The Patlak-Keller-Segel model and its variations : properties of solutions via maximum principle

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Abstract

In this paper we investigate qualitative and asymptotic behavior of solutions for a class of diffusion-aggregation equations. The challenge in the analysis consists of the nonlocal aggregation term as well as the degeneracy of the diffusion term which generates compactly supported solutions. The key tools used in the paper are maximum-principle type arguments as well as estimates on mass concentration of solutions.

1 Introduction

In this paper we study solutions of a nonlocal aggregation equation with degenerate diffusion, given by

$$\rho_t = \Delta\rho^m + \nabla\cdot(\rho\nabla(\rho * V)) \text{ in } \mathbb{R}^d \times [0,\infty) \tag{1.1}$$

with initial data $\rho_0 \in L^1(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$. Here $m > 1$, $d \geq 3$ and $*$ denotes convolution operator. In the absence of the aggregation term (when $V = 0$, our equation becomes the well-known *Porous medium equation* (PME):

$$\rho_t - \Delta(\rho^m) = 0. \tag{1.2}$$

Note that, formally, the mass of solution is preserved over time:

$$\int_{\mathbb{R}^d} \rho(\cdot,0)dx = \int_{\mathbb{R}^d} \rho(\cdot,t)dx \text{ for all } t > 0.$$

Nonlocal aggregation phenomena have been studied in various biological applications such as population dynamics ([BoCM], [BuCM], [GM], [TBL]) and Patlak-Keller-Segel (PKS) models of chemotaxis ([KS], [LL], [P],[FLP]). In the context of biological aggregation, ρ represents the population density which locally disperses by the diffusion term, while V is the interaction kernel that models the long-range attraction. Mathematically, the equation models competition between degenerate diffusion and nonlocal aggregation.

Recently, there has been a growing interest in models with degenerate diffusion to include overcrowding effects (see for example [TBL], [BoCM]). The Keller-Segel model where V is the Newtonian

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kernel and $m = 1$ (full-diffusion), and the PKS model where V is the Newtonian and $m > 1$ (degenerate diffusion), are the most widely studied models for aggregation with diffusion. On the other hand in population dynamics one often considers smooth kernels with fast decay at infinity. In this paper we consider the following two types of potentials:

(A) (PKS-model) $V(x)$ is a *Newtonian potential*:

$$V(x) = \mathcal{N} := -\frac{c_d}{|x|^{d-2}}, \tag{1.3}$$

where $c_d := \dfrac{1}{(d-2)\sigma_d}$, where σ_d is the surface area of the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$.

(B) (regularized Newtonian potential)

$$V(x) = (\mathcal{N} * h)(x), \tag{1.4}$$

where $*$ denotes convolution and $h(x)$ is a radial function in $L^1(\mathbb{R}^d : (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ which is continuous and radially decreasing. We mention that, even though the kernel given by (A)-(B) has slow decay at infinity, our results are relevant for kernels with fast decay at inifinity, since our solutions have compact support with finite propagation property (see Theorem 3.1). In fact in the subcritical case ($m > 2 - 2/d$), radial solutions starting with compact support, has their support uniformly bounded for all times (Corollary 5.5).

Note that (A)-(B) covers all attractive potentials V with its Laplacian being nonnegative and radially decreasing. The restrictions on ΔV turns out to be necessary for obtaining the preservation of radial monotonicity (see Proposition 4.3) as well as the mass comparison principle in section 5.

The dynamics in (1.1) is governed by the "free energy" functional

$$\mathcal{F}(\rho) = \int_{\mathbb{R}^d} \frac{1}{m-1}\rho^m + \frac{1}{2}\rho(\rho * V)dx. \tag{1.5}$$

Indeed (1.1) is the gradient flow for $\mathcal{F}$ with respect to the Wasserstein metric (see for example [BCalC] and [BCarC]). Depending on m, the solution of (1.1) exhibits different behavior. For $1 \leq m < 2 - 2/d$, the problem is *supercritical*: the diffusion is dominant at low concentrations and the aggregation is dominant at high concentration. As a result supercritical and critical problems with singular kernels may exhibit finite time blow-up phenomena ([DP], [HV], [S1], [BlCM]). On the other hand solutions globally exist with small mass and relatively regular initial data, and here the diffusion dominates at large length scale (see [C] and [S2]). Indeed using the entropy dissipation method ([CJMTU]) it is shown that the solutions with small L^1 and $L^{(2-m)d/2}$- norms converge to the self-similar Barenblatt profile ([LS1]-[LS2] and [B2]).

On the other hand, in the *subcritical* regime ($m > 2 - 2/d$), the diffusion is dominant at high concentration. For this reason there is a global solution for all mass sizes ([S1], [BCL], [BRB]). Since aggregation dominates in low concentration, there are compactly supported stationary solutions for any mass size (see Proposition 2.1). In fact there is no uniqueness result for stationary solutions, even for radial solutions, except the well-known result of Lieb and Yau ([LY]) for the PKS model. Furthermore, even for the PKS model, there are few results addressing the qualitative behavior of general radial solutions: this is perhaps due to the fact that entropy methods faces challenge due to the strong aggregation term and the generic presence of the free boundary. This motivates our investigation in this paper.

