Regularity of the Free Boundary for the One Phase Hele-Shaw Problem

Inwon C. Kim
Department of Mathematics, MIT

August 6, 2004

Abstract

In this paper we prove that, in a local neighborhood, Lipschitz continuous free boundary of a solution of the one-phase Hele-Shaw problem is indeed smooth if the solution is Lipschitz continuous and non-degenerate in the neighborhood.

0 Introduction

Consider a compact set $K \subset \mathbb{R}^n$ with smooth boundary ∂K. Suppose that a bounded domain Ω contains K and let $\Omega_0 = \Omega - K$ and $\Gamma_0 = \partial\Omega$ (Figure 1). Note that $\partial\Omega_0 = \Gamma_0 \cup \partial K$.

Let u_0 be the harmonic function in Ω_0 with $u_0 = f > 0$ on K and zero on Γ_0. Let $u(x,t)$ solve the one phase Hele-Shaw problem

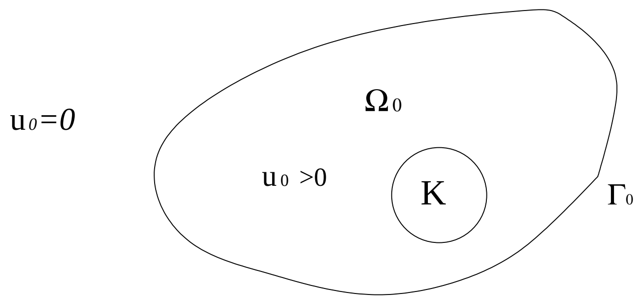

Figure 1.

$$
(HS) \qquad \begin{cases} -\Delta u = 0 & \text{in } \{u>0\}\cap Q, \\ u_t - |Du|^2 = 0 & \text{on } \partial\{u>0\}\cap Q, \\ u(x,0)=u_0(x); \quad u(x,t)=f \text{ for } x\in\partial K. \end{cases}
$$

where $Q = (I\!R^n - K)\times(0,\infty)$. We refer to $\Gamma_t(u) := \partial\{u(\cdot,t)>0\} - \partial K$ as the *free boundary* of u at time t. Note that if u is smooth up to the free boundary, then the free boundary moves with normal velocity $V = u_t/|Du|$, and hence the second equation in (HS) implies that $V = |Du|$. The classical Hele-Shaw problem models an incompressible viscous fluid which occupies part of the space between two parallel, narrowly placed plates. The short-time existence of classical solutions when Γ_0 is $C^{2+\alpha}$ was proved by Escher and Simonett [ES]. When $n=2$, Elliot and Janovsky [EJ] showed the existence and uniqueness of weak solutions formulated by a parabolic variational inequality in $H^1(Q)$.

Our goal is to prove that, in a local neighborhood, if the *free boundary* $\Gamma(u) := \cup_{t\geq 0}\Gamma_t(u)$ is a Lipschitz graph in space-time and if u is Lipschitz continuous and non-degenerate on the free boundary, then the free boundary is indeed differentiable and u satisfies the free boundary condition $V = |Du|$ in the classical sense. As for the notion of generalized solutions, we use viscosity solutions whose existence and uniqueness were proved in [K1]. For rigorous statements, see Section 1.

The above result - in short, Lipschitz free boundaries are smooth - is proved in [ACS] for the Stefan problem

$$
(St) \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } \{u>0\}, \\ u_t - |Du|^2 = 0 & \text{on } \partial\{u>0\} \end{cases}
$$

The underlying idea of such result is that the regularity of positive level sets of u 'propagates' to the free boundary over time. Concerning our problem (HS), this idea doesn't seem to apply since our solution u is harmonic at each time, and thus the regularity of u in time is not necessarily better in the positive set than on the free boundary. In other words the difficulty in our analysis lies in the hyperbolic nature of our problem. However our result holds because (a) u has strong spatial regularity since $u(\cdot,t)$ is harmonic for each $t>0$, and (b) the regularity of level sets in space and in time affect each other by the free boundary motion.

In fact if we consider the blow-up solution of (HS)

$$u(x,t) := a(t)(x_n + \int_0^t a(t)dt)_+, \quad x = (x', x_n) \in I\!R^{n-1} \times I\!R$$

where $a(t)$ is only Lipschitz continuous in time, then u may not be differentiable in time, but $\Gamma_t(u)$ is differentiable in time with normal velocity $a(t)$.

Here is a summary of the paper. The main tools used in the following sections are barrier arguments, properties of harmonic functions in Lipschitz domains (see section 1), and the iteration method developed in [ACS] and [C2]. In section 1 we introduce notations used in this paper and state our main results. In section 2 we state and prove some preliminary lemmas which will be used in later sections. In particular we show that positive level sets of u do not change their normal direction too fast over time (Lemma 2.6), which produces an 'interior gain' for the positive level sets. In section 3 we construct a family of perturbations to prove a propagation lemma (Lemma 3.4) which carries the 'interior gain' to the free boundary over time. In section 4 we adopt the iteration method and apply Lemma 3.4 to show that the free boundary is differentiable in space. In section 5 using the spatial regularity of u, we show that the free boundary is indeed differentiable in space-time. Furthermore we prove that the spatial regularity of the free boundary is strong enough for Du to exist on the free boundary, satisfying the free boundary condition $V = |Du|$. Finally in section 6, as an application of the regularity results from previous sections, we transform (HS) into an obstacle problem to prove that the free boundary of a viscosity solution of (HS) with smooth boundary data becomes analytic after a finite time.

1 Notations and main results

We first introduce several notations that are used frequently in this paper.

- Let us assume that $n > 0$ is a given integer. Then we define

(a) For $x \in I\!R^n$ let $B_r(x) = \{y \in I\!R^n : |x - y| < r\}$

(b) For $(x,t) \in I\!R^{n+1}$ let
$B_r^{n+1}(x,t) = \{(y,s) \in I\!R^{n+1} : |(x,t) - (y,s)| < r\}$.

- For $D \subset I\!R^{n+1}$ and a constant c, $cD = \{y \in I\!R^{n+1} : y = cz, z \in D\}$.

• For a unit vector $\nu \in I\!R^n$ and $0 < \theta \leq \pi/2$, we define

$$W(\theta, \nu) := \{x \in I\!R^n :< x, \nu >\geq |x| \cos\theta.\}$$

Also we denote by $\alpha(e, f)$ the angle between vectors e and f in $I\!R^n$.

• A pair of functions $u_0, v_0 : \bar{D} \to [0, \infty)$ are *(strictly) separated* (denoted by $u_0 \prec v_0$) in $D \subset I\!R^n$ if

(i) the support of u_0, $\text{supp}(u_0) = \overline{\{u_0 > 0\}}$ restricted in $\bar{D}$ is compact and

(ii) in $\text{supp}(u_0) \cap \bar{D}$ the functions are strictly ordered:

$$u_0(x) < v_0(x).$$

• For a nonnegative function $u(x, t)$ defined in $D \subset I\!R^n \times [0, \infty)$,

$u^*(x, t) = \limsup_{(y,s)\in D \to (x,t)} u(y, s)$;

$u_*(x, t) = \liminf_{(y,s)\in D \to (x,t)} u(y, s)$;

$\Omega(u) = \{(x, t) : u(x, t) > 0\}, \quad \Omega_t(u) = \{x : u(x, t) > 0\}$;

$\Gamma(u) = \partial\{(x, t) : u(x, t) = 0\}, \quad \Gamma_t(u) = \partial\{x : u(x, t) = 0\}$.

Below we state the definition of viscosity solutions introduced in [K1]: We keep the notions used in the previous section.

