# Additional exercises for Math 220C.

**Ex. A.** (Principle of induction on a wellfounded relation.) Let $r$ be a wellfounded relation in a set $a$. For each $x \in a$ let $\mathrm{pre}(x)$ denote the set of predecessors of $x$ in $r$, namely $\{z \in a \mid \langle z, x\rangle \in r\}$. Let $p$ be a subset of $a$ with the property

(∗) $(\forall x \in a)(\mathrm{pre}(x) \subset p \to x \in p)$.

Prove that $p = a$.

**Ex. B.** (Schema of definition by recursion on a wellfounded relation.) Let $F\colon V \to V$ be a class function. Prove that for each set $a$ and each wellfounded relation $r$ in $a$ there exists a unique set function $g$ so that

(1) $\mathrm{dom}(g) = a$; and
(2) $(\forall x \in a)\ g(x) = F(g\restriction \mathrm{pre}(x))$.

($r$ is *not* assumed to be transitive, and your proof should take account of this.)

**Ex. C.** Let $r$ be a relation in a set $a$. A **rank function** for $r$ is a (set) function $f\colon a \to \mathrm{ON}$ so that

(1) $(\forall u \in a)(\forall w \in a)(\langle u, w\rangle \in r \to f(u) < f(w))$.

Prove that if there is a rank function for $r$ then $r$ is wellfounded.

**Ex. D.** A rank function is **tight** if for every $w \in a$,

(2) $f(w)$ is the *least* ordinal greater than all ordinals in $f''\{u \in a \mid \langle u, w\rangle \in r\}$.

Prove that if $r$ is wellfounded then there is a tight rank function for $r$.

**Ex. E.** Prove that the range of a tight rank function $f$ is an ordinal.

**Ex. F.** Let $r$ and $s$ be relations in sets $x$ and $y$ respectively. We say that $(x, r)$ is **isomorphic** to $(y, s)$ if there is a function $f\colon x \to y$ so that

- $f$ is one-to-one and onto; and
- $(\forall u \in x)(\forall w \in x)(\langle u, w\rangle \in r \leftrightarrow \langle f(u), f(w)\rangle \in s)$.

We say that $(x, r)$ is isomorphic to an ordinal $\alpha$ iff $(x, r)$ is isomorphic to $(\alpha, <\restriction(\alpha \times \alpha))$.

Suppose that $r$ is a wellordering of $x$. Prove that $(x, r)$ is isomorphic to a unique ordinal. (Hint: for existence use D and E.)

**Ex. G.** Show, without using choice, that $(\forall x)(x$ can be wellordered$)$ implies the axiom of choice.

**Ex. H.** For a set $x$, let $\mathcal{P}_{\mathrm{wo}}(x) = \{z \subset x \mid z \text{ can be wellordered}\}$. (Assuming the axiom of choice, $\mathcal{P}_{\mathrm{wo}}(x) = \mathcal{P}(x)$. But without AC the two can be different.) Prove, without using the axiom of choice, that $\mathcal{P}_{\mathrm{wo}}(x) \not\preceq x$.

**Ex. I.** (Uses choice.) For cardinals $\lambda \le \kappa$ let $\mathcal{P}_\lambda(\kappa) = \{x \in \mathcal{P}(\kappa) \mid \mathrm{card}(x) = \lambda\}$. For $\kappa$ infinite, show that $\mathcal{P}_\lambda(\kappa) \approx {}^{\lambda}\kappa$.

**Ex. J.** (Uses Choice.) Let $\langle a_n \mid n < \omega \rangle$ be a sequence of sets. Suppose that for each $n < \omega$, $a_n \subset a_{n+1}$ and $a_{n+1} \not\preceq a_n$ (so that the sets are increasing in size). Let $a = \bigcup_{n<\omega} a_n$. Prove that ${}^{\omega}a \not\preceq a$. (Hint: It is more convenient to show that there is no surjection $g \colon a \to {}^{\omega}a$.)