The main tools in our analysis are various types of comparison principles. While maximum-principle type arguments are natural to parabolic PDEs, the classical maximum principle does not hold with (1.1) due to the nonlocal aggregation term, and therefore the standard comparison principle or the corresponding viscosity solutions theory does not apply. Instead we establish order-preserving properties of several associated quantities: the radial monotonicity (section 4), the mass concentration (section 5), and the rearranged mass concentration for non-radial solutions (section 6).

The following existence and uniqueness results will be used throughout our paper.

Theorem 1.1 (Theorem 3 and 7 in [BRB]. Also see [BS] and [S1])**.** *Let V be given by (A) and (B) and $d \geq 3$. Suppose ρ_0 be a nonegative function in $L^1(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$. Then for $m > 2 - 2/d$ there exists a unique, uniformly bounded weak solution ρ of* (1.1) *in $\mathbb{R}^d \times [0, \infty)$ with initial data ρ_0.*

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1.1 Summary of results

First we investigate properties of radial, stationary solutions of (1.1):

Theorem 1.2 (Properties of Stationary solutions)**.** *Let V be given by (A) or (B) and let $m > 2 - \frac{2}{d}$. Let ρ_A be a non-negative radial stationary solution of* (1.1) *with $\int \rho_A(x)dx = A > 0$. Then*

(a) ρ_A is radially decreasing, compactly supported and smooth in its support; (Proposition 2.1)

(b) ρ_A is uniquely determined for any given A. (Theorem 2.2 and Theorem 2.3.)

When V is given by (A), uniqueness of radial stationary solution is the well-known results of Lieb and Yau ([LY]). Their proof is based on the fact that the mass function satisfies an ODE with uniqueness properties: this property fails when V is given by (B). Instead, we look at the dynamic equation (1.1), and prove uniqueness out of asymptotic convergence towards stationary solution. A more direct proof of uniqueness and the uniqueness of general (possibly non-radial) stationary solution are interesting open questions.

Next we show several qualitative behavior of the solutions:

Theorem 1.3 (Properties of solutions)**.** *Suppose $m > 1$. Let V be given by (A) or (B), and let $\rho(x,t)$ be a weak solution to* (1.1)*, which is uniformly bounded in $\mathbb{R}^d \times [0, T)$, where T can be either finite or ∞. Then the following holds:*

(a) For any $\delta > 0$, ρ is uniformly continuous in $\mathbb{R}^d \times [\delta, T)$; (Theorem 3.1)

(b) $\{\rho > 0\}$ expands over time period τ with maximal rate of $C\tau^{-1/2}$; (Theorem 3.1)

(c) If $\rho(\cdot,0)$ is radially decreasing, then so is $\rho(\cdot,t)$ for any $t \in [0,T)$ (Theorem 4.2).

The preservation of radial monotonicity is new, to the best of the authors' knowledge, for any type of diffusion-aggregation equation. For the first-order aggregation equation ((1.1) without the diffusion term), this property has been recently shown in [BGL] for the same class of potentials, via method of characteristics.

Based on Theorem 1.2 as well as the *mass comparison* (Proposition 5.3), the following results are obtained for general (not necessarily radially decreasing) radial solutions:

Theorem 1.4 (Asymptotic behavior: subcritical regime)**.** *Let V be given by (A) or (B), $m > 2 - \frac{2}{d}$, and let $\rho(x,t)$ be the solution to* (1.1) *with radial, compactly supported initial data $\rho_0(x) \in L^1(R^d;(1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ which has mass A. Let ρ_A be a radial stationary solution with mass A. Then*

(a) The support of ρ, $\{\rho(\cdot,t) > 0\}$ stays inside of a large ball $\{|x| \le R\}$ for all $t \ge 0$, where R depends on m, d, V and the initial data ρ_0 (Corollary 5.5);

(b) ρ converges to ρ_A exponentially fast in p-Wasserstein distance for all $p > 1$ (Corollary 5.8), and $\|\rho(\cdot,t) - \rho_A\|_{L^\infty(\mathbb{R}^d)} \to 0$ as $t \to \infty$ (Corollary 5.9).

The mass comparison property have been previously observed for two-dimensional Keller-Segel model ([BKLN]) , however the property has not been taken full advantage of, perhaps because of the success of entropy method for the KS model.

Our method also provides interesting results for asymptotic behavior of radial and non-radial solutions in supercritical regime, when the solution starts from a sufficiently less concentrated initial data in comparison to a re-scaled stationary profile:

Theorem 1.5 (Asymptotic behavior:supercritical regime)**.** *Let $V(x)$ be given by (A) or (B), and let $1 < m < 2 - \frac{2}{d}$. Assume ρ_0 is compactly supported and has mass A. Then there exists a sufficiently small constant $\delta > 0$ depending on d, m, μ_0 and V, such that if*

$$\rho_0(\lambda) \prec \delta^d \mu_A(\delta\lambda)$$

where $\mu_A(\lambda)$ is given in (5.31)*, then the weak solution ρ with initial data ρ_0 exists globally and algebraicaly converges to the Barenblatt profile (Corollary 5.14).*

Even for the Newtonian potential, the asymptotic behavior of non-radial solutions in subcritical regime remains open, except in some cases with sufficiently large m (in progress by authors). In particular we do not know if the solutions lose part of their mass over time to infinity.