Definition 1.1 *(1) A nonnegative uppersemicontinuous function u defined in $\bar{Q}$ is a* viscosity subsolution *of (HS) with initial data u_0 and fixed boundary data $f > 0$ if*

(a) $u = u_0$ at $t = 0$, $u \leq f$ for $x \in K$;

(b) $\bar{\Omega}(u) \cap \{t = 0\} = \bar{\Omega}_0(u)$;

(c) for each $T \geq 0$ the set $\bar{\Omega}(u) \cap \{t \leq T\}$ is bounded; and

(d) for every $\phi \in C^{2,1}(Q)$ such that $u - \phi$ has a local maximum in $\bar{\Omega}(u) \cap \{t \leq t_0\} \cap Q$ at (x_0, t_0),

$$(i) \qquad -\Delta\phi(x_0,t_0) \leq 0 \qquad \textit{if } u(x_0,t_0) > 0.$$

$$(ii) \qquad \min(-\Delta\phi, \phi_t - |D\phi|^2)(x_0,t_0) \leq 0 \qquad \textit{otherwise.}$$

(2) A nonnegative lowersemicontinuous function v defined in $\bar{Q}$ is a viscosity supersolution *of (HS) with initial data v_0 and fixed boundary data $f > 0$ if*

(a) $v = v_0$ at $t = 0$, $v \geq f$ for $x \in K$ and

(b) if for every $\phi \in C^{2,1}(Q)$ such that $v - \phi$ has a local minimum in $\bar{\Omega}(v) \cap \{t \leq t_0\} \cap Q$ at (x_0,t_0),

$$(i) \qquad -\Delta\phi(x_0,t_0) \geq 0 \qquad \textit{if } v(x_0,t_0) > 0,$$

(ii) *If $v(x_0,t_0) = 0$ and if*

$$|D\phi|(x_0,t_0) \neq 0 \textit{ and } (x_0,t_0) \in \overline{\Omega(\phi) \cap \Omega(v)},$$

then

$$\max(-\Delta\phi, \phi_t - |D\phi|^2)(x_0,t_0) \geq 0.$$

(3) u is a viscosity solution of (HS) with boundary data u_0 and f if u^* is a viscosity subsolution and if $u = u_*$ is a viscosity supersolution of (HS) with boundary data u_0 and f.

(4) A nonnegative lower semicontinuous function u defined in the closure of a cylindrical domain $\Sigma := D \times (a,b) \subset I\!R^n \times I\!R$ where D is bounded in $I\!R^n$ is a viscosity solution of (HS) in Σ if (1)(d) holds for u^* and (2)(b) holds for $u = u_*$ with Σ replacing Q.

Remark. It follows from the above definition that if u is a continuous viscosity solution of (HS) in an open subset $\mathcal{O}$ in $I\!R^{n+1}$ then for fixed t $u(\cdot,t)$ is harmonic in $\Omega_t(u) \cap \{(x,t) \in \mathcal{O}\}$.

The following two theorems state important properties of the viscosity solutions: we refer to [K1] for proofs.

Theorem 1.2 (localized comparison principle) *Let u, v be respectively viscosity sub- and supersolutions in $D \times (0,T) \subset Q$ with initial data $u_0 \prec v_0$ in D. If $u \leq v$ on ∂D and $u < v$ on $\partial D \cap \bar{\Omega}(u)$ for $0 \leq t < T$, then $u(\cdot,t) \prec v(\cdot,t)$ in D for $t \in [0,T)$.*

Theorem 1.3 *Let u_0 be the harmonic function in Ω_0 with $u_0 = f > 0$ on K and $u_0 = 0$ on Γ_0, where $f(x,t)$ is a smooth function in $\bar{Q}$. In addition suppose that Du_0 exists at each point at Γ_0 as the limit from Ω_0 and satisfies $|Du_0| > 0$ on Γ_0. Then there exists a unique viscosity solution u of (HS) in Q with its boundary data.*

Below we state the main result of the paper:

Theorem 1 *Let u be a viscosity solution of (HS) in $S := B_2(x_0) \times (t_0 - 2, t_0 + 2)$ and suppose $(x_0, t_0) \in \Gamma(u)$. Moreover assume that u satisfies the following properties in S:*

(Pa) $\Gamma(u)$ is given by a Lipschitz graph $\{x_n = f(x', t) : (x', t) \in I\!R^{n-1} \times I\!R\}$ with Lipschitz constant L_0.

(Pb) u is continuous and $|u_t|/|Du| \leq M_0$ in $\Omega(u)$.

(Pc) $u_n := D_{e_n} u \geq m_0 > 0$ in $\Omega(u)$, where e_n is a unit vector in the direction of x_n.

Then the following conclusions hold:

(1) $\Gamma(u)$ is a C^1 graph in space and time in $S' := B_1(x_0) \times (t_0 - 1, t_0 + 1)$. Moreover, for any $\eta > 0$, there exists a positive constant $\bar{C}$ depending only on the constants in $(Pa) - (Pb)$ and n, η such that, for $(x', x_n, t), (y', y_n, s) \in \Gamma(u) \cap S'$,

$$
\begin{aligned}
|\nabla_{x'} f(x', t) - \nabla_{x'} f(y', t)| &\leq \bar{C}(-log|x' - y'|)^{-3/2+\eta}, \\
|f_t(x', t) - f_t(x', s)| &\leq \bar{C}(-log|t - s|)^{-1/2+\eta}.
\end{aligned}
$$

(2) $u(\cdot, t) \in C^1(\bar{\Omega}_t(u) \cap B_1(x_0))$ for $|t - t_0| \leq 1$ and

$$V_{x,t} = |Du|(x, t) \text{ on } \Gamma(u) \cap S'$$

where $V_{x,t}$ is the normal velocity of $\Gamma(u)$ at (x, t).

Remark

1. In addition to (Pa) and (Pc), condition (Pb) is necessary. For example see the blow-up solution given in the introduction with discontinuous $a(t)$.
2. By a barrier argument using radial solutions of (HS) and Lemma 2.2, one

can check that a viscosity solution of (HS) in S with condition (Pa)-(Pb) satisfies

$$(Pb)' \qquad |Du| \le M = M(M_0, L_0) \text{ in } B_{3/2}(x_0) \times (t_0 - 3/2, t_0 + 3/2).$$

Due to the hyperbolic nature of our problem, to obtain further regularity of $\Gamma(u)$ it is necessary to obtain more information for the regularity of boundary data on ∂S, or alternatively the global properties of u if u is a viscosity solution of (HS) in Q with $S \subset Q$. As a consequence of Theorem 1 and [K2], the following result is obtained in section 6:

Theorem 2 *Let u be a viscosity solution of (HS) with boundary data $f = 1$ and u_0. Moreover suppose that $|Du_0| > 0$ on Γ_0. Then there is $0 < T_0 < \infty$ such that $\Gamma(u)$ is analytic in $Q \cap \{t > T_0\}$.*

For the sake of simplicity, from now on we assume that $L_0 \le 1/4$ in (Pa). Hence for example if $(x_1, t_1) \in \Gamma(u) \cap S'$ then the region $B_{1/4}(x_1 + 3/4 e_n) \times [t_1 - 1, t_1 + 1]$ is contained in $\Omega(u)$. We leave it to the reader to check that a parallel argument holds for the general L_0.

2 Preliminary results

First we state several properties of harmonic functions which are important in our analysis.

Lemma 2.1 *(Dahlberg, see [D]) Let u_1, u_2 be two nonnegative harmonic functions in a domain D of $\mathbb{R}^n$ of the form*

$$D = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < 1, |x_n| < M, x_n > f(x')\}$$

with f a Lipschitz function with constant less than M and $f(0) = 0$. Assume further that $u_1 = u_2 = 0$ along the graph of f. Then, on the domain

$$D_{1/2} = \{|x'| < 1/2, |x_n| < M/2, x_n > f(x')\}$$

We have

$$0 < C_1 \le \frac{u_1(x', x_n)}{u_2(x', x_n)} \cdot \frac{u_2(0, M/2)}{u_1(0, M/2)} \le C_2$$

with C_1, C_2 depending only on M.

Lemma 2.2 *(Caffarelli, see [C2]) Let* u *be as in Lemma 2.1. Then* $u_n \geq 0$ *on* cD, $c = c(M)$, *and for* $0 < d < cM$

$$0 < C_1 \leq \frac{u_n(0,d)d}{u(0,d)} \leq C_2$$

where $C_i = C_i(M)$.

Lemma 2.3 *(Caffarelli, see [C2])*
Let $0 \leq u_1 \leq u_2$ *be harmonic functions in* $B_1(0)$. *Assume that on* $B_{1-\epsilon}(0)$

$$v_\epsilon(x) = \sup_{B_\epsilon(x)} u_1(y) \leq u_2(x)$$

and further

$$u_2(0) - v_\epsilon(0) \geq \sigma\epsilon u_2(0).$$

Then, for some $\kappa, h > 0$ *(independent of* ϵ, σ*) we have*

$$u_2(x) - v_{(1+\sigma h)\epsilon}(x) \geq \kappa\sigma\epsilon u_2(0)$$

in $B_{1/2}(0)$.

Suppose $\phi_t - |D\phi|^2(x_1, t_1) > 0$.

