Every set of first-order formulas is equivalent to an independent set

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Abstract

A set of first-order formulas, whatever the cardinality of the set of symbols, is equivalent to an independent set.

In the following we work with classical (first-order) logic. The Axiom of Choice is assumed[1].

Definition 1. *Two sets of formulas are* equivalent*, if any formula of the one set is a consequence of the other and conversely. (Equiv. they have the same models).*

A set of formulas T is independent*, if for all $\phi \in T$,*

$$T \setminus \{\phi\} \nvDash \phi.$$

(Equiv. there is a model for $(T \setminus \{\phi\}) \cup \{\neg\phi\}$).

Theorem 2. *(Tarski) Every countable set of formulas is equivalent to an independent set.*

Proof. Let $T = \{\phi_0, \phi_1, \ldots\}$ a countable set of formulas. Without loss of generality there are no valid formulas in T.

Define inductively

- $\psi'_0 = \phi_0$ and
- $\psi'_{n+1} =$ least ϕ_m such that $\psi'_0, \ldots, \psi'_n \nvDash \phi_m$.

It is not hard to see that T is equivalent to the set $\{\psi'_n | n \in \omega\}$. If this set is finite, then T is equivalent to its conjunction. So, assume it is infinite and define

- $\psi_0 = \psi'_0$ and
- $\psi_{n+1} = \bigwedge_{m \leq n} \psi'_m \to \psi'_{n+1}$.

Since $\psi'_0, \ldots, \psi'_n \nvDash \psi'_{n+1}$, there is a model $\mathcal{M}$ that satisfies $\psi'_0, \ldots, \psi'_n$ and $\neg\psi'_{n+1}$. Then $\mathcal{M} \nvDash \psi_{n+1}$, while $\mathcal{M} \models \psi_m$ for $m < n+1$. For $m > n+1$, since $\mathcal{M}$ doesn't satisfy the antecedent of ψ_m, it trivially satisfies ψ_m. Therefore

$$\mathcal{M} \models \bigwedge_{m \neq n+1} \psi_m \wedge \neg\psi_{n+1},$$

[1] This article follows closely the arguments of Reznikoff in [1] (in French), but without being a word-by-word translation.

witnessing the fact that $\{\psi_n | n \in \omega\}$ is an independent set.

Moreover, it is an easy induction to see that the sets $\{\psi'_n | n \in \omega\}$ and $\{\psi_n | n \in \omega\}$ are equivalent, which finishes the proof. $\square$

Lemma 3. *Let C, D be two disjoint sets such that:*

- $|D| \leq |C|$ *and*
- *For all $\phi \in C$, $(C \cup D) \setminus \{\phi\} \nvDash \phi$.*

Then $C \cup D$ is equivalent to an independent set.

Proof. Let f be an injection from D to C. Then

$$\{\psi \wedge f(\psi) | \psi \in D\} \cup (C \setminus f(D))$$

is an independent set equivalent to $C \cup D$. $\square$

Now, let T be a set of formulas and without loss of generality, there are no valid formulas in T (valid formulas are equivalent to the empty set). For a formula $\phi \in T$, denote by $S(\phi)$ the set of symbols that appear in ϕ and let

$$S = \bigcup_{\phi \in T} S(\phi).$$

Without loss of generality S is infinite. Otherwise T would be at most countable and equivalent to an independent set by Theorem 2. If S is infinite, then S and T have the same cardinality and let

$$|S| = |T| = \kappa \geq \omega.$$

We partition T into sets T_α, $\alpha < \kappa$ as follows:

For $\alpha = 0$, fix a formula $\phi_0 \in T$ and let $T_0 = \{\psi \in T | S(\psi) \subset S(\phi_0)\}$. For $0 < \alpha < \kappa$, assume that we have defined ϕ_β and T_β, for all $\beta < \alpha$. By a cardinality argument,

$$S \setminus \bigcup_{\beta < \alpha} S(\phi_\beta) \neq \emptyset.$$

Therefore, there exists a formula ϕ_α that contains a symbols that doesn't appear in any of the ϕ_β, $\beta < \alpha$. Define

$$N_\alpha = S(\phi_\alpha) \setminus \bigcup_{\beta < \alpha} S(\phi_\beta),$$

the set of new symbols that appear in ϕ_α. Then $N_\alpha \neq \emptyset$ and define

$$T_\alpha = \{\psi \in T | S(\psi) \subset \bigcup_{\beta \leq \alpha} S(\phi_\beta) \text{ and } S(\psi) \cap N_\alpha \neq \emptyset\},$$

i.e. T_α is the set of formulas in which appears one of the new symbols in N_α.

