega$.

Fact. For any ultrafilter $\mathcal{U}$, $\{A \in [[\omega]^\omega]^\omega : \mathcal{U} \cap D_A \neq \emptyset\}$ is stationary.

Thm. If $\mathcal{U} \cap D_A \neq \emptyset$ for club many $A \in [\omega]^\omega$, then $\mathcal{U} <_T \mathcal{U}_{\text{top}}$.

Thm. If $\mathcal{U}$ is a p-point, then $\mathcal{U} \cap D_A \neq \emptyset$ for all $A \in [[\omega]^\omega]^\omega$.

Let $P'_A = \{W \in [\omega]^\omega : \forall X \in \mathcal{X}_A (W \subseteq^* X)\}$. Let $D'_A = P'_A \cup Q_A$. D'_A is dense open in $[\omega]^\omega$.

Fact. If $\mathcal{U} \cap D'_A \neq \emptyset$ for club many A, then $\mathcal{U}$ is a p-point.

Problem. If $\mathcal{U} <_T \mathcal{U}_{\text{top}}$, does it follow that the Tukey type of $\mathcal{U}$ has size $\mathfrak{c}$?

Conjecture. Suppose there is a supercompact cardinal. If $\mathcal{U}$ is selective, then there are exactly 2 Tukey types of ultrafilters in $L(\mathbb{R})[\mathcal{U}]$.

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