Lemma 2.4 *(Caffarelli, see [C2]) Let* u *be harmonic in* B_1. *Then there exists* $\epsilon_0 > 0$ *such that if*

$$u(x + \epsilon p) \geq u(x) \text{ for } \epsilon > \epsilon_0 \text{ and } x, x + \epsilon p \in B_1$$

for a unit vector $p \in I\!R^2$, *then* $D_p u \geq 0$ *in* B_1.

Next we show that for the most cases the first term on the variational inequalities in Definition 1.1 can be omitted.

Lemma 2.5 *Let* u *be a continuous viscosity solution of* (HS) *in* S, $(x_1, t_1) \in \Gamma(u) \cap S$ *and let* ϕ *be a* $C^{2,1}$*-function in a local neighborhood of* (x_1, t_1) *such that* $u - \phi$ *has a local maximum zero at* (x_1, t_1) *in* $\bar{\Omega}(u) \cap \{t \leq t_1\}$ *and* $|D\phi|(x_1, t_1) \neq 0$. *Then it follows that*

$$(\phi_t - |D\phi|^2)(x_1, t_1) \leq 0.$$

Proof.

1. By hypothesis, for any $\epsilon > 0$ there is a space-time ball B^{n+1} with $B^{n+1} \cap \{t \leq t_1\} \subset (S - \bar{\Omega}(\phi)) \cap \{t \leq t_1\}$ and with its outward normal vector $(\nu, \phi_t/|D\phi| - \epsilon)$ at (x_1, t_1) where ν a unit vector in $I\!R^n$ such that

$$\bar{B}^{n+1} \cap \Gamma(\phi) \cap \{t \leq t_1\} = \{(x_1, t_1)\}.$$

Let $a = |D\phi|(x_1, t_1)$. Next we define a $C^{2,1}$ function $\varphi(x,t)$ in a neighborhood of $(2B^{n+1} - B^{n+1}) \cap \{|t - t_1| \leq c\}$, where $c << 1$, such that

$$\begin{cases} \varphi \geq 0, -\Delta\varphi(x,t) > 0 & \text{in } (2B^{n+1} - B^{n+1}) \cap \{|t - t_1| \leq c\}, \\ \varphi = 0 & \text{on } \partial B^{n+1} \cap \{|t - t_1| \leq c\}, \\ |D\varphi| = a + \epsilon & \text{on } \partial B^{n+1} \cap \{t = t_1\}. \end{cases}$$

(For the construction of such test function, see Appendix A of [K1].)

2. By the regularity of ϕ, $|D\phi|(x,t) \leq a + \epsilon$ in a small neighborhood of (x_1, t_1) with $\phi \leq 0$ on ∂B^{n+1}. It follows that $\phi \leq \varphi$ in a neighborhood of (x_1, t_1) in $(2B^{n+1} - B^{n+1}) \cap \{t \leq t_1\}$ with $\phi = \varphi = 0$ at (x_1, t_1). Since $u - \phi$ has a local maximum at (x_1, t_1) in $\bar{\Omega}(u) \cap \{t \leq t_1\}$, it follows that $u - \varphi$ has a local maximum at (x_1, t_1) in the set $\bar{\Omega}(u) \cap \{t \leq t_1\}$. Note that by construction $-\Delta\varphi(x_1, t_1) > 0$. Hence by definition of u we obtain

$$\frac{\varphi_t}{|D\varphi|}(x_1, t_1) \leq |D\varphi|(x_1, t_1) = |D\phi|(x_1, t_1) + \epsilon.$$

On the other hand, note that the zero level set of φ is ∂B^{n+1} in a neighborhood of (x_1, t_1) in $I\!R^{n+1}$. Hence by the construction of B^{n+1} the normal velocity of the level set $\{\varphi = \varphi(x_1, t_1)\}$ at (x_1, t_1), $\varphi_t/|D\varphi|(x_1, t_1)$, equals that of $I\!R^{n+1} - B^{n+1}$ at (x_1, t_1) and thus due to the above inequality

$$\frac{\phi_t}{|D\phi|}(x_1, t_1) - \epsilon \leq |D\phi|(x_1, t_1) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we can conclude.

$\Box$.

Lastly we give an estimate for the change of the normal directions of the positive level sets. We say that a function f has a *cone of monotonicity* $W(\theta, \nu)$ in $D \subset I\!R^n$ if f is monotone increasing along every direction $p \in W(\theta, \nu)$ in D.

Lemma 2.6 *Let u be as given in Theorem 1. Furthermore suppose $(x_1, t_1) \in \Gamma(u) \cap S'$ and*

$$A \leq \frac{u_t}{u_n} \leq B \text{ and } \alpha(Du, e_n) \leq \delta$$

in $\Sigma \cap \Omega(u)$ where $\Sigma = B_1(x_1) \times (t_1 - r, t_1 + r)$. Let us denote $\mu = \max(\delta, B - A)$. If $\delta/\mu < r$, then there are $0 < l_0, r_0 < 1$ only depending on the constants in $(Pa) - (Pc)$ and $\nu \in I\!R^n$ such that

$$\alpha(Du, \nu) \leq l_0\delta \text{ in } B_{1/8}(x_1 + \frac{3}{4}e_n) \times (t_1 - r_0\delta/\mu, t_1 + r_0\delta/\mu.)$$

Proof.

1. As before we change coordinates so that $(x_1, t_1) = (0, 0)$. Due to our hypothesis $\alpha(Du(x, 0), e_n) \leq \delta$ in $B_1(0) \cap \Omega(u)$. In other words u is increasing along the directions in the cone $W(\pi/2 - \delta, e_n)$ in $B_1(0)$. In particular in $B_{1/4}(\frac{3}{4}e_n)$ we have

$$\sup_{B_{\epsilon\lambda_0}(x)} u(x - \epsilon e_n, 0) \leq u(x, 0). \tag{2.3}$$

with small $\epsilon > 0$ and $\lambda_0 = \sin(\pi/2 - \delta)$. We also observe that

$$\begin{aligned} u(\tfrac{3}{4}e_n, 0) - \sup\nolimits_{B_{\epsilon\lambda_0}(3/4e_n)} u(y - \epsilon\eta, 0) & \\ & \geq \tfrac{\delta\epsilon}{2} Du(\tfrac{3}{4}e_n, 0) \\ \text{(by Lemma 2.2)} & \geq C\delta\epsilon u(\tfrac{3}{4}e_n, 0) \end{aligned} \tag{2.4}$$

if $\epsilon > 0$ is small enough, where $\eta := \frac{Du}{|Du|}(\frac{3}{4}e_n, 0)$ and $C = C(L_0)$.

2. Due to (2.3),(2.4) and Lemma 2.3, the technical arguments in section 5 and 6 of [C2] apply (Note that Lemma 2.3 does not directly apply since e_n may be different from η) and we obtain a unit vector $\nu \in I\!R^n$ and $\lambda = (1 + k\delta)\lambda_0$, with $k > 0$ independent of δ and μ, such that

$$\sup_{B_{\epsilon\lambda}(x)} u(x - \epsilon\nu, 0) \leq u(x, 0) \text{ in } B_{1/6}(\frac{3}{4}e_n).$$

In terms of Du this means that there is a unit vector $\nu \in I\!R^n$ and $0 < h < 1$, independent of δ and μ, such that

$$\alpha(Du(x, 0), \nu) \leq (1 - h)\delta \text{ in } B_{1/6}(\frac{3}{4}e_n). \tag{2.5}$$

3. Since $|Du| \geq m_0$ in Σ, for a unit vector p such that

$$\alpha(p, \nu(0)) \leq \pi/2 - (2 - h/2)\delta,$$

$$u(\cdot + \epsilon p, 0) \geq \frac{1}{4} m_0 h \delta \epsilon + u(\cdot, 0) \text{ in } B_{1/8}(\frac{3}{4} e_n), 0 < \epsilon < 1/16.$$

Moreover since $|u_i| \leq \delta |Du|, i = 1, ..., n-1$ in $\Omega(u)$ by our hypothesis, by interior estimates of harmonic functions

$$|u_{ii}| = O(\delta)|Du|(\frac{3}{4} e_n, t), i = 1, ..., n-1 \text{ in } B_{1/8}(\frac{3}{4} e_n) \times (-r, r).$$

Since u is harmonic in the region and $|Du| \leq M$ (see the remark below Theorem 1) we obtain $|u_{nn}| = O(\delta)$ and thus

$$|u_n(\cdot + \epsilon p, t) - u_n(\cdot, t)| = C\delta\epsilon \text{ in } B_{1/8}(\frac{3}{4} e_n),$$

where C is independent of δ, μ.

