Regularity of one-phase Stefan problem near Lipschitz initial data

Sunhi Choi* and Inwon C. Kim†

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Abstract

In this paper we show that, starting from Lipschitz initial free boundary with small Lipschitz constant, the solution of the one-phase Stefan problem (ST) immediately regularizes and is smooth in space and time, for a small positive time.

Consider $u_0(x) : I\!R^n \to [0, \infty)$. The one-phase Stefan problem is given by

$$(ST) \qquad \begin{cases} u_t - \Delta u = 0 & \text{in} \quad \{u > 0\} \\ u_t = |Du|^2 & \text{on} \quad \partial\{u > 0\} \\ u(\cdot, 0) = u_0 \end{cases}$$

The classical one-phase Stefan problem describes the phase transition between solids and liquids, for example melting of the ice ([M],[R]). In this setting u describes the temperature of the liquid, and the region $\{u = 0\}$ describes the unmelted region of ice. The main interest is then on the behavior of the *free boundary* $\partial\{u > 0\}$. The second condition of (ST) implies that the normal velocity $V_{x,t}$ at each free boundary point $(x, t) \in \partial\{u > 0\}$ is given by

$$V_{x,t} = |Du|(x, t) = Du(x, t) \cdot \nu_{x,t},$$

where $\nu_{x,t}$ denotes the spatial unit normal vector of $\partial\{u > 0\}$ at (x, t), pointing inward with respect to the *positive phase* $\{u > 0\}$.

*Department of Mathematics, University of Arizona, Tucson, AZ 85721. Partially supported by NSF 0713598

†Department of Mathematics, UCLA, LA CA 90095. Partially supported by NSF 0700732

In general, solutions of (ST) develop singularities in time, and much of work has been done to prove local regularity of "weak" solutions of (ST). Here we use the notion of viscosity solution (see [K] for its definition, existence and uniqueness). In [C1], [ACS2], [ACS3] and [CPS], local regularity of the viscosity solutions and their free boundaries has been obtained in neighborhoods away from the initial time and the initial free boundary position (see the discussion in this section). On the other hand, the free boundary behavior is more complicated near the initial time. It is shown in [CK] and [King] that if Γ_0 is locally a sharp wedge of angle $\theta \leq \theta_0$, then the wedge on the free boundary persists for a finite *waiting* time (also see Remark 4 below). Furthermore, the threshold angle θ_0 depends on the profile of u_0.

Our main result states that when Γ_0 has no sharp corner, then the free boundary immediately regularizes after $t = 0$ and the free boundary speed averages out in proportion to the distance it has moved. For a rigorous statement, first we introduce a series of assumptions on the initial data:

(I-a) $\Omega_0 = \{u_0 > 0\}$ satisfies

$$\Omega_0 \cap B_2(0) = \{(x', x_n) \in I\!R^{n-1} \times I\!R : |x'| < 2, x_n \leq f(x')\} \cap B_2(0)$$

where f satisfies $f(0) = 0$ and $|f(x') - f(y')| \leq L|x' - y'|$.

For a locally Lipschitz domain such as Ω_0, there exist growth rates $0 < \beta < 1 < \alpha$ such that the following holds: if H is a positive harmonic function in $\Omega_0 \cap B_2(0)$ with Dirichlet condition on $\Gamma_0 \cap B_2(0)$ and with value 1 at $-e_n$, then for $x \in \Gamma_0 \cap B_1(0)$ and $x - se_n \in B_1(0)$

$$s^\alpha \leq H(x - se_n) \leq s^\beta. \tag{0.1}$$

Now we precisely describe the range of L:

(I-b) $L < L_n$ for a sufficiently small dimensional constant L_n so that

$$1/2 < \beta \leq 1 \leq \alpha < 2 \text{ and } \alpha < 2\beta.$$

Observe that if L is close to 0, then α and β are close to 1 and we obtain the above inequalities.

The remaining conditions are on the regularity of u_0:

(I-c) $-C_0 \leq \Delta u_0 \leq C_0$ in $\Omega_0 \cap B_2(0)$, (I-d) $u_0(x - de_n) \geq d^{2-\delta}$ with $\delta > 0$ for $x \in \Gamma_0 \cap B_2(0)$ and for $0 < d \leq 1$.

Now we are ready to state our main theorem. For a nonnegative function $u(x,t) : I\!R^n \times [0,\infty) \to I\!R$, let us denote

$$\Omega_t(u) = \{x : u(x,t) > 0\}, \quad \Omega(u) = \{(x,t) : u(x,t) > 0\}$$

and

$$\Gamma_t(u) = \partial\Omega_t(u), \quad \Gamma(u) = \partial\Omega(u).$$

Theorem 0.1. *Suppose u is a solution of (ST) in $B_2(0) \times [0,1]$ with the initial data u_0, where u_0 satisfies (Ia)-(Id) in $B_2(0)$ with $u_0(-e_n) = 1$. If $\sup_{B_2(0)\times[0,1]} u \leq M$, then there exists a small $s > 0$ depending on C_0, δ, M and n such that u and $\Gamma(u)$ solves (ST), in the classical sense, in $B_s(0)$. Precisely,*

(A) The free boundary $\Gamma(u)$ is C^1 in space and time in $\{(x,t) : x \in \Gamma_t(u) \cap B_s(0)\}$, $|Du|$ exists on $\Gamma(u)$ as a limit from the positive region $\Omega(u)$, and the spatial normal is continuous in space and time.

(B) $\Gamma_t(u)$ is a Lipschitz graph with respect to e_n with Lipschitz constant $L' < L_n$ in $B_s(0)$.

(C) If $x \in \Gamma_0(u) \cap B_s(0)$ and $x + de_n \in \Gamma_t(u) \cap B_s(0)$, then

$$C^{-1}|Du(x - de_n, 0)| \leq |Du(x + de_n, t)| = V_{x+de_n,t} \leq C|Du(x - de_n, 0)|$$

where C depends on n and M. Hence

$$\frac{d}{t} \sim |Du(x + de_n, t)| \sim \frac{u(x - de_n, 0)}{d}.$$

Our main theorem states that if the initial data has bounded Laplacian with sub-quadratic growth rate, and if the initial free boundary is Lipschitz with small Lipschitz constant, then the Stefan free boundary regularizes in space, in a scale proportional to the distance it has traveled. We point out that the regularity results hold up to the initial time and all the regularity assumptions are imposed only on the initial data.

Remark

1. As in [ACS2]-[ACS3], our methods presented in this paper and in [CJK2] apply to free boundary motion laws of the type

$$V = G(\nu, |Du|) \text{ or } V = g(\nu)|Dv|,$$

where $G(\nu, p)$ is Lipschitz continuous with respect to ν and p with $\partial G/\partial p > c > 0$ and where $g(\nu)$ is a continuous positive function.

2. (I-c) and (I-d) are rather technical conditions, which ensures that the size of u does not change too much over time, in the regions away from the free boundary (see Lemma 2.3 and Corollary 3.3). Indeed we will see that with (I-c) and (I-d) u_0 is comparable to a harmonic function (see(0.3) below).

3. (I-d) holds when (I-c) holds with $\Delta u_0 \leq c_0 << 1$.

4. The restriction on the Lipschitz constant, $L < L_n$ is necessary. Indeed there is a waiting time phenomena of Stefan problem with initially sharp corners (see [CK]). For $n = 2$ the threshold for waiting time phenomena is 1, (i.e., if the initial free boundary has a corner with angle less than $\pi/2$ then the corner does not move for a positive amount of time), which is also our value for L_2 (For a wedge domain, we can put $\alpha = 2$ and $\beta = 1$.)

For local solutions of Hele-Shaw problem

$$(HS) \qquad \begin{cases} -\Delta u = 0 & \text{in} \quad \{u > 0\} \\ u_t = |Du|^2, & \text{on} \quad \partial\{u > 0\} \end{cases}$$

a corresponding result has been recently shown in [CJK2]. Below we state a simplified version of Theorem 1.1 in [CJK2]:

Theorem 0.2. *[CJK2] Let $Q_1 = B_1(0) \times [0, 1] \subset I\!R^n \times I\!R$. Suppose u is a solution of (HS) in Q_1 with initial positive phase Ω_0 in $B_1(0)$. Further suppose $u(-e_n, 0) = 1$, $\sup_{Q_1} u \leq M$ and that, for some $a > 0$,*

$$u(x, s) \leq Mu(x, t), \quad u_t(x, t) \geq At^{a-1}u(x, t) \text{ for } 0 \leq s < t \leq 1. \tag{0.2}$$

Then there is a constant $s > 0$ depending on n, M and A such that (A), (B) and (C) of Theorem 0.1 holds for the solution u of (HS) in $B_s(0)$.

Note that for (HS) the solution at each time is harmonic in the positive phase, including at $t = 0$ (the initial data). In particular the spatial geometry of the positive phase determines the geometry of the level sets of u. For the Stefan problem this is no longer true, which adds additional dimension of difficulty to our analysis.

For later use, we show below that, with conditions (I-c)-(I-d), the initial data u_0 is comparable to the harmonic function. Indeed with normalization $u_0(-e_n) = 1$, (I-c) and (I-d) imply that u_0 is comparable to a positive harmonic function H, which vanishes on $\Gamma_0 \cap B_2(0)$ and has value 1 at $-e_n$. Since $-C_0 \leq \Delta u_0$, $u_0 + C_0|x|^2$ is a subharmonic function with boundary values $C_0|x|^2$ on $\Gamma_0 \cap B_2(0)$ and with value $1 + C_0$ at $-e_n$. Hence one can observe that for $0 \leq s \leq 1$

$$u_0(-se_n) + C_0 s^2 \leq CH(-se_n) \leq Cs^\beta$$

where the first inequality holds for C depending on n and C_0, since H has growth rate less than 2, i.e., $\alpha < 2$. On the other hand, since $\Delta u_0 \leq C_0$ and u_0 has *sub-quadratic growth* (I-d), we can observe that $u_0 - C_0|x|^2$ is a superharmonic function with boundary values $-C_0|x|^2$ on $\Gamma_0 \cap B_2(0)$ and with a value larger than $r_0^{2-\delta}/2$ at a point $-r_0 e_n$, where r_0 is a sufficiently small constant depending on C_0 and δ. Hence for $0 \leq s \leq r_0$

$$Cs^\alpha \leq CH(-se_n) \leq u_0(-se_n) - C_0 s^2$$

where C and r_0 depend on n, C_0 and δ. Without loss of generality, we conclude that in $\{x : d(x, \Gamma_0) \leq r_0\} \cap B_1(0)$,

$$C_1 H \leq u_0 \leq C_2 H \tag{0.3}$$

where C_1, C_2 and r_0 depend on n, C_0 and δ.

We finish this section with a brief outline of the paper. For the Stefan-type free boundary problems, it has been shown in slightly different contexts (see [AC],[ACS1], [ACS2] and [CPS]) that if the level sets of u are close to Lipschitz graphs in a local space-time neighborhood, then it is indeed smooth. Therefore the key step in the proof of our main theorem is to show that the free boundary as well as nearby level sets of u remain close to a Lipschitz graph, a scale proportional to the distance the free boundary has travelled (see Proposition 5.4).

Observe that, after a hyperbolic blow-up, the solution of (ST) solves (HS). Hence the regularization phenomena of (ST) resembles that of (HS) in hyperbolic scale, while in small (parabolic) time scale, the free boundary is frozen and the solution regularizes in the positive phase. The barriers constructed in section 2-4 are illustration of this phenomena.

In section 1 some preliminary lemmas are introduced. In section 2 we establish that $\Omega(u)$ has bounded speed of propagation in parabolic scaling.

With this bound, we compare u with barriers constructed in section 3-4, apply regularity results known for caloric functions in Lipschitz domains (see section 1), and establish the ϵ-flatness of $\Gamma(u)$ in section 4. In section 5 it is shown that the positive level sets of u are also ϵ-close to Lipschitz graphs. In section 6 and in the appendix we outline further procedure to prove the main theorem, which is a rather technical combination of arguments in [ACS2] and [CJK2].

1 Preliminary lemmas and notations

We introduce some notations.
• For $x \in \mathbb{R}^n$, denote $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ where $x_n = x \cdot e_n$.
• Let $B_r(x)$ be the space ball of radius r, centered at x.
• Let $Q_r := B_r(0) \times [-r^2, r^2]$ be the parabolic cube and let $K_r := B_r(0) \times [-r, r]$ be the hyperbolic cube.
• A caloric function in $\Omega \cap Q_r$ will denote a nonnegative solution of the heat equation, vanishing along the lateral boundary of Ω.
• For $x \in \Gamma_0$, $d(x, t; u)$ is the maximal distance $\Gamma_t(u)$ has moved by time t near x. More precisely,

$$d(x, t; u) = \sup\{d : u(x + de_n, t) > 0, \quad x + de_n \in B_2(0)\}.$$

• For $x \in \Gamma_0$, $t(x + de_n; u)$ denotes the time $\Gamma_t(u)$ reaches $x + de_n$, precisely

$$t(x + de_n; u) = \inf\{t : u(x + de_n, t) > 0\}.$$

The first lemma is a direct consequence of the interior Harnack inequalities proved in [C-C].

