Qualifying Examination
LOGIC
Spring 2010

Try to answer all questions.

You may (and you will need to) use some of the "big" theorems of logic (the Gödel Completeness and Incompleteness Theorems, Tarski's Theorem, Kleene's Normal Form Theorem, the Condensation Lemma, etc.), and when you do, make sure you quote them correctly.

You may also assume that Peano arithmetic (PA) is sound (i.e., its theorems are all true in its standard interpretation) and that Zermelo-Fraenkel Set Theory with Choice (ZFC) is consistent.

Problem 1. Let $\mathrm{Fin} = \{e \mid W_e \text{ is finite}\}$ be the set of (standard) codes of all recursively enumerable sets which are finite.

(1a) Classify Fin in the arithmetical hierarchy.

(1b) Prove that there is no recursive partial function $f(e)$ which gives an upper bound for the members of every finite, r.e. set, i.e., such that

$$\text{if } W_e \text{ is finite, then } f(e)\downarrow \,\&(\forall x)[x \in W_e \implies x \leq f(e)].$$

Problem 2. A collection F of subsets of ω has the *reduction* property if for any $A, B \in F$, there exist $A^*, B^* \in F$ so that

$$A^* \subseteq A,\ B^* \subseteq B,\ A^* \cup B^* = A \cup B,\ \text{and } A^* \cap B^* = \emptyset.$$

For each total recursive function f, let

$$A_f = \{n \mid W_e \text{ is finite}\},$$

and let F be the collection of sets A_f. Prove that F has the reduction property.

Problem 3. True or false (and you must prove your answer): there is a non-standard model

$$\mathfrak{N}^* = (\mathbb{N}^*, 0, 1, +^*, \cdot^*)$$

of Peano arithmetic PA such that the set of its true sentences

$$\mathrm{True}(\mathfrak{N}^*) = \{\theta \mid \mathfrak{N}^* \models \theta\}$$

is arithmetical.

Problem 4. Let

$$\Box\theta :\equiv (\exists y)\mathbf{Proof}_{\mathrm{PA}}(\ulcorner\theta\urcorner, y), \mathbf{Proof}_{\mathrm{PA}}(\ulcorner\theta\urcorner, y),$$

where $\ulcorner\theta\urcorner$ is the numeral of the code (Gödel number) of θ and $\mathbf{Proof}_{\mathrm{PA}}(\mathsf{v}_1, \mathsf{v}_2)$ numeralwise expresses the relation

$$\mathrm{Proof}_{\mathrm{PA}}(e, y) \iff e \text{ is the code of a sentence } \theta$$
$$\text{and} y \text{ is the code of a proof of } \theta \text{ in PA.}$$

We also let $\mathsf{Con}_{\mathrm{PA}}$ be the formal sentence which expresses the consistency of PA,

$$\mathsf{Con}_{\mathrm{PA}} :\equiv \neg(\exists y)\mathbf{Proof}_{\mathrm{PA}}(\ulcorner 0 = 1\urcorner, y).$$

For each of the following four sentences, determine whether or it is *provable in* PA. You must prove your answers by reference to standard results).

(i) $\mathsf{Con}_{\mathrm{PA}} \to \Box\mathsf{Con}_{\mathrm{PA}}$.
(ii) $\mathsf{Con}_{\mathrm{PA}} \to \neg\Box\mathsf{Con}_{\mathrm{PA}}$.
(iii) $\Box\mathsf{Con}_{\mathrm{PA}} \to \mathsf{Con}_{\mathrm{PA}}$.
(iv) $\neg\Box\mathsf{Con}_{\mathrm{PA}} \to \mathsf{Con}_{\mathrm{PA}}$.

Problem 5. Work in L. For each $\alpha < \omega_1$, let $\beta(\alpha) \geq \alpha$ be least so that $L_{\beta(\alpha)+1}$ has a surjection of ω onto α. Define $X_\alpha = \{A \subseteq \alpha \mid A \in L_{\beta(\alpha)}\}$. Prove that for every $A \subseteq \omega_1$ there is a club C in ω_1 so that $A \cap \alpha \in X_\alpha$ for all $\alpha \in C$.

Problem 6. Let G be an infinite graph. Let k be a finite number. Suppose that every finite subgraph of G is k-colorable. Is G k-colorable? prove your answer.

Problem 7. Let M be a transitive model of ZFC^-, i.e., ZFC without the powerset axiom. Determine whether each of the following formulas is absolute from M to V, and prove your answers.

(a) x is a countable set.
(b) x is an uncountable set.
(c) x belongs to the transitive closure of y.

Problem 8. Let T be the theory of $M_0 = (\mathbb{Z}, S)$, where $S(n) = n + 1$.

(8a) Show that up to isomorphism there are only countably many countable models of T, and that T is categorical in every uncountable cardinality.

(8b) Give an example of a saturated countable model of T, and prove that it is saturated.

(8c) What are the complete 1-types and 2-types of T? Prove your answer.