4. Let us pick a small constant $\epsilon_0 > 0$. By the previous argument and our hypothesis, there is $r_0 > 0$ independent of δ, μ and ϵ_0 such that for $|t| \leq r_0 \epsilon_0 \delta / \mu$ the following holds:

$$u(\cdot + \epsilon p, t) \geq \frac{1}{8} m_0 h \epsilon \delta + u(\cdot, t) \text{ in } B_{1/4}(\frac{3}{4} e_n), \epsilon_0 < \epsilon < 1/16.$$

i.e., in terms of [C3] u is ϵ_0-*monotone* in the direction p in $B_{1/4}(\frac{3}{4} e_n)$.

5. Thus if we choose ϵ_0 small enough, then Lemma 2.4 applies and we obtain

$$D_p u(x + \frac{3}{4} e_n, t) \geq 0 \text{ in } B_{1/8}(\frac{3}{4} e_n) \times (-r_0 \delta/\mu, r_0 \delta/\mu)$$

which proves our assertion with $l_0 = h/2$.

□

3 Propagation of interior gain

In this section we prove a propagation lemma which carries the 'interior gain' in the positive set to the free boundary over time. For this purpose we first construct a family of test functions (perturbations) to be used for local barrier arguments.

Lemma 3.1 *(Lemma 9, [C2]) Let $\varphi(x)$ be a C^2-positive function in a domain $D \subset B_1(0)$ satisfying $|D\varphi| \leq 1$ and*

$$\Delta\varphi \geq C(n)\frac{|D\varphi|^2}{\varphi} \tag{3.1}$$

in $B_1(0)$, where $C(n)$ is a dimensional constant. Let u be continuous, defined in a domain $D \subset I\!R^n$ large enough so that the following function is defined in $B_1(0)$:

$$\omega(x) = \sup_{|\nu|=1} u(x + \varphi(x)\nu).$$

Then if u is harmonic in $\{u > 0\} \cap B_1(0)$, then ω is subharmonic in $\{\omega > 0\} \cap B_1(0)$.

Lemma 3.2 *For $r, C_0 > 0$ and for sufficiently small $h > 0$, there exist constants $k, C' > 0$ independent of r and h and a family of C^2 functions $\varphi_\eta(x,t)$, $0 \leq \eta \leq 1$, defined in*

$$D := [B_1(0) - B_{1/8}(\frac{3}{4}e_n)] \times (-r, r)$$

such that

(a) $1 \leq \varphi_\eta \leq 1 + r\eta h$ *in* D,

(b) $\varphi\Delta\varphi \geq C_0|D\varphi|^2$ *holds in* D,

(c) $\varphi_\eta \equiv 1$ *outside* $B_{8/9}(0) \times (-\frac{7}{8}r, r)$,

(d) $\varphi_\eta \geq 1 + rk\eta h$ *in* $B_{1/2}(0) \times (-\frac{1}{2}r, r)$,

(e) $0 \leq |D\varphi|, \varphi_t \leq C'\eta h$ *in* D.

Proof

It is not hard to construct a smooth function ψ in $B_1(0) - B_{1/8}(\frac{3}{4}e_n)$ such that

$$\begin{cases} 0 \leq \psi \leq (8/15)^3; \\ \psi \equiv 0 \text{ outside } B_{8/9}(0); \\ |D\psi| \leq C_1\Delta\psi, \text{ for some large } C_1; \\ \psi \geq k_0 > 0 \text{ in } B_{1/2}(0). \end{cases}$$

Next we let $\Psi(x,t) := (t+7/8)^3\psi(x)$ for $-7/8 \leq t \leq 1$ and $\Psi(x,t) := 0$ for $-1 \leq t \leq -7/8$. Then $\varphi_\eta(\cdot,t) = 1 + r\eta h\Psi(\cdot, t/r)$ is our desired function, provided that $h = h(C_0)$ is small enough .

□

Now we construct a family of test functions based on $\{\varphi_\eta\}$. For $l, \rho > 0$, we define an $(n+1)$- dimensional ellipsoid

$$E_l(x,t;\rho) = \{(y,s) : |y-x|^2 + \rho^{-2}(s-t)^2 = l^2.\}$$

We also define $E_l(x,t;0) := B_l(x) \times \{t\}$.

Lemma 3.3 *Let u be a viscosity solution of (HS) with condition (Pa)-(Pc) in 2D. Then there exists $C_0 > 0$ only depending on M_0 in (Pb) such that for φ_η as given above with C_0 and $0 < r, \epsilon < 1$*

$$v_\eta(x,t) = \sup_{E_{\epsilon\varphi_\eta(x,t)}(x,t;\rho)} u(y,s) \tag{3.2}$$

is subharmonic in D at each time for any $0 < \eta \leq 1, 0 \leq \rho \leq 1$.

Proof

1. Due to Lemma 3.1 we only have to prove the lemma for $\rho > 0$. Let us fix $t \in (-r,r)$ and show that $v_\eta(\cdot,t)$ is subharmonic in the set $\{x : (x,t) \in D\}$. Observe that from the assumption the level sets of u has normal velocity less than M_0 in $2D$, and thus u assumes its maximum in the ellipsoid $E_{\epsilon\varphi_\eta}(x,t;\rho)$ strictly away from the top and the bottom portions: that is, there exists $0 < C = C(M_0) < 1$ such that $v_\eta(x,t) = \sup_{I(x,t)} u(y,s)$, where

$$I(x,t) = E_{\epsilon\varphi_\eta(x,t)}(x,t;\rho) \cap \{(y,s) : s \in II_{(x,t)}\},$$

and

$$II_{(x,t)} = \{s : (\epsilon^2\varphi_\eta^2(x,t) - \rho^{-2}(s-t)^2)^{1/2} \geq C\epsilon\varphi_\eta(x,t)\}.$$

2. Hence $v_\eta(x,t) = \sup_{s \in II_{(x,t)}} \omega_s(x)$, where

$$\omega_s(x) = \sup_{|\nu|=1} u(x + (\epsilon^2\varphi_\eta^2(x,t) - \rho^{-2}(s-t)^2)^{1/2}\nu, s).$$

Let us fix $s \in II_{(x,t)}$ and let $\Phi(x) = (\epsilon^2\varphi_\eta^2(x,t) - \rho^{-2}(s-t)^2)^{1/2}$. Then we have $\Phi D\Phi = \epsilon^2\varphi_\eta D\varphi_\eta$ and

$$
\begin{aligned}
\Phi\Delta\Phi &= (\epsilon^2\varphi_\eta\Delta\varphi_\eta + \epsilon^2|D\varphi_\eta|^2) - |D\Phi|^2 \\
&\geq (C_0+1)\epsilon^2|D\varphi_\eta|^2 - |D\Phi|^2 \\
&\geq C_0|D\Phi|^2\tfrac{\Phi^2}{\epsilon^2\varphi_\eta^2} - |D\Phi|^2 \\
&\geq (C_0C^2-1)|D\Phi|^2
\end{aligned}
$$

where the last inequality is due to the definition of $II_{(x,t)}$. Hence with the choice of $C_0 \geq C^{-2}(C(n)+1)$ we can apply Lemma 3.1 to conclude that for any $s \in II_{(x,t)}$, $\omega_s(x)$ is subharmonic. Finally $v_\eta(\cdot,t)$, supremum of subharmonic functions, is subharmonic.

$\Box$.

Now we are ready to state our main lemma for the iteration method.

Lemma 3.4 (propagation lemma) *Let u_1 and u_2 be two viscosity solutions of (HS) in $B_1(0)\times(-r,r), 0<r\leq 1$ and assume that u_1, u_2 satisfy (Pa)-(Pc) with $(0,0)\in\Gamma(u_2)$. Suppose*

$$v_\epsilon(x,t) := \sup_{E_\epsilon(x,t;\rho)} u_1 \leq u_2(x,t) \text{ in } B_1(0)\times(-r,r)$$

with some $\rho\in[0,1]$ and there exist $\kappa,\sigma>0$ such that for some small $h>0$,

$$u_2(x,t) - v_{(1+\sigma h)\epsilon}(x,t) \geq \kappa\sigma\epsilon u_2(\frac{3}{4}e_n,t)$$

in $B_{1/8}(\frac{3}{4}e_n)\times(-r,r)$.