Lastly we state a comparison principle in terms of the symmetric rearrangement.

Let us recall that, for any nonnegative measurable function f that vanishes at infinity, the *symmetric decreasing rearrangement* f^* is given by

$$f^*(x) := \int_0^\infty \chi_{\{f(x)>t\}^*} dt \tag{1.6}$$

where Ω^* denotes the symmetric rearrangement of a measurable set Ω of finite volume in $\mathbb{R}^d$.

Theorem 1.6 (Rearrangement comparison and instant regularization)**.** *Suppose $m > 1$. Let V be given by (A) or (B). Let $d \geq 3$ and let ρ be the weak solution to* (1.1) *with initial data $\rho_0(x) \in L^1(R^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$.*

(a) Let $\bar{\rho}$ be the solution to the symmetrized problem, i.e. $\bar{\rho}$ is the weak solution to (1.1) *with initial data $\rho_0^*(x)$. Assume $\bar{\rho}$ exists for $t \in [0, T)$, where T can be either finite or ∞. Then $\rho^*(\cdot, t) \prec \bar{\rho}(\cdot, t)$ for all $0 \leq t < T$ (Theorem 6.1)*

(b) Suppose $m > 2 - \frac{2}{d}$, then for every $t > 0$ we have

$$\|\rho(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leq c(m, d, A, V)t^{-\alpha}$$

for $0 < t < 1$, where $A = \int \rho_0 dx$ and $\alpha := \frac{d}{d(m-1)+2}$. (Proposition 6.6).

Rearrangement results have been obtained before for (1.2) (Chapter 10 of [V]) and for the two-dimensional Keller-Segel model ([DNR]). We largely follow the arguments in [V]. The new component in the proof is introduction of approximate equations to deal with both the degenerate diffusion and the nonlocal aggregation term. The L^∞-regularization result is interesting on its own: similar results has been recently obtained for Keller-Segel model in [PV], by a De-Giorgi type method.

Remark 1.7. *All of our results presented in the paper extends to general degenerate-diffusion type of solutions*

$$\rho_t = \Delta f(\rho) + \nabla \cdot (\rho \nabla(\rho * V)),$$

where f is a smooth, increasing and convex function with $f'(0) = 0$.

2 Properties of the radially symmetric stationary solution

In this section we consider non-negative radially symmetric stationary solutions of (1.1), given by

$$\frac{m}{m-1}\rho^{m-1} + \rho * V = C \quad \text{in } \{\rho > 0\}, \tag{2.1}$$

where we assume $m > 2 - \frac{2}{d}$, and the constant C may be different in different positive components of ρ. When V is given by (A) or (B), for any mass $A > 0$, the existence of stationary solution ρ with mass A is proven in [L] and [B2].

Let us define the *mass function* as follows:

$$M(r) := \int_{B(0,r)} \rho(x)dx,$$

Since both ρ and V are radially symmetric, we may slightly abuse the notation and write $\rho * V$ as a function of r. When $V = \mathcal{N}$, by the divergence theorem and radial symmetry of ρ and V we have

$$\frac{\partial}{\partial r}(\rho * V)(r) = \frac{M(r)}{\sigma_d r^{d-1}}. \tag{2.2}$$

where σ_d is the surface area of the sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$.

Similarly, when V is given by (B), where $V = \mathcal{N} * h$, for all radially symmetric function ρ, we have $\rho * V$ is radially symmetric, and

$$\frac{\partial}{\partial r}(\rho * V)(r) = \frac{\tilde{M}(r)}{\sigma_d r^{d-1}}, \tag{2.3}$$

where $\tilde{M}(r) := \int_{B(0,r)} \rho * \Delta V dx$. Note that in both cases, we have $\partial_r(\rho * V) \geq 0$.

Proposition 2.1. *Let V given by (A) or (B) and suppose $m > 2 - \frac{2}{d}$. Let $\rho(x) \in L^1(\mathbb{R}^d)$ be a non-negative radially symmetric solution of (2.1). Then (a) ρ is smooth in its positive set; (b) ρ is radially decreasing; and (c) ρ is compactly supported.*

Proof. 1. To show (a) for $V = \mathcal{N}$, note that the right hand side of (2.2) is continuous since

$$f(r) := \frac{M(r)}{\sigma_d r^{d-1}}$$

is continuous at all $r > 0$, and $f(r) \to 0$ as $r \to 0$. By (2.2), $\rho * V$ is differentiable in the positive set of ρ, which implies that ρ^{m-1} (hence ρ) is also differentiable in the positive set of ρ. Therefore $\frac{M(r)}{r^{d-1}}$ is now twice differentiable, hence we can repeat this argument and conclude.

When V is given by (B), we can apply the same argument on (2.3) and conclude.

2. By differentiating (2.1) we have

$$\frac{m}{m-1}\frac{\partial}{\partial r}\rho^{m-1} = -\frac{\partial}{\partial r}(\rho * V) \quad \text{in } \{\rho > 0\}, \tag{2.4}$$

and due to (2.2) and (2.3), the right hand side of (2.4) is non-positive. Hence we conclude (b).