Lemma 1.1. *[C-C] Suppose $w(x) : \mathbb{R}^n \to \mathbb{R}$ has bounded Laplacian. Then w is Hölder continuous with its constant depending on the Laplacian bound.*

Lemma 1.2. *[DiBenedetto, Chapter V. Theorem 8.2] Suppose u satisfies heat equation in $\mathbb{R}^n \times [0, \infty)$, $u_0 = u(\cdot, 0) \in C^{\gamma}_{loc}(\mathbb{R}^n)$ and satisfies*

$$|u_0(x)| \leq A \exp B|x|^2.$$

Then there exists a constant C depending only on n, A, B, γ and the Hölder constant of u_0 over the ball $B_1(0)$ such that

$$|u_t(x, t)| + |u_{x_i x_j}(x, t)| \leq C t^{\gamma/2 - 1}$$

in Q_1.

Lemma 1.3. *[FGS1, 1984, Theorem 3] Let Ω be a domain in $I\!\!R^n \times I\!\!R$ such that $(0,0)$ is on its lateral boundary. Suppose Ω is a $Lip^{1,1/2}$ domain, i.e.,*

$$\Omega = \{(x', x_n, t) : |x'| < 1, |x_n| < 2L, |t| < 1, x_n \leq f(x', t)\},$$

where f satisfies $|f(x', t) - f(y', s)| \leq L(|x' - y'| + |t - s|^{1/2}.)$ If u is a caloric function in Ω, then there exists $C = C(n, L)$, where L is the Lipschitz constant for Ω, such that

$$\frac{u(x,t)}{v(x,t)} \leq C \frac{u(-Le_n, 1/2)}{v(-Le_n, -1/2)}.$$

for $(x, t) \in Q_{1/2}$.

Lemma 1.4. *[ACS1, 1996, Theorem 1] Let Ω be a Lipschitz domain in $I\!\!R^n \times I\!\!R$, i.e.,*

$$Q_1 \cap \Omega = Q_1 \cap \{(x,t) : x_n \leq f(x', t)\},$$

where f satisfies $|f(x,t) - f(y,s)| \leq L(|x-y| + |t-s|)$. Let u be a Caloric function in $Q_1 \cap \Omega$ with $(0,0) \in \partial\Omega$ and $u(-e_n, 0) = m > 0$ and $\sup_{Q_1} u = M$. Then there exists a constant C, depending only on n, L, $\frac{m}{M}$ such that

$$u(x, t + \rho^2) \leq C u(x, t - \rho^2)$$

for all $(x,t) \in Q_{1/2} \cap \Omega$ and for $0 \leq \rho \leq d_{x,t}$.

Lemma 1.5. *[ACS1, Lemma 5] Let u and Ω be as in Lemma 1.4. Then there exist $a, \delta > 0$ depending only on n, L, $\frac{m}{M}$ such that*

$$w_+ := u + u^{1+a} \text{ and } w_- := u - u^{1+a}$$

are subharmonic and superharmonic, respectively, in $Q_\delta \cap \Omega \cap \{t = 0\}$.

Next we state several properties of harmonic functions:

Lemma 1.6. *[Dahlberg, see [D]] Let u_1, u_2 be two nonnegative harmonic functions in a domain D of $I\!\!R^n$ of the form*

$$D = \{(x', x_n) \in I\!\!R^{n-1} \times I\!\!R : |x'| < 2, |x_n| < 2L, x_n > f(x')\}$$

with f a Lipschitz function with constant less than L and $f(0) = 0$. Assume further that $u_1 = u_2 = 0$ along the graph of f. Then in

$$D_{1/2} = \{|x'| < 1, |x_n| < L, x_n > f(x')\}$$

we have

$$0 < C_1 \leq \frac{u_1(x', x_n)}{u_2(x', x_n)} \cdot \frac{u_2(0, L)}{u_1(0, L)} \leq C_2$$

with C_1, C_2 depending only on L.

Lemma 1.7. *[Jerison and Kenig, see [JK]] Let D, u_1 and u_2 be as in Lemma 1.6. Assume further that*

$$\frac{u_1(0, L/2)}{u_2(0, L/2)} = 1.$$

Then, $u_1(x', x_n)/u_2(x', x_n)$ is Hölder continuous in $\bar{D}_{1/2}$ for some coefficient α, both α and the C^α norm of u_1/u_2 depending only on L.

Lemma 1.8. *[Caffarelli, see [C2]] Let u be as in Lemma 1.6. Then there exists $c > 0$ depending only on L such that for $0 < d < c$, $\frac{\partial}{\partial x_n} u(0, d) \geq 0$ and*

$$C_1 \frac{u(0, d)}{d} \leq \frac{\partial u}{\partial x_n}(0, d) \leq C_2 \frac{u(0, d)}{d}$$

where $C_i = C_i(M)$.

Lemma 1.9. *[[JK], Lemma 4.1] Let Ω be Lipschitz domain contained in $B_{10}(0)$. There exists a dimensional constant $\beta_n > 0$ such that for any $\zeta \in \partial\Omega$, $0 < 2r < 1$ and positive harmonic function u in $\Omega \cap B_{2r}(\zeta)$, if u vanishes continuously on $B_{2r}(\zeta) \cap \partial\Omega$, then for $x \in \Omega \cap B_r(\zeta)$,*

$$u(x) \leq C\left(\frac{|x - \zeta|}{r}\right)^{\beta_n} \sup\{u(y) : y \in \partial B_{2r}(\zeta) \cap \Omega\}$$

where C depends only on the Lipchitz *constants of Ω.*

Lastly we point out that we use the notion of viscosity solutions for our investigation. For definition as well as properties of viscosity solutions, see [K]. Below we state the central property of viscosity solutions, which is used frequently throughout the paper.

Theorem 1.10. *[Comparison principle, [K1]] Let u, v be respectively viscosity sub- and supersolutions of (ST) in $D \times (0, T) \subset Q$ with initial data $u_0 \prec v_0$ in D. If $u \leq v$ on ∂D and $u < v$ on $\partial D \cap \bar{\Omega}(u)$ for $0 \leq t < T$, then $u(\cdot, t) \prec v(\cdot, t)$ in D for $t \in [0, T)$.*

2 An upper bound on the free boundary speed

In this section we estimate how far the free boundary of u would move by time t. Our claim is the following: *The upper bound on the distance $\Gamma_t(u)$ moves by time t has to be significantly smaller than $\sqrt{t}$* (Lemma 2.2).

This would guarantee the free boundary not to be perturbed too much from its initial Lipschitz shape before the initial data regularizes in the positive phase. We also show that u does not increase too fast in parabolic cubes centered at $\Gamma(u)$ (Lemma 2.3).

Lemma 2.1. *Suppose that* $|\Delta u_0| \leq C_0$ *and* $\sup_{Q_1} u = M$*. Then there exists a constant* $C = C(C_0, M, n)$ *such that* $u_t \geq -C$ *in* $B_{1/2}(0) \times [0,1]$*. In particular for* $0 \leq t \leq 1$

$$u(-\sqrt{t}e_n, s) \geq Cu(-\sqrt{t}e_n, 0) \text{ for } 0 \leq s \leq t.$$

Proof. The first claim follows since u_t solves heat equation with $u_t(\cdot,0) = \Delta u(\cdot,0) \geq -C_0$ in $B_1(0)$, $u_t \geq 0$ on $\Gamma_t(u)$ and $0 \leq u \leq M$ on $\partial B_1(0) \times [0,1]$. Since $u_t \geq -C_0$ and $u(-\sqrt{t}e_n, 0) > t^{\alpha/2}$ with $\alpha < 2$ by our hypothesis, one obtains

$$u(-\sqrt{t}e_n, 0) \leq Cu(-\sqrt{t}e_n, s) \text{ for } 0 \leq s \leq t.$$

□

The following is the main lemma in this section.

Lemma 2.2. *There exists* $t_0 = t_0(C_0, M, n) > 0$ *such that if* $x_0 \in \Gamma_0 \cap B_1(0)$ *and* $t \leq t_0$*, then*

$$d(x_0, t; u) \leq Ct^{1/(2-\beta)} \tag{2.1}$$

where β *is given in (0.1) and* C *depends on* C_0*,* M *and* n*.*

Proof. It suffices to prove the lemma for $x_0 = 0 \in \Gamma_0$. Using a radially symmetric barrier, one can show that $d(x, t_0; u) \leq 1/10$ if $x \in \Gamma_0 \cap B_{9/10}(0)$ and t_0 is a sufficiently small time depending on n and M.

We will construct a barrier function $\breve{w} \geq u$ using a Hele Shaw flow and show that $\breve{w}$ satisfies Lemma 2.2 at $x_0 = 0$. Let $\epsilon_0 > 0$ be a sufficiently small constant. Let w be a solution of (HS) in

$$\Sigma := B_1(0) \cap \Big(\bigcup_{s \geq -\epsilon_0} \Gamma_0 + se_n \Big)$$

such that

$$\begin{cases} -\Delta w = 0 & \text{in} \quad \{w > 0\} \cap \Sigma \\ w = C_1 & \text{on} \quad \Gamma_0 - \epsilon_0 e_n \\ w = 0 & \text{on} \quad \partial B_1(0) \cap \Sigma \\ \Gamma_0(w) = \Gamma_0(u) & \text{in} \quad \Sigma \\ w_t = |Dw|^2, & \text{on} \quad \Gamma(w) \cap \Sigma \end{cases}$$

where C_1 is a sufficiently large constant depending on C_0, M and n. Since $\Delta u \geq -C_0$, $u + C_0|x - y_0|^2$ is subharmonic for any $y_0 \in \Omega_0(w)$. Also since the maximal decay rate α is less than 2, one can observe that if C_1 is sufficiently large (depending on C_0, M and n), then by Lemma 1.6

$$w_0 \geq 100 u_0, \quad w \geq 100 u \text{ on } \Gamma_0 - \epsilon_0 e_n.$$

Next, we bend up the free boundary of w (using a conformal map $\breve{\phi}$ constructed in [CJK] and [CJK2]) so that $\breve{\phi}(\Gamma_t(w))$ is located above $\Gamma_t(u)$ in $B_1(0) - B_{1/2}(0)$. Let $\breve{\psi}$ be the following conformal map

$$\breve{\psi}(x) = \frac{1}{|x|^2} \cdot (x_1, ..., x_{n-1}, 2|x|^2 - x_n)$$

and let

$$\breve{\phi} = \breve{\psi}(x + e_n) - e_n.$$

Then $\breve{\phi}$ fixes 0 and it bends up the free boundary $\Gamma_t(w)$ at least by $1/10$ in $B_1(0) - B_{1/2}(0)$. Now define $\breve{w}$ as follows:

$$\begin{cases} -\Delta \breve{w}(\cdot, t) = 0 & \text{in} \quad \breve{\phi}(\Omega_t(w)) \\ \breve{w}(\cdot, t) = C_1/50 & \text{on} \quad \breve{\phi}(\Gamma_0 - \epsilon_0 e_n) \\ \breve{w}(\cdot, t) = M & \text{on} \quad \partial B_{9/10}(0) \cap \breve{\phi}(\Omega_t(w)) \end{cases}$$

Then, one can show that $\breve{w}$ is a supersolution of (HS) in $B_{1/2}(0)$ if ϵ_0 is sufficiently small. (See [CJK] for a detailed proof.) $\breve{w}$ is also a supersolution of (ST) in $B_{1/2}(0)$ since $w_t \geq 0$. Since $\Gamma_t(\breve{w})$ is contained in the zero set of u in $B_{9/10}(0) - B_{1/2}(0)$ and $\breve{w} \geq u$ at $t = 0$ and on its lateral boundary, we conclude $\breve{w} \geq u$ in $B_{9/10}(0)$

Now it suffices to prove $d(0,t;\breve{w}) \leq Ct^{1/(2-\beta)}$. Using a speed estimate for a Hele-Shaw flow (Theorem 0.2 and Lemma 1.8), we obtain for $d_0 := d(0,t;w)$

$$\text{average speed of } \Gamma_t(w) = \frac{d_0}{t} \sim \frac{w(-d_0 e_n, t)}{d_0}.$$

Therefore, $d_0^2 \sim tw(-d_0e_n,t) \leq Ctd_0^\beta$ for C depending on C_0, M and n. This yields

$$d(0,t;w) \leq Ct^{1/(2-\beta)}.$$

¿From the construction of $\breve{w}$, one can observe that $\Gamma_t(\breve{w})$ is contained in the $40d_0^2$-neighborhood of $\Gamma_t(w)$ in $B_{2d_0}(0)$. Hence if $t \leq t_0$ and t_0 is sufficiently small, then

$$d(0,t;\breve{w}) \leq Ct^{1/(2-\beta)}.$$

□

Note that Lemma 2.2 yields us that in each parabolic scaling we have a Lipschitz domain in time. To apply Lemma 1.5 in parabolically scaled neighborhoods of each free boundary point, we need to keep track of the ratio $\frac{m}{M}$ in each scale.