Then $T = \bigcup_{\alpha < \kappa} T_\alpha$ and the different T_α's are disjoint.

Definition 4. *If $\psi \in T_\alpha$ and $S(\psi) \cap N_\beta \neq \emptyset$, for $\beta \leq \alpha$, denote this by $\beta | \psi$. In particular, for $\psi \in T_\alpha$, $\alpha | \psi$.*

If $\beta | \phi_\alpha$, with $\beta < \alpha$, denote this by $\beta || \phi_\alpha$.

Observe here that the first definition is for any $\psi \in T$, while the second one is only for the ϕ_α's. Also, for any $\psi \in T$, there are only finitely many β's with $\beta|\psi$.

Now let

$$\psi_\alpha = \bigwedge_{\beta||\phi_\alpha} \phi_\beta \to \phi_\alpha,$$

if there exists such a β. Otherwise, let $\psi_\alpha = \phi_\alpha$. Denote by C the set of all the ψ_α's.

On the other hand, for $\phi \neq \phi_\alpha$, all $\alpha < \kappa$, let

$$\phi' = \bigwedge_{\beta|\phi} \phi_\beta \to \phi.$$

As we noted, there is always such a β. Denote

$$D_\alpha = \{\phi' = \bigwedge_{\beta|\phi} \phi_\beta \to \phi | \phi \in T_\alpha \text{ and } x \neq \phi_\alpha\}$$

and let $D = \bigcup_\alpha D_\alpha$. ($D$ may be empty. We can not exclude this possibility).

Lemma 5. *Suppose that T satisfies the following condition:*

$$(\star) \qquad \textit{If } \psi, \phi_1, \ldots, \phi_n \in T \textit{ and } S(\psi) \nsubseteq \bigcup_{i=1}^{n} S(\phi_i), \textit{ then } \{\phi_1, \ldots, \phi_n\} \nvDash \psi.$$

Then C and D as defined above, satisfy the conditions of Lemma 3 and T is equivalent to an independent set.

Proof. First of all it is clear that $|C| = \kappa \geq |D|$. It also follows easily by induction on $\alpha < \kappa$ that the set $\bigcup_{\beta<\alpha} T_\beta$ is equivalent to the set $\bigcup_{\beta<\alpha}(\{\psi_\alpha\} \cup D_\alpha)$. This implies that T is equivalent to $C \cup D$ and it suffices to verify that for $\psi_\alpha \in C$, ψ_α is not a consequence of the other elements of $C \cup D$:

Let $\psi_\alpha = \bigwedge_{\beta||\phi_\alpha} \phi_\beta \to \phi_\alpha$. Then the elements of $C \cup D$ different than ψ_α are of the form $\psi_\gamma = \bigwedge_{\beta||\phi_\gamma} \phi_\beta \to \phi_\gamma$, with $\gamma \neq \alpha$, or of the form $\phi' = \bigwedge_{\beta|\phi} \phi_\beta \to \phi$, with $\phi \neq \psi_\alpha$, for all $\alpha < \kappa$.

Consider the implication

$$\models \left(\bigwedge_{i=1 \alpha_i \neq \alpha}^{m} \psi_{\alpha_i} \bigwedge_{j=1}^{n} \phi'_j \right) \to \psi_\alpha.$$

Assume that all $\psi_{\alpha_1}, \ldots, \psi_{\alpha_m}$ are different than ψ_α and that $\alpha \nmid \phi_{\alpha_i}$, for $i = 1, \ldots, p$, while $\alpha|\phi_{\alpha_i}$, for $i = p+1, \ldots, m$. Similarly, assume that $\phi'_1, \ldots, \phi'_q$ are such that $\alpha \nmid \phi_j$, $j = 1, \ldots, q$, while for $\phi'_{q+1}, \ldots, \phi'_n$, $\alpha|\phi_j$, $j = q+1, \ldots, n$.