3. It remains to check (c). Rewriting (2.1) and using the fact that ρ is radially decreasing and thus have simply connected support, ρ can be written as

$$\rho(r) = (C - \rho * V(r))^{\frac{1}{m-1}}.$$

When $V = \mathcal{N}$ the proof is similar to that of Theorem 5 in [LY]: since $\rho * V$ vanishes at infinity, we have

$$\begin{aligned} \rho * V(r) &= -\int_r^\infty \frac{M(s)}{s^{d-1}} ds \\ &= -\frac{M(r)}{(d-2)r^{d-2}} - \int_r^\infty \frac{c_d}{d-2}\rho(s) s ds \end{aligned} \tag{2.5}$$

where c_d is the volume of a ball with radius 1 in $\mathbb{R}^d$. Since ρ is radially decreasing, we have

$$\rho * V(r) \leq 0 \text{ and } -\rho * V(r) \sim \frac{1}{r^{d-2}} \text{ as } r \to \infty. \tag{2.6}$$

If $C = 0$, (2.6) implies that

$$\rho(r) = (-\rho * V(r))^{\frac{1}{m-1}} \sim r^{-\frac{d-2}{m-1}},$$

where the exponent is greater than $-d$ when $m > 2 - \frac{2}{d}$, which contradicts the finite mass property of ρ. Therefore C must be negative and thus $\rho(r)$ needs to touch zero for some r.

For the case when $V = \mathcal{N} * h$, we have

$$\rho * V = (\rho * \mathcal{N}) * h,$$

where N is the Newtonian potential. Arguing as in (2.5), we have $\rho * \mathcal{N}(x) \sim \frac{1}{|x|^{d-2}}$ as $|x| \to \infty$. Since h is integrable in $\mathbb{R}^d$ and radially decreasing, we have $\rho * V(x) \sim \frac{1}{|x|^{d-2}}$ as $|x| \to \infty$ as well, hence by same argument as above, we can conclude. $\square$

Next we state the uniqueness of the radial stationary solution when $V = \mathcal{N}$.

Theorem 2.2 ([LY])**.** *Let $V = \mathcal{N}$, and suppose $m > 2 - \frac{2}{d}$. Then for all mass $A > 0$, the radial stationary solution for* (1.1) *with mass A is unique. Moreover, the stationary solution is the global minimizer for the free energy functional* (1.5)*.*

This theorem follows from a slight modification from the proof of Theorem 5 in [LY], which proves uniqueness of the stationary solution of a slightly different problem. Roughly speaking, their proof is based on two steps: for any mass A, they first show the radial global minimizer of free energy functional (1.5) with mass A is unique, and secondly they prove every radial stationary solution is a global minimizer for some mass.

We point out that the proof in [LY] cannot be generalized when V is given by (B): the difficulty lies in the second step. When $V = \mathcal{N}$, for any radial stationary solution ρ, its mass function $M(r) = \int_{|x|\le r} \rho(x)dx$ solves a second order ODE

$$\Big(\frac{m}{m-1}\big(\frac{M'(r)}{\sigma_d r^{d-1}}\big)^{m-1}\Big)' = \frac{M(r)}{\sigma_d r^{d-1}},$$

where $M(0) = 0$ is prescribed. Thus, once $\rho(0) = \lim_{r\to 0} M'(r)/(\sigma_d r^{d-1})$ is known, $M(r)$ would be uniquely determined for all $r > 0$, which implies that ρ can be uniquely determined by $\rho(0)$. This property is crucial for the second step, since if this property holds, then both the radial stationary solutions and the radial global minimizers can be parametrized by their value at the center of mass (see Lemma 12, [LY]).

However, when V is given by (B), the mass function solves a nonlocal ODE, hence different stationary solutions may have the same center density: thus their approach cannot be applied to prove the second step. Hence when V is given by (B), it is necessary to take an alternative approach. Instead of dealing with the stationary equation (2.1) directly, we will consider the dynamic equation (1.1) and prove the uniqueness of the radial stationary solution by their asymptotic convergence (see Corollary 5.10).

Theorem 2.3. *Let V be given by (B), and suppose $m > 2 - \frac{2}{d}$. Then for any mass A, the radial stationary solution of* (1.1) *with mass A is unique.*

When $V = \mathcal{N}$, the following results can be checked with straightforward computation, making use of the uniqueness result in Theorem 2.2 (see Figure 1).

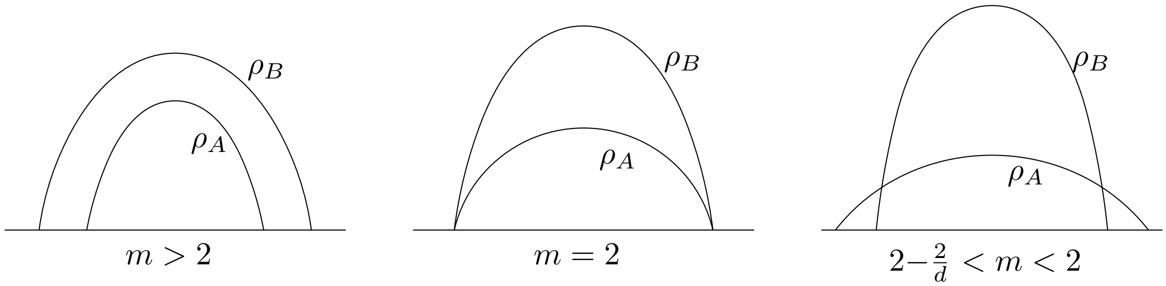

Figure 1: Stationary solutions with different mass for different m, where $\int \rho_A dx < \int \rho_B dx$