Lemma 2.3. *Suppose $(x,t) \in \Gamma(u) \cap Q_{2/3}(0)$ with $0 < t$. Then*

$$u(y,s) \leq Cu(x - \sqrt{t}e_n, 0) \text{ for } (y,s) \in B_{\sqrt{t}}(x) \times [0,t].$$

where $C = C(M, C_0, n)$.

Proof. Fix $(x,t) \in \Gamma(u) \cap Q_{2/3}(0)$. Consider the re-scaled function

$$\tilde{u}(y,s) := \frac{u(x+\sqrt{t}y, ts)}{u(x-\sqrt{t}e_n, 0)}.$$

in Q_1.

Due to (I-a), (I-b) and (2.1)

$$|\Delta \tilde{u}(y,0)| \leq C_0 \frac{t}{u(-\sqrt{t}e_n, 0)} \leq C_0.$$

Therefore $\tilde{u}(\cdot,0) \in C^\beta(I\!\!R^n)$ by Lemma 1.1, and by Lemma 1.2 $\tilde{u}(y,s) \leq C$ in Q_1, where C depends on M,C_0 ad n. Hence we conclude. □

In the following two sections we will construct barriers which approximate u better than w does, and will prove that $\Gamma_t(u)$ is very close to a Lipschitz graph. In other words, we prove the ϵ- monotonicity of the free boundary of u by constructing barriers which stays close to u, at the places $\sqrt{t}$ away from the free boundary.

3 Constructing a barrier function v with Lipschitz free boundary

In this section, we will construct an approximation of u using a parameterized solution of (HS). This enables us to apply regularity results for solutions of (HS) obtained in [CJK] and [CJK2].

Let $\epsilon > 0$ be a sufficiently small constant, and let t_1 be a small time such that

$$t_1 = \epsilon^{m_1} \tag{3.1}$$

where $m_1 > 0$ is a sufficiently large number which will be determined later. Let t_2 be a small time bigger than t_1 such that

$$t_2 = \epsilon^8 t_1^{1-\beta/2} \gg t_1. \tag{3.2}$$

Let r be a constant bigger than $\sqrt{t_2}$ such that

$$r = \epsilon^2 t_1^{1/4}.$$

Then

$$\sqrt{t_1} < \sqrt{t_2} \le \epsilon^2 r \tag{3.3}$$

and (2.1) implies that for any $x \in \Gamma_0 \cap B_1(0)$

$$d(x, t_2; u) \le C t_2^{1/(2-\beta)} \le \frac{r^2}{10} \le \epsilon^4 \sqrt{t_1} \tag{3.4}$$

if ϵ is sufficiently small. (Here t_1, t_2 and r are constructed so that they satisfy (3.3) and (3.4).)

We construct a caloric function v in $B_r(0) \times [0, t_2]$, whose free boundary does not move for $0 \le t \le t_1$ and then it moves as it does for a Hele-shaw flow (with a proper scaling) for $t_1 \le t \le t_2$. Our hope is that by time $t = t_1$ the caloric function v regularizes enough so that it is "almost harmonic" in the sense of Lemma 1.5. To this end it is important to choose t_1 and t_2 such that t_1 is neither too small compared to t_2 (so that the solution has time to regularize) nor too large compared to t_2 (so that the distance $\Gamma(u)$ has moved by t_1 is ignorable and v approximate u well enough). Equation (3.4) ensures that t_1 is not too small compared to t_2, and later (Lemma 5.1) we will see that t_1 is not too large compared to t_2.

The construction of v is as follows: Let $I = \Gamma_0(u) \cap B_1(0)$ and let

$$\Lambda_{\sqrt{t}} = \begin{cases} \bigcup_{x \in I} B_{\sqrt{t_1}}(x) & \text{if} \quad 0 \le t \le t_1 \\ \\ \bigcup_{x \in I} B_{\sqrt{t}}(x) & \text{if} \quad t_1 \le t \le t_2. \end{cases}$$

Set

$$\Pi := \bigcup_{0 \le t \le t_2} \Lambda_{\sqrt{t}} \times \{t\}$$

and let v solve

$$\begin{cases} v_t - \Delta v = 0 & \text{in} \quad \{v > 0\} \cap \Pi \\ \\ v_0 = (1 - 10\epsilon) u_0 & \text{in} \quad \Lambda_{\sqrt{t_1}} \cap B_r(0) \\ \\ v = (1 - 10\epsilon) u & \text{on} \quad \partial\Lambda_{\sqrt{t}} \cap B_r(0) \\ \\ v = 0 & \text{on} \quad \partial B_r(0) \cap \Lambda_{\sqrt{t}} \\ \\ \Gamma_t(v) = \Gamma_0(u) & \text{for} \quad 0 \le t \le t_1 \\ \\ h_t/|Dh| = |Dh| & \text{on} \quad \Gamma(v) \cap \{t_1 \le t \le t_2\} \end{cases} \tag{3.5}$$

where $h(\cdot, t)$ is the harmonic function in $\Lambda_{\epsilon\sqrt{t_1}} \cap B_r(0) \cap \{v > 0\}$ such that

$$h(\cdot, t) = \begin{cases} 0 & \text{on} \quad \Gamma_t(v) \cup \partial B_r(0) \\ \\ v(\cdot, t^-) & \text{on} \quad l := \partial\Lambda_{\epsilon\sqrt{t_1}} \cap B_r(0) \cap \{v > 0\}. \end{cases} \tag{3.6}$$

(Here $v(\cdot, t^-)$ denotes the limit from past times.)

Note that h is a solution of the Hele-Shaw problem (HS) in $\Lambda_{\epsilon\sqrt{t_1}} \cap B_r(0) \times [t_1, t_2]$. Later we will apply the regularity result for (HS) obtained in [CJK2] (Theorem 0.2) to h, to derive regularity of the free boundary $\Gamma_t(v) = \Gamma_t(h)$ for $t \in [t_1, t_2]$.

Since $u \le \breve{w}$, arguing as in section 2 it also follows that $v \le \breve{w}$. Then by (3.4)

$$d(x,t;v) \le d(x,t;\breve{w}) \le Ct^{1/(2-\beta)} \le \epsilon^2 \sqrt{t_1} \tag{3.7}$$

for $0 \le t \le t_2$. Below we state several properties of v.

Lemma 3.1. *The following properties hold for v:*

(a) $v(x,t) \le Cv(x,0)$ in $\Lambda_{\sqrt{t}} - \Lambda_{\epsilon\sqrt{t_1}}$, where $C = C(M, C_0, n)$.

(b) $v_t \ge -C$ with $C = C(M, C_0, n)$ and $v \le (1-10\epsilon)u$ for $0 \le t \le t_1$

(c) There exists $A > 0$ such that for $t_1 \le t \le t_2$ and $x \in \Lambda_{\sqrt{t}} - \Lambda_{\epsilon^2\sqrt{t_1}}$

$$v_t(x,t) \ge -At^{-\alpha/2} v(x,t).$$

(d) $\Gamma_t(v)$ as well as the level sets of $h(\cdot,t)$ is a Lipschitz graph with respect to e_n-axis in $B_{(1-\epsilon)r}(0)$ for $t \le t_2$. Furthermore the associated Lipschitz constant is close to L and less than L_n.

(e) The normal velocity $V_{x,t}$ of $\Gamma(v)$ at (x,t) satisfies

$$V_{x,t} = |Dh|(x,t) \sim |Dh|(-de_n, t) \le \frac{C}{\sqrt{t}} \text{ for } t \in [t_3, t_2] \text{ and } x \in \Gamma_t(v) \cap B_{r/2}(0),$$

where $t(d) = t(x,t;h)$

$$t_1 < t_3 := t_1 + t_1^{\frac{2-\beta}{2(1-\beta)}} < 2t_1.$$

Moreover, the average speed of $\Gamma_t(v)$ over $[t_1, t_3]$ is less than $\dfrac{C}{\sqrt{t_1}}$ in $B_{r/2}(0)$.

Proof. (a) follows from Lemma 2.3 and (3.7).

Proof of (b): $v_t \ge -C$ since v_t solves heat equation with $v_t|_{t=0} = \Delta v(\cdot,0) = \Delta u(\cdot,0) \ge -C_0$ in $\Lambda_{\sqrt{t_1}}$, $v_t = u_t \ge -C$ on $\partial\Lambda_{\sqrt{t_1}}$ due to

Lemma 2.1 and $v_t \geq 0$ on $\Gamma_t(v)$. The second claim is a consequence of the maximum principle, using the fact that the positive phase of u expands in time.

Proof of (c): Recall that $v_t \geq -C$ and $d(y,t;v) \leq \epsilon^2\sqrt{t_1}$ for any $y \in \Gamma_0 \cap B_1(0)$. Then $v_t \geq \omega$, where ω solves heat equation in

$$\Sigma := \bigcup_{0\leq t\leq t_2} ((\Omega_0 + \epsilon^2\sqrt{t_1}e_n) \cap \Lambda_{\sqrt{t}} \cap B_r(0)) \times \{t\}$$

with $\omega = -C$ at $t=0$, $\omega = -C$ on $\partial\Lambda_{\sqrt{t}} \cup \partial B_r(0)$, and $\omega = 0$ on $\Gamma_0 + \epsilon^2\sqrt{t_1}e_n$. Let $\tilde{\omega}$ solve the heat equation in Σ with $\tilde{\omega} = 0$ on $\Gamma_0 + \epsilon^2\sqrt{t_1}e_n$ and $\tilde{\omega}(x,t) = v(x - \epsilon^2\sqrt{t_1}e_n, t)$ on $\partial\Lambda_{\sqrt{t}} \cup \partial B_r(0)$ and at $t = 0$.

Then by comparison

$$\tilde{\omega}(x,t) \leq v(x - \epsilon^2\sqrt{t_1}e_n, t)$$

and

$$\tilde{\omega}(-\sqrt{t}e_n, t) \geq \tilde{\omega}(-\sqrt{t}e_n, 0) - Ct \geq \frac{1}{2}t^{\alpha/2}$$

for $t_1 \leq t \leq t_2$, since the maximal decay rate α is smaller than 2.

Hence it follows that for $t_1 \leq t \leq t_2$ and $x \in \Lambda_{\sqrt{t}} - \Lambda_{\epsilon^2\sqrt{t_1}}$

$$\begin{aligned} v_t(x,t) \geq \omega(x,t) &\geq C\frac{\omega(-\sqrt{t}e_n,t)}{\tilde{\omega}(-\sqrt{t}e_n,t)}\tilde{\omega}(x,t) \\ &\geq -Ct^{-\alpha/2}v(x - \epsilon^2\sqrt{t_1}e_n, t) \\ &\geq -Ct^{-\alpha/2}v(x,t) \end{aligned}$$

where the second inequality follows from the Lemma 1.3 applied to ω and $\tilde{\omega}$ and the last inequality follows the interior Harnack inequality for the heat equation - note that (x,t) is at least $\epsilon^2\sqrt{t_1}$-away from $\Gamma(u)$.

Proof of (d): Let h be local solution of (HS) given in (3.6). Due to (a)-(b) ,

$$h(x,t) \leq Ch(x,t_1) \text{ in } \Sigma := \Lambda_{\epsilon\sqrt{t_1}} \cap B_r(0) \times [t_1,t_2].$$

Next due to Carleson's lemma and (3.7) we have $h(x,t) \leq Ch(-\epsilon\sqrt{t_1}e_n, t_1)$ in Σ. Also, by (c) and the fact that $h_t > 0$ on $\Gamma(h)$, it follows that $h_t \geq -At^{-\alpha/2}h$ in Σ. Now (d) follows from Theorem 0.2 applied to

$$\tilde{h}(x,t) = \frac{h(\epsilon\sqrt{t_1}x, \tau t + t_1)}{h(-\epsilon\sqrt{t_1}e_n, t_1)}, \quad \tau = \frac{\epsilon^2 t_1}{h(-\epsilon\sqrt{t_1}e_n, 0)}.$$

Proof of (e): Fix $x_0 \in \Gamma_0(v) \cap B_{r/2}(0)$. For $t \in [t_1, t_2]$, let $y = y(t) = x_0 + de_n \in \Gamma_t(v)$. ¿From the construction of v,

$$V_{y,t} = \frac{v_t}{|Dv|}|_{(y,t)} = \frac{h_t}{|Dh|}|_{(y,t)}$$

where h solves (HS) for $t_1 \leq t \leq t_2$. Let $H(x,t) = h(x, t+t_1)$, then H solves (HS) for $0 \leq t \leq t_2 - t_1$ and

$$\frac{h_t}{|Dh|}|_{(y,t)} = \frac{H_t}{|DH|}|_{(y,t-t_1)}.$$

Define

$$t_3 = t_1 + t_1^{\frac{2-\beta}{2(1-\beta)}}.$$

Then for $t \in [t_3, t_2]$

$$\begin{aligned} \frac{v_t}{|Dv|}|_{(y,t)} = \frac{H_t}{|DH|}|_{(y,t-t_1)} \quad &\sim \quad |DH|(x_0 - de_n, 0) \\ &\leq \quad C\frac{d(x, t-t_1; H)}{t-t_1} \leq C(t-t_1)^{\frac{-1+\beta}{2-\beta}} \leq Ct^{-1/2}. \end{aligned}$$

where the first approximation and inequality follow by Theorem 0.2, and the second inequality follows from (2.1).