Then

$$S(\phi_\alpha) \nsubseteq \bigcup_{i=1}^{p} S(\phi_{\alpha_i}) \text{ and } S(\phi_\alpha) \nsubseteq \bigcup_{j=1}^{q} S(\phi_j).$$

Also, by the definition of ϕ_α,

$$S(\phi_\alpha) \nsubseteq \bigcup_{\beta||\phi_\alpha} S(\phi_\beta).$$

By ($\star$), there is a model $\mathcal{M}$ in which ϕ_α is false, while all of the $\phi_{\alpha_1}, \dots, \phi_{\alpha_p}$, $\phi_1, \dots, \phi_q$ and $\{\phi_\beta | \ \beta||\phi_\alpha\}$ are true. Then ψ_α is false in $\mathcal{M}$, while $\psi_{\alpha_1}, \dots, \psi_{\alpha_p}, \phi'_1, \dots, \phi'_q$ are true.

In addition, for $i = p+1, \dots, m$, ϕ_α is among the ϕ_γ's in the conjunction of $\psi_{\alpha_i} = \bigwedge_{\gamma||\phi_{\alpha_i}} \phi_\gamma \to \phi_{\alpha_i}$ and the same is true for the conjunction of $\phi'_j = \bigwedge_{\gamma|\phi_j} \phi_\gamma \to \phi_j$, for $j = q+1, \dots, n$. Since ϕ_α is false in $\mathcal{M}$, then $\psi_{\alpha_{p+1}}, \dots, \psi_{\alpha_m}, \phi'_{q+1}, \dots, \phi'_n$ are trivially true. This proves that there is a model $\mathcal{M}$ that satisfies all the $\psi_{\alpha_1}, \dots, \psi_{\alpha_m}, \phi'_1, \dots, \phi'_n$, but which does not satisfy ψ_α. In other words, ψ_α can not be a consequence of other elements of $C \cup D$. □

What remains is to prove that T can be taken to satisfy ($\star$). We use Craig's Interpolation Theorem which me mention without proof.

Theorem 6. *(Craig) If $\psi \models \phi$, then there is a formula τ such that*

- *$\psi \models \tau$ and $\tau \models \phi$, and*
- *the non-logical symbols of τ appears in both ψ and ϕ.*

τ is called the interpolant *between ψ and ϕ.*

Lemma 7. *Every set of non-valid formulas T is equivalent to a set of formulas that satisfies ($\star$).*

Proof. Let

$$E_1 = \{\phi | \ T \models \phi \text{ and } |S(\phi)| = 1\}$$

and

$$E_n = \{\phi | \ T \models \phi, \bigcup_{m<n} E_m \not\models \phi \text{ and } |S(\phi)| = n\}.$$

It is immediate that $T' = \cup_n E_n$ is equivalent to T. Let $\psi, \phi_1, \dots, \phi_n \in T'$ such that $S(\psi) \nsubseteq \bigcup_{i=1}^n S(\phi_i)$. If we assume that

$$\{\phi_1, \dots, \phi_n\} \models \psi,$$

then by Craig's Interpolation Theorem, there is a τ such that

- $\{\phi_1, \dots, \phi_n\} \models \tau$ and $\tau \models \psi$, and
- $S(\tau) \subset S(\psi) \cap (\cup_{i=1}^n S(\phi_i))$.

By the assumption on ψ, it must be $S(\tau) \subsetneq S(\psi)$ and $\psi \in T'$ would be a consequence of τ with $T \models \tau$ and $|S(\tau)| < |S(\psi)|$, contradicting the definition of T'.

Therefore, T' satisfies ($\star$). □

Putting all the previous lemmas together we conclude

Theorem 8. *(Reznikoff) Every set of formulas is equivalent to an independent set.*

References

[1] M. I. Reznikoff, *Tout ensemble de formules de la logique classique est equivalent á un ensemble independant*, C.R. Acad. Sc. Paris, **260**, 2385-2388 (1965).