Proposition 2.4. *Let $V = \mathcal{N}$ and let $m > 2 - \frac{2}{d}$, and let ρ_1 be the radial solution of (2.1) with unit mass. Then all other radial solutions ρ_A of (2.1) with mass A are of the form*

$$\rho_A(x) = a\rho_1(a^{-\frac{m-2}{2}}x) \text{ with } a := A^{\frac{2}{d(m-2+2/d))}}. \tag{2.7}$$

In particular, let $A > B$ and let ρ_A and ρ_B be the radial stationary solutions with mass A and B respectively. Then the following holds:

(a) When $m > 2$, ρ_A have larger support and smaller height than ρ_B.

(b) When $m = 2$, all stationary solutions have the same support.

(c) When $2 - \frac{2}{d} < m < 2$, ρ_A. have smaller support and bigger height than ρ_B.

3 Qualitative properties of solutions

In this section several regularity properties will be derived for general weak solutions of (1.1). We point out that the results in this section hold for general (non-radial) solutions.

Theorem 3.1. *Suppose $m > 1$. Let V given by (A) or (B), and let ρ be a weak solution of (1.1) with its initial data ρ_0: bounded with compact support. Further suppose ρ is uniformly bounded in $\mathbb{R}^d \times [0, T]$. Then*
(a) For any $\delta > 0$, ρ is uniformly continuous in $\mathbb{R}^d \times (\delta, T]$.

(b) [Finite propagation property] For given $0 < t \leq T$, if $\{x : \rho(\cdot, t) > 0\} \subset B_R(0)$, then

$$\{x : \rho(\cdot, t + h) > 0\} \subset B_{R+Ch^{1/2}}(0) \text{ for } 0 < h < 1,$$

where the constant $C > 0$ depends on m, d, ρ_0 and $||\Delta V||_1$.

Proof. Let us take our most singular potential $V = \mathcal{N}$. Parallel (and easier) arguments hold for V given by (B). Let

$$C_0 = \sup\{\rho(x, t) : (x, t) \in \mathbb{R}^n \times [0, T)\}.$$

Observe that, treating the convolution term $\Phi := V * \rho$ as a priori given, ρ solves

$$\rho_t = \Delta(\rho^m) + \nabla \cdot (\rho \nabla \Phi). \tag{3.1}$$

Note that for all $t \in [0, T)$, Φ satisfies

$$\begin{aligned} |D\Phi|(\cdot, t) &\leq C_0 \int_{|y|\leq 1} |D\mathcal{N}|(y)dy + (\|\rho(\cdot, t)\|_2)(\int_{|y|\geq 1} |D\mathcal{N}|^2(y)dy)^{1/2} \\ &\leq C_1, \end{aligned}$$

where C_1 depends on C_0, the L_1 and sup-norm of ρ (both of which depend only on ρ_0), and the dimension d. Also

$$|\Delta\Phi|(\cdot, t) \leq \|\rho\|_{L^\infty} \leq C_0 \quad \text{for all } t \in [0, T).$$

Therefore, due to Proposition 3.4 of [BH], comparison principle between weak sub- and supersolution of (3.1) holds. Moreover, Theorem 6.1 of [Dib] yields that ρ is uniformly continuous in $\mathbb{R}^d \times [\delta, T)$.

Now let us define

$$\tilde{U}(x, t) := A \inf_{|x-y|\leq C - Ct} e^{-Ct}(|x| + \omega t - B)_+,$$

where $\omega = 1 + (m-1)(d-1)A$.

Then due to Proposition 2.15 in [KL], $\tilde{U}$ satisfies

$$\tilde{U}_t \leq (m-1)\tilde{U}\Delta\tilde{U} + |D\tilde{U}|^2 + C|D\tilde{U}| + C\tilde{U}$$

in $\Sigma := \{|x| \leq 2B\} \times [0, \omega^{-1}B]$. Hence $\tilde{\rho} := (\frac{m-1}{m}\tilde{U})^{1/(m-1)}$ satisfies

$$\tilde{\rho}_t \leq \Delta(\tilde{\rho}^m) + C|D\tilde{\rho}| + \frac{C}{m-1}\tilde{\rho} \text{ in } \Sigma.$$

Moreover, observe that $\tilde{\rho}^{m-1} \sim \tilde{U}$ is Lipschitz continuous in space, and continuous in space and time. Using this regularity of $\tilde{\rho}$ as well as above estimates on the derivatives of Φ, it follows that $\tilde{\rho}$ is a weak supersolution of (1.1) in Σ, if we choose C greater than $(m-1)C_1$. More precisely the following is true: for all times $0 < t \leq \omega^{-1}B$ and for any smooth, nonnegative function $\psi(x, t) : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ with $\{\psi(\cdot, t) > 0\} \subset \{|x| \leq 2B\}$ for $0 \leq t \leq \omega^{-1}B$, we have

$$\int \tilde{\rho}(\cdot, t)\psi(\cdot, t)dx \geq \int \tilde{\rho}(\cdot, 0)\psi(\cdot, 0)dx + \int\int (\tilde{\rho}^m \Delta\psi + \tilde{\rho}\psi_t - \tilde{\rho}\nabla\Phi \cdot \nabla\psi)dxdt.$$