**Ex. K.** Prove that "$x$ is an ordinal" is absolute for any transitive $M \subset \mathrm{WF}$.

**Ex. L.** Let $M \subset \mathrm{WF}$ be a transitive class. Suppose that (the necessary finite part of) $\mathsf{ZF}$ − Powerset is true in $M$. Prove that "$r$ is a wellordering of $x$" reflects from $M$ to $V$.

Hint: Look at Exercise F. Note that only (a finite part of) $\mathsf{ZF}$ − Powerset is used there. Conclude that the statement of Exercise F holds in $M$ and use this in solving L.

**Ex. M.** Let $M \subset \mathrm{WF}$ be a transitive class. Prove that the statement "$r$ is an illfounded relation in $x$" reflects from $M$ to $V$. (You may assume that being a relation is absolute.)

**Ex. N.** Let $H$ be a set, contained in WF, but not necessarily transitive. $\in\restriction H \times H$ is then a wellfounded relation. By recursion on this relation (see Exercise B) define a function $g$ with the properties:

(1) $\mathrm{dom}(g) = H$; and
(2) $(\forall x \in H)\, g(x) = \{g(u) \mid u \in x \cap H\}$.

Let $M = \mathrm{ran}(g)$. (This $M$ is called the **Mostowski collapse** of $H$. $g$ is called the **collapse embedding**.) Prove that $M$ is transitive. Assuming that the axiom of extensionality is true in $H$, prove that $g$ is one-one. Conclude that $g$ is then an isomorphism between $(H, \in\restriction H \times H)$ and the transitive $(M, \in\restriction M \times M)$.

**Ex. O.** Suppose that $M \subset \mathrm{WF}$ is transitive and layered, and that $\omega \in M$. Suppose Comprehension is true in $M$. Prove that all axioms of $\mathsf{ZF}$ are true in $M$.

**Ex. P.** Prove that the class function $z \mapsto \mathrm{FOD}(z)$ is absolute for transitive $M \subset WF$ which satisfy $\mathsf{ZF}{-}\mathsf{Powerset}$.

**Ex. Q.** The *cofinality* of a limit ordinal $\delta$ is the least $\tau$ so that there is a function $f \colon \tau \to \delta$ with $\mathrm{ran}(f)$ unbounded in $\delta$. A limit ordinal $\delta$ is *regular* if $\mathrm{cof}(\delta) = \delta$. Prove the following:

(1) For each $\delta$, $\mathrm{cof}(\delta)$ is regular.
(2) For limit $\lambda$, $\mathrm{cof}(\aleph_\lambda) = \mathrm{cof}(\lambda)$.
(3) (Uses Choice.) For successor $\alpha$, $\aleph_\alpha$ is regular.

**Ex. R.** (Uses Choice.) Let $\kappa$ be an infinite cardinal. Prove that $\kappa^{\mathrm{cof}(\kappa)} > \kappa$.

Hint: When $\mathrm{cof}(\kappa) = \omega$ this is a consequence of Exercise J.

**Ex. S.** $C \subset \kappa$ is *club* in $\kappa$ if it is unbounded in $\kappa$, and every limit point $\alpha < \kappa$ of $C$ belongs to $C$.

(1) Suppose $\mathrm{cof}(\kappa) > \omega$. Prove that the intersection of two clubs in $\kappa$ is club in $\kappa$.
(2) Suppose $\kappa$ is regular and uncountable. Let $f \colon \kappa \to \kappa$. Prove that the set $\{\alpha < \kappa \mid f''\alpha \subset \alpha\}$ is club in $\kappa$.

**Ex. T.** Let $\alpha$ be infinite. Prove that $\operatorname{card}(L_\alpha) = \operatorname{card}(\alpha)$.

**Ex. U.** Let $\kappa$ be a regular uncountable cardinal. Prove that all axioms of $\mathsf{ZF}-\mathsf{Powerset}$ hold in $L_\kappa$.