Then there exists $0<\epsilon_0,h_0,c_0<1$ such that if $0<\epsilon<\epsilon_0$ and $0<h<h_0$ then

$$v_{(1+c_0r\sigma h)\epsilon}(x,t) \leq u_2(x,t) \quad \text{in } B_{1/2}(0)\times(-r/2,r).$$

Moreover ϵ_0, h_0 and c_0 only depends on κ and the constants given in $(Pa)-(Pc)$.

Proof.

1. We only prove the lemma when $r=1, \rho>0$. A parallel argument can be applied to prove the general case. Unless noted otherwise, we denote by C positive constants depending only the constants given in (Pa)-(Pc).

2. Below we make use of the comparison principle (Theorem 1.2). Let us set

$$v(x,t) := \sup_{E_{\epsilon\varphi_\sigma}(x,t;\rho)} u_1(y,s),$$

where φ_σ is as defined in Lemma 3.2 with C_0 obtained in Lemma 3.3, and let $\omega(x,t)$ satisfy

$$\begin{cases} -\Delta\omega(\cdot,t) = 0 & \text{in } D\cap\Omega(u); \\ \\ \omega = 0 & \text{in } \{u_2 = 0\}\cup[\partial B_1(0)\times(-1,1)]; \\ \\ \omega = u_2(\frac{3}{4}e_n, t) & \text{on } \partial B_{1/8}(\frac{3}{4}e_n)\times(-1,1). \end{cases}$$

Note that by hypothesis and maximal principle of harmonic functions

$$v + \kappa\sigma\epsilon\omega \le u_2 \text{ in } D - D', D' = [B_{8/9}(0)\times(-7/8,1)]. \tag{3.3}$$

On the other hand, due to the Harnack inequality and by the boundary condition of ω, $\omega \ge Cu_2$ at $(\frac{3}{4}e_n, t), -1 \le t \le 1$. Thus Lemma 2.1 applied to ω and u_2 at each fixed $t \in (-1,1)$ implies that $\omega \ge Cu_2$ in D'. It then follows that $\tilde{v} := (1 + C\kappa\sigma\epsilon)v \le u_2$ on the parabolic boundary of D'.

3. Next we prove that $\tilde{v}$ is a viscosity subsolution of (HS) in D'. Suppose that there is a $C^{2,1}$ function $\phi(x,t)$ such that $\tilde{v} - \phi$ has a local maximum zero at (x_1,t_1) in $\bar{\Omega}(v)\cap\{t \le t_1\}\cap D'$. Since v is subharmonic, we only have to consider the case $(x_1,t_1)\in\Gamma(v)$. By definition of v, there is a point $(y_1,s_1)\in\Gamma(u_1)$ such that

$$(y_1,s_1) = \bar{\Omega}(u_1)\cap\bar{E}_{\epsilon\varphi_\sigma(x_1,t_1)}(x_1,t_1;\rho).$$

and for (x,t) close to (x_1,t_1)

$$v(x,t) \ge u_1(f(x,t)), \quad f(x,t) := (x + \nu\bar{\varphi}_\sigma(x,t), t + s_1 - t_1)$$

where $\nu = y_1 - x_1/|y_1 - x_1|$ and

$$\bar{\varphi}_\sigma(x,t) = \sqrt{\epsilon^2\varphi_\sigma^2(x,t) - \rho^{-2}(s_1 - t_1)^2}.$$

Hence $u_1 - \tilde{\phi}$ has a local maximum zero at (y_1,s_1) in the set $\bar{\Omega}(u_1)\cap\{s \le s_1\}$ where $\tilde{\phi}(y,s) := (1 - C\kappa\sigma\epsilon)\phi(f^{-1}(y,s))$. Recall that due to (Pb) and (Pb)' we have

$$|Du|, u_t/|Du| \le M. \tag{3.4}$$

Formally speaking (3.4) implies that the normal velocity of $\Gamma(u)$, $u_t/|Du|$, is finite. More precisely, a barrier argument based on (3.4) and the definition of u yields that the ellipsoid $E_{\epsilon\varphi_\sigma(x_1,t_1)}(x_1,t_1;\rho)$ touches $\Gamma(u)$ at (y_1,s_1) from outside of $\Omega(u)$ uniformly away from the top and bottom of the ball. In terms of φ and $\bar{\varphi}$ this means that $\bar{\varphi}(x_1,t_1)$ stays away from zero and moreover

$$\frac{\epsilon\varphi_\sigma(x_1,t_1)}{\bar{\varphi}_\sigma(x_1,t_1)} \leq C. \tag{3.5}$$

It follows from (3.5), the regularity of φ_σ, and a straightforward computation that

$$0 \leq |D\bar{\varphi}_\sigma|, (\bar{\varphi}_\sigma)_t \leq C\epsilon|D\varphi_\sigma|, C\epsilon(\varphi_\sigma)_t \leq Ch\sigma\epsilon$$

near (x_1,t_1). Hence $\nabla_{(x,t)}f = I + O(\epsilon)$ and is in particular non-singular near (x_1,t_1) if ϵ is small enough. Hence f is invertible in a neighborhood of $f(x_1,t_1) = (y_1,s_1)$ if ϵ is small enough. Furthermore f is C^2 in space-time near (x_1,t_1) since φ is C^2 and $\bar{\varphi}(x_1,t_1) > 0$. Hence by the inverse function theorem it follows that f^{-1} is C^2 in a neighborhood of (y_1,s_1). Therefore $\tilde{\phi}$ is C^2 in space-time in a neighborhood of (y_1,s_1).

4. Moreover since $u_1(\cdot,s_1) \leq \tilde{\phi}(\cdot,s_1)$ in $\bar{\Omega}_{s_1}(u_1)$ in a neighborhood of y_1 and with $u_1(y_1,s_1) = \tilde{\phi}(y_1,s_1)$, (Pc) yields that

$$|D\tilde{\phi}|(y_1,s_1) \geq m_0 > 0. \tag{3.6}$$

Hence Lemma 2.5 applies to u_1 and we obtain

$$\tilde{\phi}_t \leq |D\tilde{\phi}|^2 \text{ at } (y_1,s_1).$$

Once again using (3.5), a straightforward computation leads to

$$\phi_t - Ch\sigma\epsilon|D\phi| \leq (1 - C\kappa\sigma\epsilon)(1 + Ch\sigma\epsilon)^2|D\phi|^2 \text{ at } (x_1,t_1).$$

Rearranging terms, we obtain

$$\phi_t - |D\phi|^2 \leq \sigma\epsilon|D\phi|[(Ch - C\kappa)|D\phi| + Ch + O(\epsilon)] \leq 0 \text{ at } (x_1,t_1)$$

if $\epsilon << h$ and $h \leq C\kappa m_0$.

4. Thus $\tilde{v}$ is a viscosity subsolution of (HS) and we can apply Theorem 1.2 to $\tilde{v}(x,t)$ and $u_2(x+\epsilon e_n,t)$ in D' for every $\epsilon > 0$ to yield that

$$\tilde{v} \le u_2 \text{ in } D',$$

which yields our assertion.

$\square$

4 Regularity in space

In this section we use Lemma 2.6 and Lemma 3.4 to prove the spatial regularity of $\Gamma(u)$ given in Theorem 1. First we state a basic iteration lemma. Note that due to Lemma 2.2, $u(\cdot,t)$ given in Theorem 1 has a cone of monotonicity $W(\theta,e_n)$ in a neighborhood of $\Gamma(u)\cap B_{3/2}(x_0)$ for $|t-t_0|\le 3/2$, where the size of the neighborhood and θ depends on L_0.

Lemma 4.1 *Let u be a viscosity solution of (HS) in $B_5(0)\times(-5r,5r)$ with $(0,0)\in\Gamma(u)$, and with conditions (Pa)-(Pc). Furthermore suppose that $A\le u_t/u_n\le B$ and $u(\cdot,t)$ has a cone of monotonicity $W(\theta,e_n)$ in $B_1(0)$ with $\theta\ge\theta_0$ for $|t|\le r$. Let us denote*

$$\delta=\pi/2-\theta \quad ; \quad \mu=\max(\delta,B-A).$$

and suppose that $r\ge\delta/\mu$. Then there exist constants $0<\bar{c},\bar{r}<1$ and a unit vector ν_1 such that $u(\cdot,t)$ is monotone increasing in $B_1\times(-\bar{r}\delta/\mu,\bar{r}\delta/\mu)$ along every direction $\eta\in W(\theta_1,\nu_1)$ with

$$\theta_1\ge\theta+\bar{c}\delta^2/\mu.$$

Moreover $\bar{c},\bar{r}$ only depends on θ_0 and the constants given in $(Pa)-(Pc)$.