Here the first inequality follows from Theorem 1.2 of [CJK2], the second inequality is due to $d(x, 2(t-t_1); H) \leq C(t-t_1)^{\frac{1}{2-\beta}}$ and the last inequality holds since

$$(t-t_1)^{\frac{-1+\beta}{2-\beta}} \leq \begin{cases} (t_3 - t_1)^{\frac{-1+\beta}{2-\beta}} \quad = \quad t_1^{-1/2} & \text{for} \quad t \in [t_3, 2t_1] \\ \\ Ct^{\frac{-1+\beta}{2-\beta}} \qquad\qquad < \quad t^{-1/2} & \text{for} \quad t \in [2t_1, t_2]. \end{cases}$$

Lastly,

$$\text{The average speed on } [t_1, t_3] := \frac{d(x, t_3 - t_1; H)}{t_3 - t_1} \leq C(t_3 - t_1)^{\frac{-1+\beta}{2-\beta}} = Ct_1^{-1/2}.$$

□

For the proof of main theorem, we will approximate u by a function v, whose free boundary is Lipschitz in parabolic scaling. More precisely, we would like v to satisfy

$$v_t/|Dv| \leq Ct^{-1/2} \text{ for } 0 \leq t \leq t_2$$

so that one can apply strong properties of caloric functions in Lipschitz domains such as Lemma 1.4. By (e) of Lemma 3.1, this upper bound holds for $t \in [0, t_1] \cup [t_3, t_2]$ and it holds for the average speed on $[t_1, t_3]$. Hence we modify v as follows: Let $\tilde{v}$ solve

$$\begin{cases} \tilde{v}_t - \Delta \tilde{v} = 0 & \text{in} \quad \{\tilde{v} > 0\} \cap \Pi \\ \tilde{v} = v & \text{at} \quad t = 0 \text{ and on } \partial\Lambda_{\sqrt{t}}. \\ \Gamma_t(\tilde{v}) = \Gamma_t(v) & \text{for} \quad t \in [0, t_1] \cup [t_3, t_2] \\ \Gamma_t(\tilde{v}) = (1-b)\Gamma_{t_1}(v) + b\Gamma_{t_3}(v) & \text{for} \quad t = (1-b)t_1 + bt_3 \in [t_1, t_3] \end{cases}$$

Parallel arguments as in the proof of Lemma 3.1 show that (a), (b) and (d) of Lemma 3.1 also hold for this modified v. (c) and (e) of Lemma 3.1 can be improved as follows.

Lemma 3.2. *(c′) Let $0 \le t \le t_2$ and let $x \in \Omega_t(\tilde{v})$, $\operatorname{dist}(x, \Gamma_t(\tilde{v})) \le \sqrt{t}$. Then*

$$\tilde{v}_t(x, t) \ge -Ct^{-\alpha/2}\tilde{v}(x, t).$$

(e′) For $0 \le t \le t_2$ the normal velocity of $\Gamma_t(\tilde{v})$ satisfies

$$\tilde{v}_t / |D\tilde{v}| \le Ct^{-1/2}$$

on $\Gamma_t(\tilde{v}) \cap B_{r/2}(0)$. Furthermore for $t_3 \le t \le t_2$ we have

$$(1 - 2\epsilon)|D\tilde{v}| \le \tilde{v}_t / |D\tilde{v}| \le (1 + 2\epsilon)|D\tilde{v}| \tag{3.8}$$

on $\Gamma_t(\tilde{v}) \cap B_{r/2}(0)$.

Proof. *Proof of (c′)*: The proof is similar to that of (c). Let ω solve heat equation in $\Omega(\tilde{v}) \cap \Pi$ with $\omega(\cdot, 0) = -C$ in $\Omega_0(\tilde{v}) \cap \Pi$, $\omega = -C$ on $\partial\Pi \cap \Omega(\tilde{v})$ and $\omega = 0$ on $\Gamma(\tilde{v})$, where C is the constant given in Lemma 3.1, (a). Then by maximum principle of the heat equation $\tilde{v}_t \ge \omega$.

Observe that $\Omega(\tilde{v})$ is Lipschitz in parabolic scaling, more precisely, for any $y \in \Gamma_s(\tilde{v})$ and $s \in [0, t_2]$

$$w(x, t) := \tilde{v}(\sqrt{\frac{s}{2}}(x - y) + y, \frac{s}{2}(t - s) + s)$$

has uniformly Lipschitz free boundary in $Q_1(0)$. Hence if $x \in \Omega_t(\tilde{v})$ and $\mathrm{dist}(x, \Gamma_t(\tilde{v})) \leq \sqrt{t}$, then

$$\tilde{v}_t(x,t) \geq \omega(x,t) \geq C\frac{\omega(-\sqrt{t}e_n,t)}{\tilde{v}(-\sqrt{t}e_n,t)}\tilde{v}(x,t) \geq -\frac{C}{t^{\alpha/2}}\tilde{v}(x,t)$$

where the second inequality follows from Lemma 1.3 applied to ω and $\tilde{v}$ and the last inequality is due to Lemma 2.1.

Proof of (e′):

1. We first claim that

$$(1 - C\epsilon)v \leq \tilde{v} \leq (1 + C\epsilon)v \text{ on } \partial\Lambda_{\epsilon\sqrt{t_1}} \times [t_1, t_2]. \tag{3.9}$$

Observe that due to Lemma 2.2 and (3.6),

$$w_1 \leq u, v, \tilde{v} \leq w_2 \text{ in } \Pi \cap \{s \leq t\},$$

where w_1 and w_2 are caloric functions respectively in cylindrical domains $\Omega_0 \cap B_1(0) \times [0,t]$ and $(\Omega_0 + \epsilon^2\sqrt{t_1}e_n) \cap B_1(0) \times [0,t]$, with initial data u_0 and lateral boundary data u on $\partial B_1(0)$. Due to their kernel representation, for $x \in \partial\Lambda_{\epsilon\sqrt{t_1}}$ and $t_1 \leq t \leq t_2$,

$$w_1(x,t) \leq w_2(x,t) \leq (1+\epsilon)w_1(x - \epsilon^2\sqrt{t_1}e_n, t).$$

But due to Lemma 1.5, there exists $0 < a < 1$ depending on M, C_0 and n such that $w_1 + w_1^{1+a}$ is subharmonic. Also due to the properties of harmonic functions in Lipschitz domain, one can check that

$$\begin{aligned}(w_1 + w_1^{1+a})(x - \epsilon^2\sqrt{t_1}e_n, t) &\leq (1 + C\frac{\epsilon^2\sqrt{t_1})}{d(x,\Gamma_0)})(w_1 + w_1^{1+a})(x,t) \\ &\leq (1 + C\epsilon)w_1(x,t)\end{aligned}$$

where $C = C(M,n)$. Thus $w_1 \leq w_2 \leq (1 + C\epsilon)w_1$ and which implies (3.9).

2. Next, by (e) of Lemma 3.1 and the modification of v, for $t_1 \leq t \leq t_2$

$$\tilde{v}_t/|D\tilde{v}| \leq Ct^{-1/2}$$

on $\Gamma_t(\tilde{v}) \cap B_{r/2}(0)$. Also for $0 \leq t \leq t_1$,

$$\tilde{v}_t/|D\tilde{v}| = 0$$

on $\Gamma_t(\tilde{v}) \cap B_{r/2}(0)$.

This upper bound on speed and (d) of Lemma 3.1 imply that $\Omega(\tilde{v})$ is Lipschitz in parabolic scaling. Now due to Lemma 3.1 (a), Lemma 2.3 and Lemma 1.5, there exists $0 < a < 1$ depending on M, C_0 and n such that $\tilde{v} + \tilde{v}^{1+a}$ is subharmonic in $\Lambda_{\epsilon\sqrt{t_1}} \cap B_r(0)$ for $t_1 \leq t \leq t_2$. Moreover, by (3.9)

$$\tilde{v} + \tilde{v}^{1+a} \leq (1 + C\epsilon)v = (1 + C\epsilon)h$$

on $\partial\Lambda_{\epsilon\sqrt{t_1}} \times [t_1, t_2]$. Since $r = o(t_1^{1/4}) >> \sqrt{t_1}$, due to Lemma 1.7 we obtain that for $t_3 \leq t \leq t_2$

$$\tilde{v} \leq \tilde{v} + \tilde{v}^{1+a} \leq (1 + 2\epsilon)h \text{ in } \Lambda_{\epsilon^2\sqrt{t_1}} \cap B_{r/2}(0).$$

Hence

$$|D\tilde{v}| \leq (1 + C\epsilon)|Dh| \text{ on } \Gamma_t(\tilde{v}) \cap B_{r/2}(0).$$

Similarly using the fact that $\tilde{v} - \tilde{v}^{1+a}$ is superharmonic we obtain that

$$(1 - C\epsilon)h \leq \tilde{v} \text{ in } \Lambda_{\epsilon^2\sqrt{t_1}} \cap B_{r/2}(0)$$

and thus

$$(1 - C\epsilon)|Dh| \leq |D\tilde{v}| \text{ on } \Gamma_t(\tilde{v}) \cap B_{r/2}(0).$$

Now we can conclude since $\Gamma(\tilde{v}) = \Gamma(v)$ for $t_3 \leq t \leq t_2$. In other words,

$$\tilde{v}_t/|D\tilde{v}| = v_t/|Dv| = |Dh| \text{ on } \Gamma(\tilde{v}) \cap \{t_3 \leq t \leq t_2\}.$$

□

For the rest of the paper, we will replace v with $\tilde{v}$ and denote it by v for convenience. In the following corollary, we show that v is increasing in time in the ϵ-scale.

Corollary 3.3. *Let $0 \leq t \leq t_2$, then for any $s \in [0, t]$ and $x \in \Omega_s(v)$*

$$v(x, s) \leq (1 + \epsilon)v(x, t).$$

Proof. Assume that

$$t_0 := \inf\{t \in [0, t_2] : v(x, s) > (1+\epsilon)v(x, t) \text{ for some } s \in [0, t],\ x \in \Omega_s(v)\} < t_2.$$

Then there exist $s_0 \in [0, t_0]$ and $x \in \Omega_{s_0}(v)$ solving $v(x, s_0) = (1+\epsilon)v(x, t_0)$.

Case 1. If $\mathrm{dist}(x, \Gamma_{s_0}(v)) \le \sqrt{s_0}$, then

$$\begin{aligned} v(x, t_0) &\ge v(x, s_0) - C\int_{s_0}^{t_0} t^{-\alpha/2} v(x, \tau) d\tau \\ &\ge v(x, s_0) - C v(x, t_0) t_0^{-\alpha/2+1} \\ &\ge v(x, s_0) - \epsilon^2 v(x, t_0) \end{aligned}$$

where the first inequality follows from (c$'$) of Lemma 3.2, the second inequality is due to the definition of t_0. But this contradicts our definition of t_0.

Case 2. Suppose $\sqrt{s_0} \le d := \mathrm{dist}(x, \Gamma_{s_0}(v)) \le \sqrt{t_0}$. First observe that, by the almost harmonicity of v,

$$d(x, \Gamma_{s_0}(v))^\alpha \le v(x, s_0) \tag{3.10}$$

where $0 < \alpha < 1$. Therefore

$$\begin{aligned} v(x, t_0) &\ge v(x, s_0) - C(d^2 - s_0) - C\int_{d^2}^{t_0} t^{-\alpha/2} v(x, \tau) d\tau \\ &\ge v(x, s_0) - Cd^2 - C\int_{s_0}^{t_0} t^{-\alpha/2} v(x, \tau) d\tau \\ &\ge v(x, s_0) - C(v(x, s_0))^{2/\alpha} - \epsilon^2 v(x, t_0) \\ &\ge v(x, s_0) - \epsilon^2 v(x, s_0) - \epsilon^2 v(x, t_0) \end{aligned}$$

where the first inequality follows from $v_t \ge -C$ and (c$'$) of Lemma 3.2, the third inequality follows from (3.10) and the last inequality holds since $v(x, s_0) \le C t_0^{\beta/2} \le t_2^{\beta/2} \le \epsilon^m$ for a sufficiently large m. Again we get a contradiction.

Case 3. Lastly suppose $\sqrt{t_0} \le \mathrm{dist}(x, \Gamma_{s_0}(v))$. Then similarly as in Case 2,

$$v(x, t_0) \ge v(x, s_0) - Ct_0 \ge v(x, s_0) - Cd(x, \Gamma_{s_0}(v))^2 \ge v(x, s_0) - \epsilon^2 v(x, s_0)$$

and we conclude. $\square$

Recall that $\Gamma_t(v)$ is a Lipschitz graph with respect to e_n-axis, with Lipschitz constant close to L (in particular less than L_n) for $0 \leq t \leq t_2$. This and (e′) Lemma 3.2 invoke Lemma 1.5 to yield the following statement.