Hence by comparing ρ with $\tilde{\rho}$ in $\Sigma = \{|x| \leq 2B\} \times [0, \omega^{-1}B]$, with $B = R + h^{1/2}$ and $A = 2C_0h^{-1/2}$, we conclude that (c) holds. □

Remark 3.2. *Due to [BRB], when $m > 2 - \frac{2}{d}$, there exists a global weak solution ρ of (1.1) with initial data ρ_0. Moreover, ρ is uniformly bounded in $\mathbb{R}^d \times (0, \infty)$ due to Theorem 10 in [BRB], so in that case we may let $T = \infty$.*

We finish this section with an approximation lemma which links case (A) and (B). Let

$$h^\epsilon := \epsilon^{-d}h(\frac{x}{\epsilon})$$

with h being the standard mollifier in $\mathbb{R}^d$ with unit mass, and let ρ^ϵ be the corresponding solution of (1.1) with $V = \mathcal{N} * h^\epsilon$ and with initial data ρ_0. Then Lemma 8 in [BRB] yields that $\{\rho^\epsilon\}_{\epsilon>0}$ are uniformly bounded for $t \in [0,T]$ for some T. This bound as well as Theorem 6.1 of [Dib] yields that the family of solutions $\{\rho^\epsilon\}$ are equi-continuous in space and time. This immediately yields the following result:

Proposition 3.3. *Let ρ_0 be as given in Theorem 3.1. Let $V = \mathcal{N} * h^\epsilon$ and let ρ^ϵ be the corresponding weak solution of (1.1) with initial data ρ_0. Let ρ be the unique solution to (1.1) with $V = \mathcal{N}$ and initial data ρ_0, and assume ρ exists for $t \in [0,T)$, where T can be either finite or ∞. Then the solutions ρ^ϵ locally uniformly converge to ρ in $\mathbb{R}^d \times [0,T)$.*

4 Monotonicity-preserving properties of solutions

In this section, we show that when V is given by (A) or (B), solution with radially decreasing initial data remains radially decreasing for all time. The central observation is that maximum principle type arguments works for the double-variable function

$$\Phi(x,y;t) = \rho(x,t) - \rho(y,t) \text{ in } \{|x| \geq |y|\} \times [0,\infty)$$

to ensure that Φ cannot achieve a positive maximum at a positive time.

We begin with an observation on the convolution term:

Lemma 4.1. *Let $V(x)$ be given by (B). Let $u(x)$ be a bounded non-negative radially symmetric function in $\mathbb{R}^d$ with compact support. Further suppose $u(x)$ is not radially decreasing, then there exists $a_1 = (\alpha, 0, ..., 0)$ and $b_1 = (\beta, 0, ..., 0)$ with $\alpha, \beta > 0$ such that*

$$u(b_1) - u(a_1) = \sup_{|a|<|b|} u(b) - u(a) > 0. \tag{4.1}$$

Then we have

$$(u * \Delta V)(b_1) - (u * \Delta V)(a_1) \leq \|\Delta V\|_{L^1}(u(b_1) - u(a_1)).$$

Proof. Observe that ΔV is nonnegative and radially decreasing, and thus it can be approximated in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ by the sum of bump functions of the form $c\chi_{B(0,r)}$, where $c > 0$. By linearity of convolution, it suffices to prove that for each bump function $\chi_{B(0,r)}$, where r is any positive real number, we have

$$(u * \chi_{B(0,r)})(b_1) - (u * \chi_{B(0,r)})(a_1) \leq \|\chi_{B(0,r)}\|_1 (u(b_1) - u(a_1)). \tag{4.2}$$

Observe that

$$\begin{aligned}(u * \chi_{B(0,r)})(b_1) - (u * \chi_{B(0,r)})(a_1) &= \int_{B(b_1,r)} u(x)dx - \int_{B(a_1,r)} u(x)dx && (4.3)\\ &= \int_{\Omega_B} u(x)dx - \int_{\Omega_A} u(x)dx && (4.4)\end{aligned}$$

Where $\Omega_A := B(a_1,r)\backslash B(b_1,r)$ and $\Omega_B := B(b_1,r)\backslash B(a_1,r)$ (see Figure 2).