Proof.

1. As before, unless noted otherwise, we denote by C positive constants depending only on the constants in (Pa)-(Pc). Due to Lemma 2.6 there is a unit vector ν and $b<1$ such that, for $\theta^*=\pi/2-b\delta$, u has a cone of monotonicity $W(\theta^*,\nu)$ in $D_1:=B_{1/8}(\frac{3}{4}e_n)\times(-\bar{r}\delta/\mu,\bar{r}\delta/\mu)$. Consider $p\in W(\theta/2,e_n)-\mathcal{N}$ where $\mathcal{N}$ denotes a neighborhood of the touching line (if they touch) $\partial W(\theta/2,e_n)\cap\partial W(\theta^*-\theta/2,\nu)$.

2. For each $t\in(-\bar{r}\delta/\mu,\bar{r}\delta/\mu)$, let

$$u_1(x,t)=u(x-p,t) \text{ and } \epsilon=|p|\sin\theta/2.$$

Now define σ as

$$\sigma = [\pi/2 - (\alpha(p,\nu) + \theta/2)] \geq (b + c_0)\delta$$

where $c_0 > 0$ depends on the size of the deleted neighborhood $\mathcal{N}$.

Since $p + \epsilon\rho \in W(\theta, e_n)$ for any unit vector $\rho \in I\!R^n$, it follows that

$$v_\epsilon \leq u \text{ in } B_1(0) \times (-r, r),$$

where $v_\epsilon(x,t) := \sup_{y \in B_\epsilon(x)} u_1(y,t)$.

3. Note that for $\bar{p} = p + \eta$, $|\eta| = 1$,

$$\alpha(\bar{p}, p) \leq \theta/2.$$

Hence it follows that in $B_{1/6}(\frac{3}{4}e_n) \times (-\bar{r}\delta/\mu, \bar{r}\delta/\mu)$,

$$\begin{aligned} D_{\bar{p}}u(x,t) &\geq C D_{\bar{p}}u(\tfrac{3}{4}e_n, t) \\ &\geq C u_n(\tfrac{3}{4}e_n, t)|\bar{p}|\alpha(Du(\tfrac{3}{4}e_n, t), \bar{p}) \\ &\geq C u(\tfrac{3}{4}e_n, t))|\bar{p}| \cos[\alpha(\bar{p}, \nu) + b\delta] \\ &\geq c\sigma\epsilon u(\tfrac{3}{4}e_n, t), \end{aligned}$$

where $c = Cc_0$. The third equality is due to Lemma 2.2 and the cone of monotonicity $W(\theta^*, \nu)$ of u with $\theta^* = \pi/2 - b\delta$ introduced in step 1. Thus in D_1 we have

$$u(x - \bar{p}, t) \leq u(x,t) - D_{\bar{p}}u(\tilde{x}, t) \leq u(x,t) - c\sigma\epsilon u(x,t)$$

where $\tilde{x} = x - \lambda\bar{p}$ for some $0 \leq \lambda \leq 1$.

4. Now we are ready to apply Lemma 3.4 with $\rho = 0$ to u_1 and u in the domain in $B_1(0) \times (-\bar{r}\delta/\mu, \bar{r}\delta/\mu)$, which yields $\bar{c}, \bar{r} > 0$ independent of δ and μ such that

$$v_{(1+\bar{c}h\sigma\delta/\mu)\epsilon}(x, 0) \leq u_2(x, 0) \text{ in } B_1(0) \times (-\bar{r}\delta/2\mu, \bar{r}\delta/2\mu).$$

if $0 < \epsilon < \epsilon_0$. Since $\sigma \geq b\delta$ the last inequality implies that, along any direction of the form $p + (1 + \bar{c}h\delta^2/\mu)\epsilon\eta$, η a unit vector in $I\!R^n$ and $0 < \epsilon < \epsilon_0$, u is monotone increasing. The convex envelope of this family of directions and the original cone $W(\theta, e_n)$ is readily seen to contain a new cone $W(\theta_1, \nu_1)$ with ν_1 depending on the direction of ν and

$$\theta_1 - \theta \geq C\delta^2/\mu.$$

$\square$

Recall that (HS) is invariant under the hyperbolic scaling

$$u_n(x,t) := 2^n u(2^{-n}x, 2^{-n}t).$$

Using this scaling, an iteration of Lemma 4.1 centered at each point of $\Gamma(u) \cap S'$ (see the proof of Theorem 1 in [C2]) yields the following:

Corollary 4.2 *Let u be given as in Theorem 1. Then the free boundary $\Gamma_t(u)$ is C^1 in $B_1(x_0)$ for $|t - t_0| \leq 1$.*

5 Regularity in time

Using the spatial regularity of $\Gamma(u)$ obtained in section 4, we now proceed to show that $\Gamma(u)$ is C^1 in time in S'. First we prove that the free boundary condition is satisfied in the classical sense almost everywhere on the free boundary at each time.

Lemma 5.1 *Let u be a viscosity solution of (HS) in a local neighborhood $\mathcal{O}$ of $(x_1, t_1) \in \Gamma(u)$ with conditions (Pa)-(Pc) in $\mathcal{O}$. Suppose $\Gamma_{t_1}(u)$ is differentiable at x_1 with the inward unit normal vector ν. Furthermore suppose that $D_\nu u(\cdot, t_1)$ has its nontangential limit a_0 from $\Omega_{t_1}(u)$ at x_1. Then $\Gamma(u)$ is differentiable at (x_1, t_1) with*

$$V_{(x_1,t_1)} = a_0.$$

Proof

For $\delta > 0$ let us define

$$u^\delta(x,t) = \frac{1}{\delta} u(x_1 + \delta x, t_1 + \delta t) \text{ in } B_{1/\delta}(0) \times [-1/\delta, 1/\delta].$$

Without loss of generality we may assume that $\nu = e_n$. Since $|Du|, |u_t|/|Du| \leq M$ (see the remark below Theorem 1), As $\delta \to 0$ along a subsequence u^δ converges locally uniformly to u^0 in $I\!R^{n+1}$ with

$$Du^\delta, (u^\delta)_t \to Du^0, (u^0)_t \quad w^* \text{ - weakly in } L^\infty.$$

An iteration of Lemma 4.1, as mentioned before Corollary 4.2, yields that there is a sequence of unit vectors $\nu_k \in I\!R^n$, $k = 1, 2, ..$ such that $\alpha(Du^{1/k}, \nu_k) \to 0$ in $B_R(0)$ for any fixed $R > 0$ as $k \to \infty$. Since $\Gamma_{t_1}(u)$ is differentiable with the inward unit normal vector ν, it follows that $\alpha(\nu_k, e_n) \to 0$. In particular

$$< Du^\delta, e_k > \to 0 \quad \text{if } k \neq n$$

locally uniformly in $I\!R^{n+1}$. Since $|Du^\delta|$ is bounded, it follows that $D_{e_k}u^0 = 0$ if $k \neq n$ and

$$u^0(x,t) = u^0(x_n,t), \quad \Omega_t(u^0) = \{x : x_n + b(t) > 0\}, b(0) = 0.$$

Moreover due to the stability property of viscosity solutions one can check that u^0 is a viscosity solution of (HS) in $I\!R^{n+1}$. In particular $u^0(\cdot,t)$ is harmonic in $\Omega_t(u_0)$ for each t and thus

$$u^0(x,t) = a(t)(x_n + b(t))_+,$$

where $a(t), b(t)$ are Lipschitz continuous. Hence if we consider the limit (up to a subsequence) of $(u^0)^\delta$ we obtain u^{00}: a viscosity solution of (HS) in $I\!R^{n+1}$ given as

$$u^{00}(x,t) = a_0(x_n + b_0(t))_+ \tag{5.1}$$

where $a_0 = a(0)$ and $b_0(0) = 0$, b_0: Lipschitz continuous. From barrier arguments with classical solutions of the form

$$a_\epsilon(x_n + a_\epsilon(t - t_\epsilon))_+; \quad a_\epsilon = a_0 \pm \epsilon, t_\epsilon = \pm\epsilon,$$

it follows that viscosity solutions of form (5.1) are uniquely determined with $b_0(t) = a_0 t$. Hence

$$u^0(x,t) = a_0(x_n + a_0 t)_+ + o(|x| + |t|).$$

Finally notice that from the hypothesis a_0, the nontangential limit of $D_\nu u$ at (x_1, t_1), is unique. Thus our assertion is proved.

