Additional exercises for Math 220C.

Ex. A. (Principle of induction on a wellfounded relation.) Let $r$ be a wellfounded relation in a set $a$. For each $x \in a$ let $\text{pre}(x)$ denote the set of predecessors of $x$ in $r$, namely $\{z \in a \mid \langle z, x \rangle \in r\}$. Let $p$ be a subset of $a$ with the property

$$(\forall x \in a)(\text{pre}(x) \subset p \rightarrow x \in p).$$

Prove that $p = a$.

Ex. B. (Schema of definition by recursion on a wellfounded relation.) Let $F: V \rightarrow V$ be a class function. Prove that for each set $a$ and each wellfounded relation $r$ in $a$ there exists a unique set function $g$ so that

1. $\text{dom}(g) = a$; and
2. $(\forall x \in a) g(x) = F(g|\text{pre}(x))$.

($r$ is not assumed to be transitive, and your proof should take account of this.)

Ex. C. Let $r$ be a relation in a set $a$. A rank function for $r$ is a (set) function $f: a \rightarrow \text{ON}$ so that

1. $(\forall u \in a)(\forall w \in a)(\langle u, w \rangle \in r \rightarrow f(u) < f(w))$.

Prove that if there is a rank function for $r$ then $r$ is wellfounded.

Ex. D. A rank function is tight if for every $w \in a$,

2. $f(w)$ is the least ordinal greater than all ordinals in $f''\{u \in a \mid \langle u, w \rangle \in r\}$.

Prove that if $r$ is wellfounded then there is a tight rank function for $r$.

Ex. E. Prove that the range of a tight rank function $f$ is an ordinal.

Ex. F. Let $r$ and $s$ be relations in sets $x$ and $y$ respectively. We say that $(x, r)$ is isomorphic to $(y, s)$ if there is a function $f: x \rightarrow y$ so that

- $f$ is one-to-one and onto; and
- $(\forall u \in x)(\forall w \in x)(\langle u, w \rangle \in r \leftrightarrow \langle f(u), f(w) \rangle \in s)$.

We say that $(x, r)$ is isomorphic to an ordinal $\alpha$ iff $(x, r)$ is isomorphic to $(\alpha, <\restriction(\alpha \times \alpha))$.

Suppose that $r$ is a wellordering of $x$. Prove that $(x, r)$ is isomorphic to a unique ordinal. (Hint: for existence use D and E.)

Ex. G. Show, without using choice, that $\langle \forall x \rangle(x \text{ can be wellordered})$ implies the axiom of choice.

Ex. H. For a set $x$, let $\mathcal{P}_{\text{wo}}(x) = \{z \subset x \mid z \text{ can be wellordered}\}$. (Assuming the axiom of choice, $\mathcal{P}_{\text{wo}}(x) = \mathcal{P}(x)$. But without AC the two can be different.) Prove, without using the axiom of choice, that $\mathcal{P}_{\text{wo}}(x) \not\subseteq x$.

Ex. I. (Uses choice.) For cardinals $\lambda \leq \kappa$ let $\mathcal{P}_\lambda(\kappa) = \{x \in \mathcal{P}(\kappa) \mid \text{card}(x) = \lambda\}$. For $\kappa$ infinite, show that $\mathcal{P}_\lambda(\kappa) \approx \lambda \kappa$. 1
Ex. J. (Uses Choice.) Let \( \langle a_n \mid n < \omega \rangle \) be a sequence of sets. Suppose that for each \( n < \omega \), \( a_n \subset a_{n+1} \) and \( a_{n+1} \not\subseteq a_n \) (so that the sets are increasing in size). Let \( a = \bigcup_{n<\omega} a_n \). Prove that \( a \not\subseteq a \). (Hint: It is more convenient to show that there is no surjection \( g : a \to \omega a \).)

Ex. K. Prove that “\( x \) is an ordinal” is absolute for any transitive \( M \subset WF \).

Ex. L. Let \( M \subset WF \) be a transitive class. Suppose that (the necessary finite part of) \( ZF - \text{Powerset} \) is true in \( M \). Prove that “\( r \) is a wellordering of \( x \)” reflects from \( M \) to \( V \).

Hint: Look at Exercise F. Note that only (a finite part of) \( ZF - \text{Powerset} \) is used there. Conclude that the statement of Exercise F holds in \( M \) and use this in solving L.

Ex. M. Let \( M \subset WF \) be a transitive class. Prove that the statement “\( r \) is an illfounded relation in \( x \)” reflects from \( M \) to \( V \). (You may assume that being a relation is absolute.)

Ex. N. Let \( H \) be a set, contained in \( WF \), but not necessarily transitive. \( \in \restriction H \times H \) is then a wellfounded relation. By recursion on this relation (see Exercise B) define a function \( g \) with the properties:

1. \( \text{dom}(g) = H \); and
2. \( (\forall x \in H) \ g(x) = \{ g(u) \mid u \in x \cap H \} \).

Let \( M = \text{ran}(g) \). (This \( M \) is called the Mostowski collapse of \( H \). \( g \) is called the collapse embedding.) Prove that \( M \) is transitive. Assuming that the axiom of extensionality is true in \( H \), prove that \( g \) is one-one. Conclude that \( g \) is then an isomorphism between \( (H, \in \restriction H \times H) \) and the transitive \( (M, \in \restriction M \times M) \).

Ex. O. Suppose that \( M \subset WF \) is transitive and layered, and that \( \omega \in M \). Suppose Comprehension is true in \( M \). Prove that all axioms of \( ZF \) are true in \( M \).

Ex. P. Prove that the class function \( z \mapsto \text{FOD}(z) \) is absolute for transitive \( M \subset WF \) which satisfy \( ZF - \text{Powerset} \).

Ex. Q. The cofinality of a limit ordinal \( \delta \) is the least \( \tau \) so that there is a function \( f : \tau \to \delta \) with \( \text{ran}(f) \) unbounded in \( \delta \). A limit ordinal \( \delta \) is regular if \( \text{cof}(\delta) = \delta \). Prove the following:

1. For each \( \delta \), \( \text{cof}(\delta) \) is regular.
2. For limit \( \lambda \), \( \text{cof}(\aleph_\lambda) = \text{cof}(\lambda) \).
3. (Uses Choice.) For successor \( \alpha \), \( \aleph_\alpha \) is regular.

Ex. R. (Uses Choice.) Let \( \kappa \) be an infinite cardinal. Prove that \( \kappa^{\text{cof}(\kappa)} > \kappa \).

Hint: When \( \text{cof}(\kappa) = \omega \) this is a consequence of Exercise J.

Ex. S. \( C \subset \kappa \) is club in \( \kappa \) if it is unbounded in \( \kappa \), and every limit point \( \alpha < \kappa \) of \( C \) belongs to \( C \).

1. Suppose \( \text{cof}(\kappa) > \omega \). Prove that the intersection of two clubs in \( \kappa \) is club in \( \kappa \).
2. Suppose \( \kappa \) is regular and uncountable. Let \( f : \kappa \to \kappa \). Prove that the set \( \{ \alpha < \kappa \mid f^n\alpha \subset \alpha \} \) is club in \( \kappa \).
**Ex. T.** Let $\alpha$ be infinite. Prove that $\text{card}(L_\alpha) = \text{card}(\alpha)$.

**Ex. U.** Let $\kappa$ be a regular uncountable cardinal. Prove that all axioms of $\text{ZF} - \text{Powerset}$ hold in $L_\kappa$. 