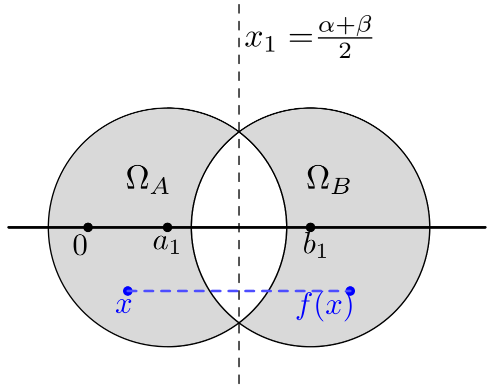

Figure 2: The domain Ω_A and Ω_B

Note that Ω_A and Ω_B are symmetric through the hyperplane $H = \{x : x_1 = \frac{\alpha+\beta}{2}\}$. For any $x \in \Omega_A$, use $f(x)$ to denote the reflection point of x with respect to H, then we have

$$\int_{\Omega_B} u(x)dx - \int_{\Omega_A} u(x)dx = \int_{\Omega_A} u(f(x)) - u(x)dx.$$

Since $|x| < |f(x)|$ for $x \in \Omega_A$, we can use the assumption (4.1) to obtain

$$\int_{\Omega_A} u(f(x)) - u(x)dx \leq \int_{\Omega_A} u(b) - u(a)dx \leq |B(0,r)|(u(b) - u(a)),$$

which completes the proof. $\square$

Theorem 4.2. *Let $V(x)$ be given by (A) or (B). Suppose that the initial data $\rho(x,0) \in L^1(\mathbb{R}^d;(1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ is* radially decreasing, *i.e. $\rho(x,0)$ is radially symmetric and is a decreasing function of $|x|$. We assume ρ exists for $t \in [0,T)$, where T can be either finite or ∞. Then $\rho(x,t)$ is radially decreasing for all $t \in [0,T)$.*

Proof. 1. Without loss of generality we assume that V is given by (B), and $\rho(x,0)$ is positive and smooth. Then $\rho(\cdot,t)$ exists for all $t \geq 0$, and we want to show $\rho(\cdot,t)$ is radially decreasing for all $t \geq 0$. When $V = \mathcal{N}$, we can use mollified Newtonian kernel to approximate $\mathcal{N}$; and for general radial decreasing initial data, we can use positive and smooth functions to approximate $\rho(x,0)$. Then the result follows via Proposition 3.3.

2. Radial symmetry of ρ for all $t > 0$ directly follows from the uniqueness of weak solution. Let us define

$$w(t) := \sup_{|a| \leq |b|} \rho(b,t) - \rho(a,t).$$

then since ρ is uniformly bounded and uniformly continuous in $\mathbb{R}^d \times [0,\infty)$, $w(t)$ is continuous in t, and uniformly bounded for $t \in [0,\infty)$. Moreover, $\rho(x,0)$ being radially decreasing implies $w(0) = 0$. We will use a maximum principle-type argument to show that $w(t) = 0$ for all $t \geq 0$, which proves the theorem.

To prove the claim, suppose not. Then for any $\lambda > 0$ the function $w(t)e^{-\lambda t}$ has a positive maximum at $t = t_1$ for some $t_1 > 0$. We will show that this cannot happen when we choose $\lambda := 4\|\rho\|_\infty \|\Delta V\|_1$.

At $t = t_1$ there exists $a_1 = (\alpha, 0, ..., 0)$ and $b_1 = (\beta, 0, ..., 0)$ such that $\alpha < \beta$ and

$$\rho(b_1, t_1) - \rho(a_1, t_1) = w(t_1) > 0 \quad \text{(see Figure 3)}. \tag{4.5}$$

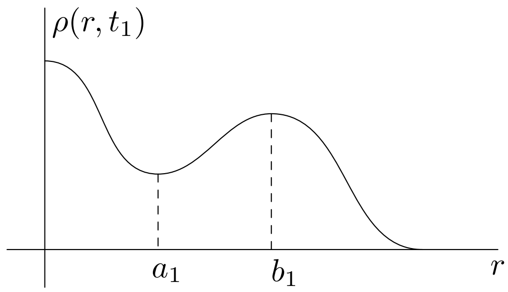

Figure 3: Graph of ρ at time t_1

Moreover by definition $\rho(b_1, t) - \rho(a_1, t) \leq w(t)$, and thus

$$\frac{d}{dt}((\rho(b_1, t) - \rho(a_1, t))e^{-\lambda t}) = 0 \quad \text{at } t = t_1,$$

which means

$$\rho_t(b_1, t_1) - \rho_t(a_1, t_1) = \lambda(\rho(b_1, t_1) - \rho(a_1, t_1)). \tag{4.6}$$

Further observe that $\rho(x, t_1)$ has a local minimum (in space only) at a_1 and $\rho(x, t_1)$ has a local maximum at b_1. This yields

$$\nabla\rho(a_1, t_1) = 0, \nabla\rho(b_1, t_1) = 0,$$

and

$$\Delta\rho^m(a_1, t_1) \geq 0, \Delta\rho^m(b_1, t_1) \leq 0,$$

Let us now make use of the equation (1.1) that ρ satisfies to get a contradiction: we have

$$\begin{aligned}
\rho_t(b_1, t_1) - \rho_t(a_1, t_1) &= \Delta\rho^m(b_1, t_1) + \nabla \cdot (\rho\nabla(\rho * V))(b_1, t_1) \\
&\quad -\Delta\rho^m(a_1, t_1) - \nabla \cdot (\rho\nabla(\rho * V))(a_1, t_1) \\
&\leq \rho(b_1, t_1)(\rho * \Delta V)(b_1, t_1) - \rho(a_1, t_1)(\rho * \Delta V)(a_1, t_1) \\
&= \rho(b_1, t_1)((\rho * \Delta V)(b_1, t_1) - (\rho * \Delta V)(a_1, t_1)) \\
&\quad +(\rho(b_1, t_1) - \rho(a_1, t_1))(\rho * \Delta V)(a_1, t_1).
\end{aligned}$$