□

Remark

We mention that, for u given as in Theorem 1, the hypotheses of Lemma 5.1 is indeed satisfied almost everywhere on $\Gamma_t(u) \cap B_2(x_0)$ for $|t - t_0| \leq 2$ with respect to surface measure (for example see Theorem 2.3 in [JK].)

The following lemma, a parallel statement of Lemmas 6 and 7 in [ACS], can be proved with a slight modification of arguments in [ACS] using interior estimates of harmonic functions, condition $(Pa)-(Pc)$ and Lemma 2.2, and thus we omit the proof.

Lemma 5.2 *Let $u, \bar{r}$ be as in Lemma 4.1. with $\delta << \mu$. Then*

$$(a) \qquad u(x,t) = u(\frac{3}{4}e_n, 0) + \alpha(x_n - \frac{3}{4}) + \alpha O(\delta/\mu)$$

in $B_{1/6}(\frac{3}{4}e_n) \times (-\delta/\mu, \delta/\mu)$, where $\alpha = u_n(\frac{3}{4}e_n, 0)$.
(b) For all $|t| < \delta/\mu$,

$$\oint_{B_{1/6}(0)\cap\Gamma_t(u)} |D_n u - \alpha|^2 dS \le \alpha^2 O(\delta/\mu).$$

Using Lemma 5.1 and 5.2, we are able to show that μ can decrease if we stay away from the free boundary.

Lemma 5.3 *Let u, α, δ, μ be as in Lemma 5.2 and suppose $\delta << \mu^3$. If $\alpha \ge b := -\frac{1}{2}A + \frac{3}{2}B$ (or $\alpha \le b$), then there exists $c_1 > 0$ such that*

$$\frac{u_t}{u_n} \ge A + c_1\mu \quad (\text{ or } \frac{u_t}{u_n} \le B - c_1\mu)$$

in $(x,t) \in B_{1/6}(\frac{3}{4}e_n) \times (-\delta/\mu, \delta/\mu)$. Here c_1 depends only on the constants in (Pa)-(Pc).

.

Proof.

Suppose $\alpha \ge b$. For $|t| \le \delta/\mu$, let $\omega_t^{(x)}$ be the harmonic measure in $\Omega_t(u) \cap B_2(0)$ evaluated at x. Due to Lemma 5.1 , on $\Gamma_t(u) \cap B_1(0)$ almost everywhere with respect to surface measure we have

$$\frac{u_t}{u_n} = |Du|(1 + O(\delta)).$$

By Lemma 5.2(b), if we define

$$\Sigma_t = \{p \in \Gamma_t(u) \cap B_{1/6}(0) : u_n(p) = \alpha(1 + O(\delta^{1/3}))\}$$

then $|\Sigma_t| \ge \frac{1}{2}|\Gamma_t(u) \cap B_{1/6}(0)|$ for any $|t| \le \delta/\mu$. Since (the restriction of) $\omega_t^{(x)}$ on $\Gamma_t(u)$ is an A_∞ weight with respect to surface measure, we have

$$\omega_t^{(x)}(\Sigma_t) \ge c.$$

On the other hand, on Σ_t,

$$|Du| = \alpha + O(\delta^{1/3}) \ge b + O(\delta^{1/3}) \ge A + \mu/2 + O(\delta^{1/3}) \ge A + c'\mu.$$

Therefore we can write, for $(x,t) \in B_{1/6}(\frac{3}{4}e_n) \times (-\delta/\mu, \delta/\mu)$

$$(u_t - Au_n)(x,t) \geq \int_{\Gamma_t(u)B_2(0)} (u_t - Au_n) d\omega_t^{(x)} \geq \bar{c}\mu\alpha\omega_t^{(x)}(\Sigma_t) \geq C\mu\alpha$$

which yields our assertion.

Similarly, we prove the complementary statement, too. $\square$

Next we show that, using Lemma 3.4, the interior gain we obtained from the previous lemma propagates to the free boundary over time. Let us denote $e_t := (0,..,0,1) \in I\!R^n \times I\!R$.

Lemma 5.4 *Let $u, \delta, \mu, \bar{r}, \nu, \nu_1$ be as in Lemma 4.1. If $\delta << \mu^3$, then in $B_1(0) \times (-\bar{r}\delta/\mu, \bar{r}\delta/\mu)$ there exists $c_2 > 0$ independent of δ and μ such that u is monotone increasing along the directions $e_t - A_1\nu_1$ and $-e_t + B_1\nu_1$ with*

$$0 < B_1 - A_1 \leq \mu_1 \text{ and } \mu_1 \leq \mu - c_2\delta.$$

Proof.

1. First observe that the new axis ν_1 of the enlarged cone obtained in Lemma 4.1 is shifted from e_n by order less than δ^2/μ. Since $\delta << \mu^3$ Lemma 5.3 applies to yield

(5.2)
$$u_t - Au_{\nu_1} \geq c\mu u_{\nu_1} \text{ or } -u_t + Bu_{\nu_1} \geq c\mu u_{\nu_1} \text{ in } B_{1/6}(\frac{3}{4}e_n) \times (-2\bar{r}\delta/\mu, 2\bar{r}\delta/\mu).$$

For simplicity suppose that the first inequality holds. Let now $\rho = e_t - C\nu_1 := e_t - C(\nu_1, 0) \in I\!R^{n+1}$, $C < A$ which makes an angle μ with $e_t - A\nu_1$. Let p be any small vector in the ρ direction and set $\epsilon = |p| \sin \mu$ and $u_1(x,t) = u((x,t) - p)$. Then

$$\sup_{B_\epsilon(x)} u_1(y,t) \leq u(x,t) \text{ in } Q_1.$$

Moreover due to (5.2), we obtain that for $|t| \leq \bar{r}\delta/\mu$ and $\bar{p} = p + \epsilon\xi$, ξ a unit vector in $I\!R^n$,

$$u((\frac{3}{4}e_n, t) - \bar{p}) \leq u(\frac{3}{4}e_n, t) - D_{\bar{p}}u(\tilde{x}, t) \leq (1 - c\mu\epsilon)u(\frac{3}{4}e_n, t)$$

where $\tilde{x} \in B_{1/6}(\frac{3}{4}e_n)$. Hence Lemma 2.3 applies and there is $\kappa, h > 0$ such that

$$\sup_{B_{(1+c\mu h)\epsilon}(x)} u_1(x,t) \leq (1 - c\kappa\mu\epsilon)u(x,t) \text{ in } B_{1/6}(\frac{3}{4}e_n) \times (-\bar{r}\delta/\mu, \bar{r}\delta/\mu).$$

2. Now if we define

$$v_\epsilon(x,t) = \sup_{B_\epsilon^{(n+1)}(x,t)} u_1(y,s),$$

then $v_\epsilon(x,t) \leq u(x,t)$ in $B_1 \times (-2\bar{r}\delta/\mu, 2\bar{r}\delta/\mu)$.

Moreover by (5.2) if we choose c small enough - depending on the size of u_t/u_n- then we can proceed as in Lemma 4.1 to obtain

$$v_{(1+c\mu h)\epsilon}(x,t) \leq (1 - \frac{\kappa}{2}c\mu\epsilon)u(x,t) \text{ in } B_{1/8}(\frac{3}{4}e_n) \times (-2\bar{r}\delta/\mu, 2\bar{r}\delta/\mu).$$

Hence Lemma 3.4 yields that there is $\bar{c} > 0$ independent of the choice of μ such that

$$v_{(1+\bar{c}\delta)\epsilon}(x,t) \leq u(x,t) \quad \text{ in } B_{1/2} \times (-\bar{r}\delta/\mu, \bar{r}\delta/\mu).$$

This implies that u is monotone increasing along the direction

$$e_t - (A + \bar{c}\delta)\nu_1.$$

Therefore the theorem holds with $A_1 = A + \bar{c}\delta$, $B_1 = B$. □

Proof of Theorem 1.

1. Suppose $(x_1, t_1) \in \Gamma(u) \cap S'$. Now combining Lemma 4.1 and Lemma 5.4, we can use an iteration argument using the hyperbolic scaling $u_n(x,t) = 2^n u(2^{-n}(x - x_1), 2^{-n}(t - t_1))$ to obtain

$$\delta_{n+1} = \delta_n - c\delta_n^2/\mu_n, \quad \mu_{n+1} = \mu_n - c\delta_n; \quad n = 1, 2, 3, ..$$

for u_n in $B_1(0) \times (-\bar{r}\delta_n/\mu_n, \bar{r}\delta_n/\mu_n)$ (see the proof of the main theorem in [ACS].) From this relations we obtain the continuity mode of $\nabla_{x'} f$ and f_t as stated in Theorem 1.