Corollary 3.4. *If $x \in \Omega_t(v)$ and $d := \mathrm{dist}(x, \Gamma_t(v)) \leq \sqrt{t}$, then*

$$d^\alpha \leq v(x,t) \leq d^\beta. \tag{3.11}$$

4 Constructing a subsolution w^{sub} and a supersolution w^{sup}

In section 3 v is constructed as an "approximation" of u. Based on v, we construct a subsolution w^{sub} and a supersolution w^{sup} such that $w^{sub} \leq u \leq w^{sup}$ and that w^{sub} and w^{sup} have sufficiently close Lipschitz free boundaries.

1. Construction of a subsolution: Recall that v is a caloric function in $\Lambda_{\sqrt{t}} \cap B_r(0) \times [0, t_2]$ with $v_0 = (1-10\epsilon)u_0$, $v = (1-10\epsilon)u$ on $\partial\Lambda_{\sqrt{t}} \cap B_r(0)$, and with free boundary satisfying (3.8) in $B_{r/2}(0)$.

Due to (3.8), $(1+2\epsilon)v(x,t)$ is a subsolution of (ST) in $B_{r/2}(0) \times [t_3, t_2]$, and one can prove $(1+2\epsilon)v \leq u$ at $t = t_3$, and on $\partial\Lambda_{\sqrt{t}} \cap B_{r/2}(0)$. However, this function $(1+2\epsilon)v$ is not guaranteed to be smaller than u on the sides $\partial B_{r/2}(0) \cap \Lambda_{\sqrt{t}}$. Hence to control the values of v on the sides, we will use a conformal map which bends down the free boundary of v and locates it below the free boundary of u in the annulus $(B_r(0) - B_{r/2}(0)) \cap \Lambda_{\sqrt{t}}$.

Let $\hat{\psi}$ be the following conformal map

$$\hat{\psi}(x) = \frac{1}{|x - 2e_n|^2} \cdot (x_1, ..., x_{n-1}, 2|x - 2e_n|^2 - x_n)$$

and let $\hat{\phi} = \hat{\psi}(x + e_n) - e_n$. Then for $x \in \Gamma_t(v) \cap (B_r(0) - B_{r/2}(0))$,

$$\hat{\phi}(x) \in x - r^2/10 e_n + W(\pi/3, -e_n)$$

where $W(\theta, \nu) := \{y \in I\!R^n : (y, \nu) \geq |y| \cos(\theta)\}$ is a cone with axis ν and opening angle 2θ. Figuratively speaking, $\hat{\phi}$ bends down the free boundary $\Gamma_t(v)$ toward $-e_n$-direction at least by $r^2/10$ in $B_r(0) - B_{r/2}(0)$. Also in $B_r(0)$, $\hat{\phi}(\Gamma_t(v))$ is contained in the $10r^2$-neighborhood of $\Gamma_t(v)$, and

$$|\nabla \hat{\phi} - I| \leq Cr$$

where I is the identity matrix. (Due to this bound on $\nabla\hat{\phi}$, the speed of $\hat{\phi}(\Gamma_t(v))$ will be sufficiently close to the speed of $\Gamma_t(v)$ in $B_r(0)$ if r is small enough.)

Next define $\hat{v}$ in $B_r(0) \times [0, t_2]$ as follows:

$$\begin{cases} \hat{v}_t - \Delta \hat{v} = 0 & \text{in} \quad \{\hat{v} > 0\} \\ \Gamma_t(\hat{v}) = \hat{\phi}(\Gamma_t(v)) & \text{in} \quad B_r(0) \\ \hat{v} = v & \text{on} \quad (\partial \Lambda_{\sqrt{t}} \cap B_r(0)) \cup (\Lambda_{\sqrt{t}} \cap \partial B_r(0)) \\ \hat{v}_0 = v_0 & \text{in} \quad \{\hat{v}_0 > 0\} \cap B_r(0) \end{cases}$$

Note that since $\hat{\phi}(\Gamma_0(v))$ is located below $\Gamma_0(v)$, the initial positive set $\{\hat{v}_0 > 0\}$ is a subset of $\{v_0 > 0\}$ and the initial value $\hat{v}_0$ has discontinuities on the initial free boundary $\Gamma_0(\hat{v})$.

Now define our barrier function w^{sub} as below:

$$w^{sub}(x,t) = (1+5\epsilon)\hat{v}(x,t).$$

Our first task is to prove that $w^{sub} \leq u$ for $0 \leq t \leq t_2$. Let us begin by enlisting some properties of w^{sub}.

Lemma 4.1. *(a) $w^{sub}_t - \Delta w^{sub} = 0$ in $\Omega(w^{sub})$.*

(b) $w^{sub} \leq (1-5\epsilon)u$ for $0 \leq t \leq t_3$.

(c) For $t_3 \leq t \leq t_2$, the free boundary velocity of w^{sub} satisfies

$$w^{sub}_t / |Dw^{sub}| \leq |Dw^{sub}| \text{ on } \Gamma_t(w^{sub}) \cap B_{r/2}(0). \tag{4.1}$$

(d) For $0 \leq t \leq t_2$

$$(\Gamma_t(w^{sub}) \cap (B_r(0) - B_{r/2}(0))) \subset \Omega_t(u).$$

(e) For $0 \leq t \leq t_2$

$$w^{sub} \leq u \text{ on } (\partial \Lambda_{\sqrt{t}} \cap B_{3r/4}(0)) \cup (\partial B_{3r/4}(0) \cap \Lambda_{\sqrt{t}}). \tag{4.2}$$

Proof. Proof of (b): Let $x \in \Gamma_0(u)$. Using a barrier argument with $u_0(-se_n) \geq s^\alpha$, one can prove that

$$d(x, t_1; u) \geq t_1^{\frac{1}{2-\alpha}}.$$

Also observe that

$$d(x,t_3;v) \le (t_3-t_1)^{\frac{1}{2-\beta}} = t_1^{\frac{1}{2(1-\beta)}} \le t_1^{\frac{1}{2-\alpha}} \le d(x,t_1;u),$$

where the second inequality follows from the assumption $\alpha < 2\beta$. Hence we conclude that $\Omega_t(v)$ is contained in $\Omega_t(u)$ for $0 \le t \le t_3$. (b) then follows since $\Omega_t(w^{sub})$ is contained in $\Omega_t(v)$ and $w^{sub} \le (1+5\epsilon)v \le (1-5\epsilon)u$ on the parabolic boundary of Π.

Proof of (c): By (3.8), $v_t/|Dv| \le (1+2\epsilon)|Dv|$ on $\Gamma_t(v)\cap B_{r/2}(0)$. Since $\Gamma_t(w^{sub}) = \hat{\phi}(\Gamma_t(v))$ with $|\nabla\hat{\phi} - I| \le r \le \epsilon$ in $B_r(0)$, the normal velocity $V_{x,t}$ of $\Gamma_t(w^{sub})$ at (x,t) satisfies the following inequalities: For $x \in \Gamma_t(w^{sub})\cap B_{r/2}(0)$ and $t_3 \le t \le t_2$

$$\begin{aligned}
V_{x,t} &= w_t^{sub}(x,t)/|Dw^{sub}(x,t)| \\
&\le (1+\epsilon)v_t(\hat{\phi}^{-1}(x),t)/|Dv(\hat{\phi}^{-1}(x),t)| \\
&\le (1+3\epsilon)|Dv(\hat{\phi}^{-1}(x),t)|
\end{aligned}$$

where the last inequality follows from (3.8). Hence it suffices to prove

$$(1+3\epsilon)|Dv(\hat{\phi}^{-1}(x),t)| \le |Dw^{sub}(x,t)|$$

for $x \in \Gamma_t(w^{sub})\cap B_{r/2}(0)$ and $t_3 \le t \le t_2$. Let $\zeta = (1+5\epsilon)v - w^{sub}$ in $\{w^{sub} > 0\}$. Then ζ is a positive caloric function with the following boundary values:

$$\zeta = \begin{cases} 0 & \text{on} \quad (\partial\Lambda_{\sqrt{t}}\cap B_r(0)) \cup (\Lambda_{\sqrt{t}}\cap \partial B_r(0)) \\ (1+5\epsilon)v & \text{on} \quad \Gamma_t(w^{sub}). \end{cases}$$

Recall that $\Gamma_t(w^{sub})$ is contained in the $10r^2$-neighborhood of $\Gamma_t(v)$ in $B_r(0)$, and by (3.4)

$$10r^2 \le \epsilon^2\sqrt{t_1}.$$

On the other hand, by Lemma 1.5, $v(\cdot,t)$ and $w^{sub}(\cdot,t)$ are almost harmonic in $\sqrt{t}/2$-neighborhood of $\Gamma_0(v)\cap B_{(1-\epsilon)r}(0)$ because the free boundaries of v and w^{sub} are Lipschitz in parabolic scaling. Since $\Gamma_t(w^{sub})$ is contained in the $\epsilon^2\sqrt{t_1}$-neighborhood of $\Gamma_t(v)$, Lemma 1.9 implies that for $x \in \Gamma_t(w^{sub})\cap B_{(1-2\epsilon)r}(0)$

$$v(x,t) \le C\epsilon^{2\beta} v(x - \frac{\sqrt{t_1}}{2}e_n, t). \tag{4.3}$$

By definition ζ has boundary values $(1+5\epsilon)v$ on $\Gamma_t(w^{sub})$ and $\zeta = 0$ elsewhere on its parabolic boundary. Therefore (4.3) implies that for $z \in \partial\Lambda_{\sqrt{t_1}/2} \cap B_{(1-2\epsilon)r}(0)$

$$\zeta(z,t) \leq C \sup\{\zeta(x,t) : x \in B_{\sqrt{t_1}}(z) \cap \Gamma_t(w^{sub})\} \leq C\epsilon^{2\beta} v(z,t).$$

Hence

$$\zeta \leq C\epsilon^{2\beta} v \leq \epsilon v \text{ on } \partial\Lambda_{\sqrt{t_1}/2} \cap B_{(1-2\epsilon)r}(0).$$

Since $\zeta = (1+5\epsilon)v - w^{sub}$, the above inequality implies

$$(1+4\epsilon)v \leq w^{sub} \text{ on } \partial\Lambda_{\sqrt{t_1}/2} \cap B_{(1-2\epsilon)r}(0). \tag{4.4}$$

Using the property $|\nabla\hat{\phi} - I| \leq \epsilon$, (4.4) and the almost harmonicity of v and w^{sub}, we conclude

$$(1+3\epsilon)|Dv(\hat{\phi}^{-1}(x),t)| \leq |Dw^{sub}(x,t)|$$

for $x \in \Gamma_t(w^{sub}) \cap B_{r/2}(0)$ and $t_3 \leq t \leq t_2$.

Proof of (d): In $B_r(0) - B_{r/2}(0)$, $\hat{\phi}$ bends down the free boundary $\Gamma_t(v)$ at least by $r^2/10$. Also, (3.4) implies that $\Gamma_t(v)$ moves at most by $r^2/10$ in $B_r(0)$ for $t \leq t_2$. Hence

$$(\Gamma_t(w^{sub}) \cap (B_r(0) - B_{r/2}(0))) \subset \Omega_0(v) \subset \Omega_t(u).$$

Proof of (e): On $\partial\Lambda_{\sqrt{t}} \cap B_{3r/4}(0)$, the inequality follows from the construction of v and w^{sub}. On $\partial B_{3r/4}(0) \cap \Lambda_{\sqrt{t}}$, we prove (e) by showing that

$$w^{sub} \leq u_1 \leq u,$$

where u_1 solves

$$\begin{cases} (u_1)_t - \Delta u_1 = 0 & \text{in} \quad \{u_1 > 0\} \\ u_1(\cdot,0) = u(\cdot,0) & \text{in} \quad \Lambda_{\sqrt{t}} \cap B_r(0) \\ u_1 = u & \text{on} \quad \partial\Lambda_{\sqrt{t}} \cap B_r(0) \\ u_1 = 0 & \text{on} \quad \partial B_r(0) \cap \Lambda_{\sqrt{t}} \\ \Gamma_t(u_1) = \Gamma_0(u) & \text{for} \quad 0 \leq t \leq t_2. \end{cases}$$

By maximum principle for the heat equation $u_1 \leq u$. Next we show $w^{sub} \leq u_1$ on $\partial B_{3r/4}(0) \cap \Lambda_{\sqrt{t}}$. Observe that w^{sub} and u_1 are almost harmonic with

$$w^{sub} \leq (1-5\epsilon)u = (1-5\epsilon)u_1 \text{ on } \partial\Lambda_{\sqrt{t}} \cap B_{9r/10}(0).$$

Moreover by construction

$$\{w^{sub} > 0\} \cap (B_{9r/10}(0) - B_{r/2}(0)) \subset \{u_1 > 0\}.$$

Since the length r of the strip $\Lambda_{\sqrt{t}} \cap (B_{9r/10}(0) - B_{r/2}(0))$ is much larger than its width $\sqrt{t}$, i.e., since $r \geq \sqrt{t}/\epsilon$, it suffices to prove the following inequality : for some $c > 0$

$$\begin{aligned} &\sup\{v(x,t) : x \in (\partial B_{9r/10}(0) \cup \partial B_{r/2}(0)) \cap \Lambda_{\sqrt{t}}\} \\ &\qquad \leq \epsilon^{-c} \sup\{v(x,t) : x \in \partial B_{3r/4}(0) \cap \Lambda_{\sqrt{t}}\}. \end{aligned}$$

But this is a direct consequence of (3.11). $\square$

Now we prove the following corollary of Lemma 4.1.

