(A topic distantly related to) Natural ideals under PFA

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Motivation

(Viale/Weiß): In ZFC there is a naturally-defined ideal on $\wp_{\omega_2}(\theta)$ that:

- is trivial in many models of ZFC;
- when not trivial, has powerful consequences;
- is not trivial when the Proper Forcing Axiom holds.

There are similar ideals which are non-trivial when Martin's Maximum holds and have powerful consequences (Foreman).

Outline

- Forcing Axioms
- Ideals
- Stationary set reflection
  - characterization in terms of ideals whose completeness is ω_2
- Some consistency results
- Open questions

Forcing Axioms

Let Γ be a class of posets.

Definition

$MA(\Gamma)$ means: for every $\mathbb{Q} \in \Gamma$: for every ω_1-sized collection $\mathcal{D}$ of dense subsets of $\mathbb{Q}$, there is a filter $F \subset \mathbb{Q}$ which meets every element of $\mathcal{D}$.

- MA_{ω_1} is $MA(ccc)$
- PFA is $MA(proper)$
- MM is $MA(\text{stationary set preserving posets})$.

Ideals

EXAMPLE 1:

κ regular uncountable. $NS_\kappa = \{A \subset \kappa | A \text{ is nonstationary}\}$.

- dual is the *club filter* (on κ).
- $< \kappa$ complete and normal

EXAMPLE 2 (the one we'll use):

$\wp_{\omega_2}(H_\theta) := \{M \subset H_\theta | |M| < \omega_2 \text{ and } M \cap \omega_2 \in \omega_2\}$.

- If $\mathcal{A} = (H_\theta, \in, ...)$ is structure in countable language, $C_\mathcal{A} := \{M | M \prec \mathcal{A}\}$.
- $B \subset \wp_{\omega_2}(H_\theta)$ is called (weakly) *nonstationary* iff there is a structure $\mathcal{A} = (H_\theta, \in, f_0, f_1, ...)$ such that $B \cap C_\mathcal{A} = \emptyset$.
- $NS \restriction S$ is the collection of nonstationary subsets of S (dual is the club filter).
  - It is $< \omega_2$-complete and normal

Generic ultrapowers

Let I be an ideal over S (so $I \subset \wp(S)$).

$\mathbb{P}_I$ denotes the boolean algebra $\wp(S)/I$ without the 0 element.

(NOTATION: $\Vdash_I$ means $\Vdash_{\mathbb{P}_I}$)

Let G be generic for $\mathbb{P}_I$.

- G is essentially a V-ultrafilter which extends the dual of I.
- Inside $V[G]$ you can define $j : V \to_G ult(V, G)$
- Genericity ensures that G inherits nice properties of I
  - normality
  - completeness (e.g. if $I = NS \restriction \wp_{\omega_2}(H_\theta)$ then j has critical point ω_2)

A few strong properties that ideals may possess

- precipitous ($ult(V, G)$ is wellfounded)
- saturated (that $P(S)/I$ has small chain-condition; implies precipitousness)
- decisive (a portion of j_G is independent of G, and more)

Stationary set reflection

If $S \subset \kappa$ is stationary, we say "S reflects" iff there is some $\gamma < \kappa$ such that $S \cap \gamma$ is stationary in γ.

EXAMPLES:

If κ is measurable then:

- every stationary $S \subset \kappa$ reflects
- $V^{Col(\mu,<\kappa)} \models$ "every stationary subset of $\mu^+ \cap cof(\omega)$ reflects." (at a point of cofinality μ)

Reflection at small cofinalities

Arguments from above yield reflection at the *largest possible cofinality*. Contrast with:

Theorem

(Minor variation of an argument of Foreman): Assume MM and let $\kappa \geq \omega_2$ be regular. There are stationarily many $M \prec H_{\kappa^+}$ such that:

- *$cf(\kappa_M) = \omega_1$, where $\kappa_M := sup(M \cap \kappa)$*
- *For every $R \in M \cap \{stationary\ subsets\ of\ \omega_3 \cap cof(\omega)\}$: R reflects at κ_M.*

Definition

Ref$(3, 0, 1)$: Every stationary subset of S^3_0 reflects at a point of cofinality ω_1.

Reflection at small cofinality

Let $Unif(\wp_{\omega_2}(\omega_3)) :=$ the collection of $M \in \wp_{\omega_2}(\omega_3)$ such that $M \cap \omega_2$ and $sup(M \cap \omega_3)$ both have uncountable cofinality.

Lemma

TFAE:

1. $Ref(3, 0, 1)$
2. *For every stationary* $R \subset S^3_0$ *there is a normal ideal* I_R *over* $Unif(\wp_{\omega_2}(\omega_3))$ *such that* $\Vdash_{I_R}$ *"*$\check{R}$ *remains stationary in* $ult(V, \dot{G})$*"*
3. *For every stationary* $R \subset S^3_0$ *there is a stationary* $S_R \subset Unif(\wp_{\omega_2}(\omega_3))$ *such that* $S_R \Vdash_{NS}$ *"*$\check{R}$ *remains stationary in* $ult(V, \dot{G})$*".*

Some comments

Ways to strengthen the properties of the ideals in that characterization: require

- that R remains stationary in $V[G]$, rather than just in $ult(V, G)$.
- that there is a single ideal which works for all R
- that the ideals be precipitous

At least one of these properties holds in all known models of $Ref(3, 0, 1)$

Consistency strength: lower bounds

Theorem

- *CON(ZFC + Ref(3,0,1))* $\implies$ *CON(ZFC + "almost" a measurable* κ *of Mitchell order* κ^+*)*
- *CON(ZFC + "simultaneous version of Ref(3,0,1)")* $\implies$ *CON(ZFC + there is a* κ *of Mitchell order* κ^+*)*

However, if in addition there is a precipitous ideal on ω_2 then there is an inner model with a Woodin cardinal (due to theorem of Schindler).

Consistency strength: upper bounds

Known models of $Ref(3,0,1)$:

- Any model of $MA^+(\{Col(\omega_1,\omega_3)\})$.
- Any model of MM gives highly simultaneous version
- $V^{Col(\omega_1,<\kappa)}$ where κ is a quasicompact cardinal
  - Gives simultaneous versions of $Ref(3,0,1)$
  - The forcings associated with the ideals I_R are *proper*
    - so you also get precipitousness and preservation of stationary sets in $V[G]$ rather than just in $ult(V,G)$.

Open Problems

What is the consistency strength of:

1. $Ref(3, 0, 1)$?
2. $Ref(3, 0, 1)$ + "there is a precipitous ideal on ω_2"?
3. $Ref(3, 0, 1)$ + "there is an ideal on ω_2 whose forcing is proper"?