2. Next the spatial regularity of the free boundary and Theorem 2.4 of [W] yields the existence Du up to $\bar{\Omega}(u)$ in S'. Lastly Lemma 5.1 leads to the last assertion $V = |Du|$ on $\Gamma(u)$.

□

6 Long time regularity of the free boundary

Let u be the viscosity solution of (HS) in Q with its boundary data $f = 1$ on K and $u(x,0) = u_0(x)$. In addition let us suppose u_0 satisfies

$$\Omega \subset B_R(0) \quad ; \quad |Du_0| > 0 \text{ on } \Gamma_0.$$

With the above assumptions it is proved in [K2] that

$$T_0 := \inf\{t : B_R(0) \subset \Omega_t(u) \cup K\} < \infty$$

and that for any point $(x_0, t_0) \in \Gamma(u)$, $t_0 > T_0$, there is a neighborhood of (x_0, t_0) in Q where $(Pa) - (Pc)$ holds. Theorem 1 then yields that (i) $\Gamma(u) \cap \{t > T_0\}$ is differentiable, (ii) Du exists up to $\bar{\Omega}(u)$ and (iii) the free boundary condition $V = |Du|$ is satisfied on $\Gamma(u)$ for $t > T_0$. As we show below, (i)-(iii) provide enough regularity for us to apply the transformation of [EJ] to obtain further regularity of $\Gamma(u)$ for $t > T_0$.

Proof of Theorem 2.

1. Let us define $l(x)$ in K by

$$l(x) = \begin{cases} 0 & \text{if } x \in \bar{\Omega}_0 \\ t : (x,t) \in \Gamma(u) & \text{otherwise.} \end{cases}$$

Then it follows that

$$\Gamma_t(u) = \{(x,t) : \quad S(x,t) \equiv t - l(x) = 0\} \text{ for } t \leq T_0.$$

Due to the additional assumption $|Du_0| > 0$ on Γ_0 one can easily check that $\Gamma(u)$ strictly expands in time and $l(x)$ is well defined. Moreover since $\Gamma_t(u)$ is differentiable for $t > T_0$ with its normal velocity bigger than $m_0 > 0$, $l(x)$ is differentiable in $\mathbb{R}^n - K$ for $t > T_0$.

2. Since $\Gamma(u)$ is the zero level set of $S(\cdot, t)$ with normal velocity $|Du|$ for $t > T_0$, it follows that

$$\frac{S_t}{Du/|Du| \cdot DS} = |Du| \text{ on } \Gamma(u) \cap \{t > T_0\}$$

and thus

$$Du \cdot Dl = 1 \text{ on } \Gamma(u) \cap \{t > T_0\}. \tag{5.1}$$

3. Next let us apply the transformation introduced in [EJ]:

$$\begin{cases} v(x,t) = 0 & \text{for } x \in \mathbb{R}^n - K; \quad 0 \leq t \leq l(x), \\ v(x,t) = \int_{l(x)}^t u(x,\tau)d\tau & \text{for } x \in \mathbb{R}^n - K; \quad l(x) \leq t, \end{cases}$$

Due to (5.1), Lemma 5.1 and the Lipschitz continuity of u the computation in [EJ] holds for our solution for $t > T_0$ and the function $v^t(x) = v(x,t)$ solves the following obstacle problem in $I\!R^n - K$:

$$\begin{cases} -\Delta v^t - f \geq 0, \quad v^t \geq 0; \\ -\Delta v^t - f = 0 \qquad \text{in } \{v^t > 0\} = \Omega_t(u); \\ v^t = Dv^t = 0 \qquad \text{on } \partial\{v^t = 0\} = \Gamma_t(u); \\ v^t = t \qquad \text{on } \partial K \end{cases}$$

where $f(x) = 1\!\!\mathrm{l}_{\Omega_0} - 1$.

3. Due to Theorem 3 of [C1], the C^1 regularity of $\Gamma_t(u)$ yields that $v^t \in C^2(\bar{\Omega}_t(u) \cap \mathcal{N})$ where $\mathcal{N}$ is a neighborhood of $\Gamma_t(u)$. Now we can apply the Hodograph method (Theorem 1.1 of [F], also see [KN]) to v^t to conclude that $\Gamma_t(u)$ is analytic for each $t > T_0$.

4. Finally using the analytic semigroup theory (see [A]), [ES] proved the short time existence of classical solutions of (HS) when the initial free boundary is analytic. Hence for any $t_0 > T_0$ there exists $\epsilon > 0$ and a classical solution $h(x,t)$ with initial free boundary $\Gamma_{t_0}(u)$ and fixed boundary data 1 on Γ_1 for $t_0 \leq t < t_0 + \epsilon$. Furthermore it is proven in [ES] that for $t_0 < t < t_0 + \epsilon$ the free boundary $\Gamma(h)$ of h is analytic in time. On the other hand by the uniqueness result of [K1] $h = u$ for $t_0 \leq t < t_0 + \epsilon$. Therefore, $\Gamma(u)$ is analytic in time for $t_0 < t < t_0 + \epsilon$. Since $\Gamma(u) \cap \{t > T_0\}$ is analytic in space and the normal vector of $\Gamma(u) \cap \{t > T_0\}$ changes continuously in time (this follows from the proof of Lemma 2.6), one can verify from the arguments of [ES] and [A] that $\epsilon = \epsilon(t) > 0$ can be chosen uniformly for compact subsets of time interval (T_0, ∞). Thus we conclude that $\Gamma(u)$ is analytic in time for $t > T_0$. $\square$

References

[A] S. B. Angenent, Nonlinear analytic semiflows,*Proc. Roy. Soc. Edinburgh*, **115A**(1990), 91-107.

[ACS] I. Athanasopoulos, L. Caffarelli and S. Salsa, Regularity of the free boundary in parabolic phase-transition problems, *Acta Math.*, **176**(1996), 245-282.

[C1] L. Caffarelli, The regularity of free boundaries in higher dimension, *Acta math.*, **139**(1977), 155-184.

[C2] L. Caffarelli, A Harnack inequality approach to the regularity of free boundaries, Part I: Lipschitz free boundaries are $C^{1,\alpha}$, *Rev. Mat. Iberoamericana***3** (1987), no. 2, 139-162

[C3] L. Caffarelli, A Harnack inequality approach to the regularity of free boundaries, Part II: Flat free boundaries are Lipschitz,*Comm. Pure Appl. Math.,* **42**(1989), 55-78.

[D] B.E. Dahlberg, *Harmonic functions in Lipschitz domains,* Harmonic analysis in Euclidean spaces, Part 1, pp. 313–322, Proc. Sympos. Pure Math., XXXV, Part, Amer. Math. Soc., Providence, R.I., 1979

[F] A. Friedman, *Variational principles and free boundary problems,* Pure and applied mathematics, John Wiley & Sons, Inc., 1982.

[EJ] C. M. Elliot and V. Janovsky, A variational inequality approach to Hele-Shaw flow with a moving boundary, *Proc. Roy. Soc. Edinburgh. Sect. A* **88** (1981), no.1-2, 93-107.

[ES] J. Escher, G. Simonett, Classical solutions of multidimensional Hele-Shaw models, SIAM J. Math. Anal **28** (1997), no.5, pp. 1028-1047.

[JK] D. S. Jerison and C. E. Kenig, Boundary value problems on Lipschitz domains, *Studies in Partial Differential Equations* edited by W. Littman, 1982, The Mathematical Association of America., 1-68.

[K1] I. C. Kim, Uniqueness and Existence result of Hele-Shaw and Stefan problem, *Arch. Rat. Mech. Anal,* **168** (2003), 299-328.

[K2] I. C. Kim, Long time regularity of Hele-Shaw problem, submitted.

[KN] D. Kinderlehrer and L. Nirenberg, Hodograph methods and the smoothness of the free boundary in the one phase Stefan problem, *Moving Boundary problems* edited by D. G. Wilson, A. D. Solomon, P. T. Boggs, 1978, Academic Press Inc., 57-69.

[W] K. -O. Widman.,Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations, *Math. Scand.,* **21** (1967), 17-37.