Corollary 4.2. *$w^{sub} \leq u$ in $B_{3r/4}(0) \times [0, t_2]$.*

Proof. By Lemma 4.1 (b), $w^{sub} \leq u$ for $0 \leq t \leq t_3$. For $t \geq t_3$, Lemma 4.1 (b) and (e) imply that $w^{sub} \leq u$ on the parabolic boundary of $\Lambda_{\sqrt{t}} \cap B_{3r/4}(0) \times [t_3, t_2]$.

Lemma 4.1 (d) implies that $\Gamma_t(w^{sub})$ cannot hit $\Gamma_t(u)$ in $B_r(0) - B_{r/2}(0)$, and Lemma 4.1(c) implies that $\Gamma_t(w^{sub})$ cannot hit $\Gamma_t(u)$ first in $B_{r/2}(0)$. Hence we conclude $w^{sub} \leq u$ in $B_{3r/4}(0) \times [0, t_2]$. $\square$

2. Construction of a supersolution: The function v will be again used in the construction of w^{sup} as in the construction of w^{sub}, so that the free boundary of w^{sup} is very close to that of w^{sub}.

To make w^{sup} larger than u on $\partial B_{3r/4}$, this time we will bend up the free boundary of v toward e_n-direction so that it is located above $\Gamma_t(u)$ in $B_r(0) - B_{r/2}(0)$. Let $\breve{\phi}$ be the conformal map constructed in section 2. Then $\breve{\phi}$ bends up the free boundary $\Gamma_t(v)$ at least by $r^2/10$ in $B_r(0) - B_{r/2}(0)$. $\breve{v}$

is then defined in $B_r(0) \times [0, t_2]$ as follows.

$$\begin{cases} \breve{v}_t - \Delta \breve{v} = 0 & \text{in} \quad \{\breve{v} > 0\} \\ \Gamma_t(\breve{v}) = \breve{\phi}(\Gamma_t(v)) & \text{in } B_r(0) \\ \breve{v} = v & \text{on} \quad (\partial\Lambda_{\sqrt{t}} \cap B_r(0)) \cup (\Lambda_{\sqrt{t}} \cap \partial B_r(0)) \\ \breve{v}_0 = v_0 & \text{in} \quad \Lambda_{\sqrt{t}} \cap B_r(0) \end{cases}$$

To construct a supersolution based on $\breve{v}$, we state some properties of $\breve{v}$ in Lemma 4.3. Proof of the lemma is parallel to that of Lemma 4.1.

Lemma 4.3. *(a) For $t_3 \leq t \leq t_2$, the normal velocity of $\Gamma_t(\breve{v})$ satisfies*

$$\breve{v}_t/|D\breve{v}| \geq (1-5\epsilon)|D\breve{v}| \text{ on } \Gamma_t(\breve{v}) \cap B_{r/2}(0). \tag{4.5}$$

(b) $\Gamma_0(\breve{v}) \cap (B_r(0) - B_{r/2}(0))$ is located above $\Gamma_t(u)$ for $0 \leq t \leq t_2$, i.e.,

$$\Gamma_0(\breve{v}) \cap (B_r(0) - B_{r/2}(0)) \subset \{x : u(x, t_2) = 0\}.$$

(c) On $\partial\Lambda_{\sqrt{t}} \cap B_{3r/4}(0)$ and $\partial B_{3r/4}(0) \cap \Lambda_{\sqrt{t}}$,

$$\breve{v} \geq (1-15\epsilon)u. \tag{4.6}$$

(a) of Lemma 4.3 yields in particular that the free boundary of $\omega := \breve{v}(x, (1+6\epsilon)t)$ moves with its normal velocity faster than $|D\omega|$ for $t_3 \leq t \leq t_2$. However ω is not quite comparable to u, since ω is not a supersolution of the heat equation. Hence we still need more (rather technical) work to construct our supersolution w^{sup}. This is done below.

Corollary 4.4. *For $t_3 \leq t \leq t_2$ let*

$$w^{sup}(x,t) = \alpha(t)\breve{v}(x, t + \epsilon g(t))$$

where $\alpha(t) = 1 + \epsilon f(t)$, $f(t) = 20(\frac{t}{t_3})^{(1-\beta)/m_1}$ and $g(t) = \int_{t_3}^{t} 10 f(s) ds$. Then w^{sup} is a supersolution in $\Lambda_{\sqrt{t}} \cap B_{3r/4}(0) \times [t_3, t_2 + t_3]$ and

$$u(\cdot, t) \leq w^{sup}(\cdot, t + t_3).$$

Proof. 1. In $\Lambda_{\sqrt{t}} \cap B_r(0) \times [t_3, t_2 + t_3]$

$$\begin{aligned}
w_t^{sup} - \Delta w^{sup} &= \alpha'(t)\breve{v} + (1 + \epsilon g'(t))\alpha(t)\breve{v}_t - \alpha(t)\Delta\breve{v} \\
&= \alpha'(t)\breve{v} + (1 + \epsilon g'(t))\alpha(t)\breve{v}_t - \alpha(t)\breve{v}_t \\
&= \epsilon f'(t)\breve{v} + 10\epsilon f(t)(1 + \epsilon f(t))\breve{v}_t \\
&\geq \frac{\epsilon(1-\beta)f\breve{v}}{m_1 t} - 10\epsilon f(1 + 20\epsilon(\frac{t}{t_3})^{(1-\beta)/m_1})Ct^{-\alpha/2}\breve{v} \\
&\geq \frac{\epsilon f\breve{v}}{t}(\frac{1-\beta}{m_1} - 10C(1 + 20\epsilon(\frac{t_2}{t_3})^{(1-\beta)/m_1})t_2^{(2-\alpha)/2}) \\
&\geq \frac{\epsilon f\breve{v}}{t}(\frac{1-\beta}{m_1} - 10C(1 + 20\epsilon^{1/2})t_2^{(2-\alpha)/2}) \\
&\geq 0
\end{aligned}$$

where the first inequality follows from (c$'$) of Lemma 3.2, the last inequality is due to (3.1) and (3.2), and the third inequality holds since

$$(\frac{t_2}{t_3})^{(1-\beta)/m_1} < t_1^{(\beta^2-2\beta)/2m_1} = \epsilon^{(\beta^2-2\beta)/2} < \epsilon^{-1/2}.$$

Now on the free boundary $\Gamma_t(w^{sup}) \cap B_{3r/4}(0)$,

$$\begin{aligned}
w_t^{sup} &= (1 + 10\epsilon f)\alpha(t)\breve{v}_t \\
&\geq (1 + 10\epsilon f)(1 - 5\epsilon)\alpha(t)|D\breve{v}|^2 \\
&= \frac{(1 + 10\epsilon f)(1 - 5\epsilon)}{1 + \epsilon f}|Dw^{sup}|^2 \\
&\geq |Dw^{sup}|^2
\end{aligned}$$

where (4.5) yields the first inequality, the construction of w^{sup} yields the second equality and $f(t) > 20$ yields the last. It is verified now that w^{sup} is a supersolution in $\Lambda_{\sqrt{t}} \cap B_{3r/4}(0) \times [t_3, t_2 + t_3]$.

2. We will compare $u(\cdot, t)$ and $w^{sup}(\cdot, t + t_3)$, at $t = 0$ and on the lateral boundary

$$\Sigma := (\partial\Lambda_{\sqrt{t}} \cap B_{3r/4}(0)) \cup (\Lambda_{\sqrt{t}} \cap \partial B_{3r/4}(0)).$$

In $\Lambda_{\sqrt{t_3}} \cap B_{3r/4}(0)$

$$\begin{aligned} u(\cdot,0) = \frac{v(\cdot,0)}{1-10\epsilon} &\leq \frac{v(\cdot,t_3)}{(1-10\epsilon)(1-\epsilon)} \\ &\leq \frac{\breve{v}(\cdot,t_3)}{(1-10\epsilon)(1-\epsilon)} \\ &= \frac{w^{sup}(\cdot,t_3)}{(1-10\epsilon)(1-\epsilon)(1+20\epsilon)} \\ &\leq w^{sup}(\cdot,t_3) \end{aligned}$$

where the first inequality follows from Corollary 3.3. Therefore $u(\cdot,t) \leq w^{sup}(\cdot,t+t_3)$ at $t=0$.

Next on Σ

$$\begin{aligned} u(\cdot,t) \leq \frac{v(\cdot,t)}{1-15\epsilon} &\leq \frac{v(\cdot,t+t_3+\epsilon g(t+t_3))}{(1-15\epsilon)(1-\epsilon)} \\ &\leq \frac{\breve{v}(\cdot,t+t_3+\epsilon g(t+t_3))}{(1-15\epsilon)(1-\epsilon)} \\ &= \frac{w^{sup}(\cdot,t+t_3)}{(1-15\epsilon)(1-\epsilon)\alpha(t)} \\ &\leq w^{sup}(\cdot,t+t_3) \end{aligned}$$

where the first inequality follows from (4.6), the second inequality from Corollary 3.3 and the last inequality due to the fact $\alpha(t) \geq 1+20\epsilon$.

3. By step 1., $w^{sup}(x,t+t_3)$ is a supersolution in $\Lambda_{\sqrt{t}} \cap B_{3r/4}(0) \times [0,t_2]$. By 2, $u(x,t) \leq w^{sup}(x,t+t_3)$ on the parabolic boundary. Hence we conclude that in $\Lambda_{\sqrt{t}} \cap B_{3r/4}(0) \times [0,t_2]$,

$$u(x,t) \leq w^{sup}(x,t+t_3).$$

$\square$

5 ϵ-monotonicity of u

In sections 3 and 4, we construct a subsolution $w^{sub}(x,t)$ and a super solution $w^{sup}(x,t+t_3)$ such that

$$w^{sub}(x,t) \leq u(x,t) \leq w^{sup}(x,t+t_3) \text{ in } B_{3r/4}(0) \times [0,t_2].$$

Hence we can locate the free boundary $\Gamma_t(u)$ of u between the Lipschitz free boundaries $\Gamma_t(w^{sub})$ and $\Gamma_t(w^{sup}(\cdot,\cdot+t_3))$. Recall that $\Gamma_t(w^{sub})=\Gamma_t(\hat{v})=\hat{\phi}(\Gamma_t(v))$ and

$$\Gamma_t(w^{sup}(\cdot,\cdot+t_3))=\Gamma_{t+t_3+\epsilon g(t+t_3)}(\breve{v})=\breve{\phi}(\Gamma_{t+t_3+\epsilon g(t+t_3)}(v)).$$

Here one can observe

$$\begin{aligned}\epsilon g(t+t_3)\leq C\epsilon t_3(\frac{t+t_3}{t_3})^{\frac{1-\beta}{m_1}+1} &\leq C\epsilon(\frac{t+t_3}{t_3})^{\frac{1-\beta}{m_1}}(t+t_3)\\ &\leq C\epsilon(\frac{t_2}{t_3})^{\frac{1-\beta}{m_1}}(t+t_3)\\ &\leq C\epsilon^{1-(\beta(1-\beta))/2}(t+t_3)\\ &\leq \epsilon^{7/8}(t+t_3).\end{aligned}$$

Hence, $\Gamma_t(u)$ is located between $\hat{\phi}(\Gamma_t(v))$ and $\breve{\phi}(\Gamma_{(1+\epsilon^{7/8})(t+t_3)}(v))$.

Clearly our hope now is to show that these two trapping boundaries are close to each other, in comparison to the distance it has moved. For this purpose it is necessary to first show that the distance $\Gamma(v)$ has moved by time $t=t_3$ is small.

For $x\in\Gamma_0$ denote $d(x,t)$ to be the distance $\Gamma(v)$ has traveled in the direction of e_n by time t: i.e., $d(x,t)=\{d: x+de_n\in\Gamma_t(v)\}$. (Note that such d is unique since $\Gamma_t(v)$ is Lipschitz with respect to e_n-axis.)

Lemma 5.1. *For any $x\in\Gamma_0\cap B_1(0)$ and for $t_1\leq t\leq t_2$,*

$$d(x,t)\leq C(\frac{t}{t_2})^{\gamma}d(x,t_2),$$

where C depends on M and $0<\gamma<1$ depends on L and n.

Proof. Let $x\in\Gamma_0\cap B_1(0)$. Recall that Lemma 2.3 and Corollary 3.3 imply

$$C^{-1}\leq\frac{v(x-\sqrt{t}e_n,t)}{v(x-\sqrt{t}e_n,0)}\leq C.$$

Also due to Lemma 3.2 (e′), $\Gamma(v)$ is Lipschitz in space and time, and hence v is almost harmonic. Since $d(x,t_2)\leq\epsilon^2\sqrt{t_1}$ (see (3.7)), we have

$$C^{-1}\leq\frac{v(x-\epsilon\sqrt{t_1}e_n,t)}{v(x-\epsilon\sqrt{t_1}e_n,0)}\leq C. \tag{5.1}$$

Recall that $\Gamma(v)$ is determined by $\Gamma(h)$, where h (constructed in (3.5)-(3.6)) solves the Hele-Shaw equation on $[t_1, t_2]$. Since $h = v$ on $\Lambda_{\epsilon\sqrt{t_1}}$, (5.1) enable us to apply Theorem 0.2. Hence for $d := d(x,t)$,

$$t \sim \frac{d^2}{h(x - de_n, t)}. \tag{5.2}$$

Choose $c = c(x,t) < 1$ such that $d := d(x,t) = cd(x,t_2) := cd_2$. Then by Hölder continuity of harmonic functions in Lipschitz domains (Lemma 1.7)

$$h(x - d_2 e_n, t) \geq c^{-\beta} h(x - de_n, t).$$

Combining this inequality with (5.2),

$$t \sim \frac{d^2}{h(x - de_n, t)} \geq \frac{c^2 d_2^2}{c^\beta h(x - d_2 e_n, t)} \sim c^{2-\beta} t_2$$

where the last approximation follows from Lemmas 2.1 and 2.3, the fact that $h(x - d_2 e_n, t) \sim v(x - d_2 e_n, t)$ $(0 \leq t \leq t_2)$ and (5.2). Hence we obtain

$$\frac{d(x,t)}{d(x,t_2)} = c \leq C(\frac{t}{t_2})^{1/(2-\beta)}.$$

□

Lemma 5.2. *Let s be the distance $\Gamma_t(v)$ moves by $t = t_2$ near 0, i.e.,*

$$s = d(0, t_2; v).$$

Then for $0 \leq t \leq t_2$ the free boundaries $\hat{\phi}(\Gamma_t(v))$ and $\breve{\phi}(\Gamma_{(1+\epsilon^{7/8})(t+t_3)}(v))$ are $\sqrt{\epsilon}s$-close in $B_{2s}(0)$. In particular, $\Gamma_t(u)$ is $\sqrt{\epsilon}s$-close to $\Gamma_t(v)$ in $B_{2s}(0)$.

Proof. Since $|D\hat{\phi} - I| \leq \epsilon$ and $|D\breve{\phi} - I| \leq \epsilon$ in $B_r(0)$, it suffices to prove the lemma for $\Gamma_t(v)$ and $\Gamma_{\tilde{t}}(v)$, where $\tilde{t} = (1 + \epsilon^{7/8})(t + t_3)$. First we prove that for $x \in \Gamma_0(v)$, $0 < \delta < 1$ and $t_3 \leq t \leq t_2$

$$d(x, (1+\delta)t) - d(x,t) \leq C\delta d(x,t). \tag{5.3}$$

To prove (5.3), recall that v remains comparable in time on $\partial\Lambda_{\epsilon\sqrt{t_1}} \times [t_3, t_2]$. Then by Theorem 0.2, for $y := x + de_n \in \Gamma_t(v)$ and $z \in \Gamma_s(v) \cap B_{d/2}(y)$

$$V_{z,s} = |Dh(z,s)| \sim |Dh(y,t)| = V_{y,t} \sim d/t \tag{5.4}$$

where $V_{z,s}$ and $V_{y,t}$ denote the normal velocities of $\Gamma(v)$ at (z,s) and (y,t). This implies (5.3).

Now fix $t \in [0, t_2]$. Then

$$\tilde{t} = (1 + \epsilon^{7/8})(t + t_3) \leq t + 2t_3 + 2\epsilon^{7/8} t. \tag{5.5}$$

We divide into two cases.

Case 1. If $t_3 \leq \epsilon^{7/8} t$, then for $x \in \Gamma_0(v) \cap B_{2s}(0)$

$$d(x, \tilde{t}) - d(x, t) \leq d(x, t + 4\epsilon^{7/8} t) - d(x, t) \leq C\epsilon^{7/8} d(x, t) \leq C\epsilon^{7/8} d(x, t_2) \leq \sqrt{\epsilon} s$$

where the second inequality follows from (5.3) and the last inequality follows from $d(x, t_2) \sim d(0, t_2)$.

Case 2. If $\epsilon^{7/8} t \leq t_3$, then

$$\tilde{t} \leq \frac{t_3}{\epsilon^{7/8}} + 4t_3 \leq \frac{t_3}{\epsilon^{6/7}}.$$

Hence for $x \in \Gamma_0(v) \cap B_{2s}(0)$

$$d(x, \tilde{t}) - d(x, t) \leq d(x, \tilde{t}) \leq d(x, \frac{t_3}{\epsilon^{6/7}}) \leq C(\frac{t_3}{\epsilon^{6/7} t_2})^{\gamma} d(x, t_2) \leq \sqrt{\epsilon} s$$

where the third inequality follows from Lemma 5.1 and the last inequality follows since t_3/t_2 is sufficiently small. $\square$

We use the corollary in the proof of Proposition 5.4.

Corollary 5.3. *There exists $0 < \gamma(n, L) < 1$ such that for $s = d(0, t_2; v)$ and $0 \leq t \leq t_2$, the free boundaries $\hat{\phi}(\Gamma_t(v))$ and $\breve{\phi}(\Gamma_{(1+\epsilon^{7/8})(t+t_3)}(v))$ are $\epsilon^{1/4} s$-close in $B_{\epsilon^{-\gamma} s}(0)$.*

Proof. Let $\gamma > 0$ be a sufficiently small constant and let $x_0 \in \Gamma_0 \cap B_{\epsilon^{-\gamma} s}(0)$. Recall that $\Gamma_t(v)$ is well approximated by a free boundary $\Gamma_t(h)$ of a Hele-Shaw flow, with $v = h$ on $\partial \Lambda_{\sqrt{\epsilon} t_1}$. Since v remains comparable in time on $\partial \Lambda_{\sqrt{\epsilon} t_1}$, one can prove that

$$d(y, t_2; v) \leq \epsilon^{-\gamma\alpha} d(0, t_2; v)$$

where $0 < \alpha < 1$ is a constant depending on n and L. (See Lemma 2.5 of [CJK] for a detailed proof.) Combining this inequality dswith Lemma 5.2, we obtain that $\hat{\phi}(\Gamma_t(v))$ and $\breve{\phi}(\Gamma_{(1+\epsilon^{7/8})(t+t_3)}(v))$ are $\epsilon^{1/2-\gamma\alpha} s$-close in $B_{\epsilon^{-\gamma} s}(0)$. By choosing $\gamma < \dfrac{1}{4\alpha}$, we conclude. $\square$

Now we show that the positive level sets of u is close to those of h.

Proposition 5.4. *The level sets of u are $\epsilon^{1/4}s$-close to those of v and therefore to those of h defined in the construction of v in $B_{2s}(0) \times [t_1, t_2]$.*

Proof. Recall that $\Gamma_t(u)$ is trapped between Γ_t^1 and Γ_t^2 where

$$\Gamma_t^1 = \hat{\phi}(\Gamma_t(v)) \text{ and } \Gamma_t^2 = \breve{\phi}(\Gamma_{(1+\epsilon^{7/8})(t+t_3)}(v)).$$

Also note that Γ_t^1 and Γ_t^2 are $\epsilon^{1/4}s$-close in $B_{\epsilon^{-\gamma}s}(0)$ by Corollary 5.3. Now to prove the positive level sets of u are close to Lipschitz graphs, we invoke the arguments used in the proof of Lemma 3.2 (e$'$): more precisely two caloric functions w_1 and w_2 will be constructed with Dirichlet boundaries Γ_t^1 and Γ_t^2, to confine u in between.

Construct parabolic domains Π_1 and Π_2 as below: for any $t \in [0, t_2]$, define

$$\Pi_1 \cap \{t = t\} = (\bigcup_{\eta \geq 0} (\Gamma_t^1 - \eta e_n)) \cap \Lambda_s \cap B_{\epsilon^{-\gamma}s}(0)$$

and

$$\Pi_2 \cap \{t = t\} = (\bigcup_{\eta \geq 0} (\Gamma_t^2 - \eta e_n)) \cap \Lambda_s \cap B_{\epsilon^{-\gamma}s}(0).$$

For $i = 1, 2$ let w_i be the caloric function in Π_i with $w_i = 0$ on Γ_t^i and $w_i = u$ on the rest of the parabolic boundary of Π_i. Observe that Γ_t^1 and Γ_t^2 are $\epsilon^{1/4}s$-close in a ball $B_{\epsilon^{-\gamma}s}(0)$, whose radius $\epsilon^{-\gamma}s$ is much larger than $\mathrm{dist}(\Gamma_0 - se_n, \Gamma_t^2)$. Using this fact, one can compute as in step 1. of the proof of Lemma 3.2 (e$'$) to show that

$$w_1(x,t) \leq u(x,t) \leq w_2(x,t) \leq (1+C\epsilon^{1/4})w_1(x-\epsilon^{1/4}se_n, t) \text{ in } B_{2s}(0). \quad (5.6)$$

The above inequality concludes the proof of Proposition 5.4.

□

Corollary 5.5. *(Harnack-type inequality) Let $x \in \Gamma_0 \cap B_{2s}(0)$, $\frac{t_2}{2} \leq t \leq t_2$ and let $d = d(x, t; u)$. Then*

$$t \sim \frac{d^2}{u(x - de_n, t)} \sim \frac{d^2}{u(x - de_n, t_1)}. \quad (5.7)$$

Proof. For $x \in \Gamma_0 \cap B_{2s}(0)$ and $t \in [\frac{t_2}{2}, t_2]$, let $d = d(x, t; u)$. Then by Corollary 5.3 and (5.4)

$$d \sim d(0, t; v) \sim d(x, t_2; v) = d(x, t_2; h).$$

Combining above approximation with (5.2), we get

$$t \sim \frac{d^2}{h(x-de_n,t)}.$$

Then Corollary 5.5 follows since

$$h(x-de_n,t) \sim h(x-de_n,t_1) \sim v(x-de_n,t_1) \sim u(x-de_n,t_1)$$

where the first follows from (5.1) and Theorem 0.2, the second follows from the almost harmonicity of v, and the last follows from (5.6). □

Due to Proposition 5.4, the level sets of u is $\epsilon^{1/4}s$-close to those of h in $B_{2s}(0)$ by the time $\Gamma(u)$ reaches se_n. We mention that ϵ can be chosen as small as we need: indeed $0<\epsilon<\epsilon_0(n)$, and s depends on ϵ, C_0 and M.

Now define a re-scaled function in Q_1:

$$\tilde{u}(x,t) = \rho^{-1}u(sx, s^2\rho^{-1}t+t_1), \quad \rho = u(-se_n,t_1) \tag{5.8}$$

where $s = d(0,t_2;u)(1+O(\epsilon))$ and $s^2\rho^{-1} \sim t_2$ by Corollary 5.5. Then $\tilde{u}$ solves

$$\begin{cases} \rho\tilde{u}_t - \Delta\tilde{u} = 0, & \text{in } \Omega(\tilde{u}) \\ \tilde{u}_t = |D\tilde{u}|^2 & \text{on } \Gamma(\tilde{u}). \end{cases} \tag{5.9}$$

Next Corollary is due to Proposition 5.4 and the non degeneracy of h. The proof is parallel to that of Proposition 5.1 and Corollary 5.2 in [CJK2].

Corollary 5.6. *$\tilde{u}(x,t)$ is non-degenerate on its free boundary in $\epsilon^{1/4}$-scale:*

$$\sup_{y\in B_{\epsilon^{1/4}}(0)} \tilde{u}(y,t) \geq C\epsilon^{1/4},$$

where $C = C(L,M,n)$.

Let $\tilde{h}$ be the correspondingly scaled version of h, i.e,

$$\tilde{h}(x,t) = \rho^{-1}u(sx, s^2\rho^{-1}t+t_1), \quad \rho = u(-se_n,t_1).$$

The next corollary is a re-scaled version of Proposition 5.4, combined with properties of the level sets of $\tilde{h}$. Below $(\tilde{h})_n$ denotes $D\tilde{h}\cdot(-e_n)$.

Corollary 5.7. *The level sets of $\tilde{u}$ is $\epsilon^{1/4}$- close to those of $\tilde{h}$. Moreover* *(a) The positive level sets of $\tilde{h}$ are Lipschitz graphs in space with Lipschitz constant less than L_n.*

(b) $\tilde{h} \leq M$ in Q_1;

(c) $\Gamma(\tilde{h})$ is a Lipschitz graph in space and time, with spatial Lipschitz constant less than L_n.

(d) $\rho \tilde{h}_t \leq C \tilde{h}_n$.

(e) $\tilde{h}_t \geq -A\tilde{h}_n$, where A is given in Lemma 3.1 (c).

Proof. 1. The first statement is a direct consequence of Proposition 5.4, thus it remains to check the properties of $\tilde{h}$. (a) is due to Lemma 3.1 (d).

2. (b) follows from Lemma 3.1 (a), (3.7), the Harnack inequality in Theorem 1.2 of [CJK2] and the Carlson Lemma for harmonic functions.

3. (c) follows from (b) and Lemma 3.2 (e').

4. (d) follows from Theorem A of [ACS], Lemma 3.1 (a) and Lemma 3.2(e').

5. (e) is due to (b), Lemma 3.1 (b) and Lemma 1.8. □

6 Free boundary regularity based on flatness

Based on Proposition 5.4 and Corollary 5.5, the proof of Theorem 0.1 follows from the iteration method developed in [ACS2] and [CJK2].

Let $\tilde{u}$ be as in (5.8). Since $s \to 0$ as $\epsilon \to 0$, the following proposition yields the main theorem (Theorem 0.1):

Proposition 6.1. *In $B_1(0) \times [1/2, 1]$ the free boundary $\Gamma(\tilde{u})$ is a C^1 graph in space and time. Moreover, $\tilde{u} \in C^1(\bar{\Omega}(\tilde{u}))$ and*

$$C^{-1} \leq |D\tilde{u}|(x,t) \leq C,$$

where C only depends on M and n.

Theorem 1.4 in [ACS2] yields the corresponding result for $\tilde{u}$ solving (5.9) with $\rho = 1$, and Theorem 1.1 in [CJK2] for $\tilde{u}$ solving (5.9) with $\rho = 0$. For general case $0 \leq \rho \leq 1$, the proof of Proposition 6.1 follows by interpolating the proof of Theorem 1.4 in [ACS2] and Theorem 1.1 in [CJK2]. In the appendix we outline the series of technical modifications which is necessary to prove the proposition.

A Regularity of the free boundary

Let $\tilde{u}$ as defined in (5.8) and let $e_t = (0, ..., 1)$ be the unit vector in the time direction. Following the notations in [ACS2] we define ϵ-monotonicity as below:

Definition A.1. *(a) Given $\epsilon > 0$, a function w is called ϵ-monotone in the direction τ if*

$$u(p + \lambda\tau) \geq u(p) \text{ for any } \lambda \geq \epsilon.$$

(b) $W_x(\theta^x, e)$ and $W_t(\theta^t, \nu)$ with $e \in I\!R^n$ and $\nu \in span(e_n, e_t)$ respectively denote a spatial circular cone of aperture $2\theta^x$ and axis in the direction of e, and a two-dimensional space-time cone in (e_n, e_t) plane of aperture $2\theta^t$ and axis in the direction of ν.

(c) w is ϵ-monotone in a cone of directions if w is ϵ-monotone in every direction in the cone.

Due to Proposition 5.4 and Corollary 5.7, $\tilde{u}$ is $\epsilon^{1/4}$-monotone in

$$\mathcal{C} = \Gamma_x(\theta_x, e_n) \cup \Gamma_t(\theta_t, \nu)$$

with $\cot\theta_x \leq L_n$,$\nu \in span(e_n, e_t)$.

Moreover due to Corollary 5.7 (a),

$$\tilde{u}(-e_n, 0) = 1, \quad \sup_{Q_1} \tilde{u} \leq M.$$

A.1 Lipschitz in space

First we follow the arguments in [ACS2] to show that $\tilde{u}$ is fully monotone in a smaller cone in $Q_{1/2}$. Without loss of generality we replace $\epsilon^{1/4}$ by ϵ.

Let us consider $w(x,t) := \tilde{u}(x, \rho t)$. Then w solves the heat equation in $B_1(0) \times [0, \rho^{-1}]$, and w is ϵ-monotone in $W_x(\theta_x, e_n)$ (the same space cone as $\tilde{u}$), and $\epsilon\rho$-monotone in a time cone $W_t(\tilde{\theta}_t, e_n)$ with aperture bigger than a constant $c(M, n)$.

For later purpose we state a modification of Lemma 9 in [ACS], with the minimal assumptions on the caloric function v necessary in the proof:

Lemma A.2. *Let v satisfy $av_t - \Delta v \leq 0$ in $\{v > 0\}$ and v is monotone for the space cone $\Gamma_x(\theta_0, e_n)$. Moreover suppose $|av_t| \leq Cv_n$. Then for the* *function defined in Lemma 9 in [ACS] with its constants depending on n, θ_0 and C,*

$$\tilde{v}(x,t) := \sup_{B_{\varphi(x,t)}} \tilde{u}(y,s)$$

is sub-caloric in $\{v > 0\} \cap D'$ and in $\{v < 0\} \cap D'$.

We apply above lemma to w and proceed as in the proof of Lemma 7.2 in [ACS2] to derive full monotonicity (space-time) in half of the original domain. Note that w satisfies the free boundary velocity law

$$w_t = \rho |Dw|^2 \text{ on } \Gamma(w),$$

which is slower than the one in (ST) by a factor of ρ and thus will slow down the regularization of the free boundary in the proof of Lemma 7.2. On the other hand we also have a long time interval $[\rho^{-1}, \rho^{-1}]$ for the regularization over time. Hence we obtain the following lemma:

Lemma A.3. *There exist constants $0 < \epsilon_0, \lambda < 1$ depending on n, δ, μ such that if $\epsilon < \epsilon_0, \delta$ then w is $\lambda\epsilon\rho$-monotone in the cone of directions $\Gamma_x(\theta_x - \bar{c}\epsilon^\beta, e_n)$ and $\Gamma_t(\theta_t - \bar{c}\epsilon^\beta, \nu)$ in the domain $B_{1-\epsilon^\alpha}(0) \times [\rho^{-1}(1-\epsilon^\alpha), \rho^{-1}]$ where $0 < \alpha < \beta < 1/2$.*

The proof of Lemma A.3 is parallel to that of Lemma 7.2 in [ACS2].

One can then iterate above lemma to improve the ϵ-monotonicity to full monotonicity, and state the result in terms of $\tilde{u}$:

Lemma A.4. *$\tilde{u}$ is fully monotone in $Q_{1/2}$ for the cone*

$$\mathcal{C}_1 := \Gamma_x(\theta_x - C\epsilon^\beta, e_n) \cup \Gamma_t(\theta_t - C\rho\epsilon^\beta, \nu),$$

for some constants $C, \alpha, \beta > 0$.

Corollary A.5. *$\tilde{u}$ satisfies*

$$|\rho \tilde{u}_t| \leq \tilde{u}_n \quad \text{in } Q_1.$$

A.2 Regularity in space

To Proceeding from Lipschitz to C^1 regularity in space, we first estimate the change of $\dfrac{Du}{|Du|}$ away from the boundary:

Lemma A.6 (Interior enlargement of the space cone). *Let $\tilde{u}$ as above. In addition suppose that $u(\cdot,t)$ is monotone for the cone $W_x(\nu_l,\theta_l)$ in $B_{2^{-l}+1}(0)\times(0,2^{-l}\delta_l)$ with $\tilde{u}(-2^{-l}e_n,0)=2^{-l}$ and $\delta_l=\pi/2-\theta_l>l^{-2}$. Then for $l>l_0$ where l_0 depending only on θ_0 and n, there exists a unit vector $\nu_{l+1}\in I\!R^n$, $0<h_0(n)<1$ and $0<r_0(n)<1$ such that $\tilde{u}(\cdot,t)$ is monotone increasing in $B_{2^{-l}-5}(-2^{-l-1}e_n)\times(0,r_0 2^{-l}\delta_l)$ for the cone $\eta\in W_x(\nu_{l+1},\theta_{l+1})$ with*

$$\delta_{l+1}\le h_0\delta_l.$$

The proof of above lemma is parallel to that of Lemma 6.1 in [CJK2], and uses Proposition 5.4 instead of Proposition 4.1 in [CJK2].

The second lemma, combined with the one above, states that for any free boundary point (x_0,t_0), $\frac{D\tilde{u}}{|D\tilde{u}|}$ converges as we approach the point. The rate of this convergence determines the regularity of $\Gamma(u)$ in space.

Lemma A.7 (basic iteration). *Let $\tilde{u}$ solve (5.9) in Q_1 with $(0,0)\in\Gamma(\tilde{u})$, $\tilde{u}(-\frac{3}{4}e_n,0)=1$ and $|D\tilde{u}|>m_0$ in $\Gamma(\tilde{u})$. In addition suppose that $|a\tilde{u}_t|\le C\tilde{u}_n$ and there exists a unit vector $\nu\in I\!R^n$ and $0<b_0<1$ such that*

$$\alpha(D\tilde{u},-e_n)\le\delta \text{ in } B_1(0)\times(-2r,2r)$$

and

$$\alpha(D\tilde{u},\nu)\le b_0\delta \text{ in } B_{1/16}(-e_n)\times(-2r,2r).$$

Then there exists a unit vector $\nu_1\in I\!R^n$ and a constant $0<c<1$ depending on n,m_0 and b_0 such that

$$\alpha(D\tilde{u}(x,t),\nu_1)\le\delta_1 \text{ in } B_{1/2}(0)\times(-r,r)$$

where $\delta_1\le\delta-c\delta r$.

Now proceeding as in section 6 in [CJK2] yields the C^1 regularity of the free boundary:

Theorem A.8. *$\Gamma(\tilde{u})$ is C^1 in space in $Q_{1/2}$. In particular, three exist constants $l_0,C_0>0$ depending only on L,n and M such that for a free boundary point $(x_0,t_0)\in\Gamma(\tilde{u})$, $\Gamma(\tilde{u})\cap B_{2^{-l}}(x_0,t_0)$ is a Lipschitz graph with Lipschitz constant less than $\dfrac{C_0}{l}$ if $l\ge l_0$.*

.

A.3 Regularity in time

Lastly, proceeding as in section 7-8 of [CJK2] yields the differentiability of $\Gamma(u)$ in time. The main step in the argument is the following proposition: the statement and its proof is parallel to those of Theorem 7.2 in [CJK2].

Proposition A.9. *There exist constants $l_0 > 0$ and $1 < \gamma < 2$ depending only on L, n, M such that for $(x_0, t_0) \in \Gamma(\tilde{u}) \cap Q_1$, if $l > l_0$ then $\Gamma(\tilde{u}) \cap B_{2^{-l}}(x_0, t_0)$ is a Lipschitz graph with Lipschitz constant less than $l^{-\gamma}$.*

Above proposition and the blow-up argument in section 8 of [CJK2] yields the desired result:

Theorem A.10.

$$C^{-1} \leq |D\tilde{u}|(x,t) \leq C \text{ in } \bar{\Omega}(\tilde{u}) \cap Q_{1/2},$$

where $C = C(M, n)$. Moreover $\Gamma(\tilde{u})$ is differentiable in time with its normal velocity $V_{x,t} = |D\tilde{u}|$ on $\Gamma(\tilde{u})$.

References

[ACS1] I. Athanasopoulos, L. Caffarelli and S. Salsa, Caloric functions in Lipschitz domains and the Regularity of Solutions to Phase Transition Problems, *Ann. of Math. (2)***143** (1996) pp 413–434.

[ACS2] I. Athanasopoulos, L. Cafferlli and S. Salsa, Phase Transition problems of parabolic type: flat free boundaries are smooth, *Comm. Pure Appl. Math.***51** (1998) pp 77–112.

[ACS3] I. Athanasopoulos, L. Cafferlli and S. Salsa, Regularity of the free boundary in parabolic phase-transition problems. *Acta Math.* **176** (1996), pp 245–282.

[C1] L. Caffarelli, The regularity of free boundaries in higher dimensions,*Acta Math***139** (1977), pp 155-184.

[C2] L. Caffarelli, A Harnack inequality approach to the regularity of free boundaries, Part I: Lipschitz free boundaries are $C^{1,\alpha}$, *Rev. Mat. Iberoamericana***3** (1987), no. 2, 139–162

[CJK1] S. Choi, D. S. Jerison and I. C. Kim, Regularity for the One-Phase Hele-Shaw problem from a Lipschitz initial surface, *Amer. J. Math.* **129** (2007) pp 527–582.

[CJK2] S. Choi, D. S. Jerison and I. C. Kim, A local regularization theorem on one-phase Hele-Shaw flow, submitted.

[CK] S. Choi and I. Kim, Waiting time phenomena of the Hele-Shaw and the Stefan problem, *Indiana U. Math. Journal,* **55** (2006), pp 525-552.

[CPS] L. Caffarelli, A. Petrosyan and H. Shahgholian, Regularity of a free boundary in parabolic potential theory, Journal of AMS **17** (2004) pp 827–869.

[FGS1] E. B. Fabes, N. Garofalo and S. Salsa, Comparison Theorems for Temperatures in non-cylindrical Domains, *Atti Accad. Naz. Lincei, Read. Ser. 8,***78** (1984) pp. 1–12

[FGS2] E. B. Fabes, N. Garofalo and S. Salsa, *Illinois Journal of Mathematics***30** (1986) pp 536–565.

[JK] D. S. Jerison and C. E. Kenig, Boundary behavior of Harmonic functions in Non-tangentially Accessible Domains, *Advan. in Math.* **46** (1982), 80-147.

[K] I. C. Kim, *Uniqueness and Existence result of Hele-Shaw and Stefan problem,* Arch. Rat. Mech. Anal, **168** (2003), 299-328.

[King] J. R. King, Development of singularities in some moving boundary problems, European J. Appl. Math. **6** (1995)

[M] A. M. Meirmanov, The Stefan Problem, de Gruyter Expositions in Mathematics, vol. 3, Walter de Gruyter & Co., Berlin, 1992, ISBN

[R] L. Rubinstein, The Stefan Prolem, American Mathematical Society, Providence, R.I., 1971.