By Lemma 4.1 we have

$$(\rho * \Delta V)(b_1, t_1) - (\rho * \Delta V)(a_1, t_1) \leq \|\Delta V\|_1(\rho(b_1, t_1) - \rho(a_1, t_1)).$$

Since $(\rho * \Delta V)(a_1, t_1) \leq \|\rho\|_\infty \|\Delta V\|_1$, we have

$$\begin{aligned}
\rho_t(b_1, t_1) - \rho_t(a_1, t_1) &\leq 2\|\rho\|_\infty \|\Delta V\|_1(\rho(b_1, t_1) - \rho(a_1, t_1)) \\
&\leq \frac{\lambda}{2}(\rho(b_1, t_1) - \rho(a_1, t_1)),
\end{aligned}$$

which contradicts (4.6). $\square$

The following proposition sates that in the previous theorem, the condition that ΔV is radially decreasing is indeed necessary.

Proposition 4.3. *Let $V(x)$ be radially symmetric, $\Delta V \geq 0$, and ΔV is continuous, but not radially decreasing. Then there exists a radially decreasing initial data ρ_0 such that the solution $\rho(x,t)$ of (1.1) with initial data ρ_0 is not radially decreasing for all sufficiently small $t > 0$.*

Proof. Since ΔV is not radially decreasing, we can find $x_1, x_2 \in \mathbb{R}^d$, such that $0 < |x_1| < |x_2|$, and $\Delta V(x_1) < \Delta V(x_2)$.

For a small $\epsilon > 0$, let $\rho_0(x)$ be given as below:

$$\rho_0(x) = \epsilon \chi_{B(0,x_2+1)} * \phi(x) + \frac{1}{\epsilon}\phi(\epsilon x),$$

where χ is the indicator function and ϕ is a radially symmetric mollifier with unit mass and supported in $B(0, \min\{1, |x_1|/2\})$. Note that in a small space-time neighborhood of x_1 and x_2, ρ solves a uniformly parabolic equation, and thus is smooth.

Since $\Delta \rho^m(x_i, 0) = \nabla \rho(x_i, 0) = 0$ for $i = 1, 2$, we have

$$\rho_t(x_i, 0) = \rho_0(x_i)(\rho_0 * \Delta V)(x_i), \quad i = 1, 2.$$

Since $\rho_0(x_1) = \rho_0(x_2)$, to show $\rho_t(x_1, 0) < \rho_t(x_2, 0)$, it suffices to prove

$$(\rho_0 * \Delta V)(x_1) < (\rho_0 * \Delta V)(x_2). \tag{4.7}$$

Note that $\rho_0 * \Delta V$ locally uniformly converges to $\Delta V(x)$ as $\epsilon \to 0$. Since $\Delta V(x_1) < \Delta V(x_2)$, if we let ϵ be sufficiently small, we would have (4.7). In particular $\rho(x_1, t) < \rho(x_2, t)$ for small $t > 0$, which means $\rho(x,t)$ becomes non-radially decreasing as soon as $t > 0$. $\square$

5 Mass Comparison and asymptotic behavior for radial solutions

Recall that there is no classical comparison principle for (1.1), due to the nonlocal term; however, we will prove that a comparison principle actually hold for the *mass function*

$$M(r,t) = M(r,t;\rho) := \int_{B(0,r)} \rho(x,t)dt \tag{5.1}$$

(see Proposition 5.3). We point out that the corresponding property has been observed for (1.2) ([V]) and also for the Keller-Segel model ([BKLN]). It turns out that mass comparison holds for (1.1) if the potential V satisfies $\Delta V \geq 0$ in the distribution sense (Proposition 5.3.) As we will see later, mass comparison effectively describes asymptotic behavior of radial solutions in both sub- and supercritical regime.

5.1 Mass comparison

First note that, in the positive set, ρ is at least C^1 in space and time variable, since it is a bounded solution of uniformly parabolic, divergence-type equation with continuous coefficients. Therefore it follows that $M(r,t)$ is C^2 in space and C^1 in time in $\{\rho > 0\}$.

Next we compute the PDE that $M(r,t)$ satisfies in $\{r < R(t)\}$:

$$\begin{aligned}\frac{\partial M}{\partial t}(r,t) &= \int_{\partial B(0,r)} \vec{n}\cdot(\nabla\rho^m + \rho\nabla(\rho * V))dx \\ &= \sigma_d r^{d-1}(\frac{\partial}{\partial r}((\frac{\partial M}{\partial r}\frac{1}{\sigma_d r^{d-1}})^m) + (\frac{\partial M}{\partial r}\frac{1}{\sigma_d r^{d-1}})(\frac{\tilde{M}}{\sigma_d r^{d-1}})) \\ &= \sigma_d r^{d-1}\frac{\partial}{\partial r}\big((\frac{\partial M}{\partial r}\frac{1}{\sigma_d r^{d-1}})^m\big) + (\frac{\partial M}{\partial r}\frac{1}{\sigma_d r^{d-1}})\tilde{M}, \end{aligned} \tag{5.2}$$

where

$$\tilde{M}(r,t) = \tilde{M}(r,t;\rho) := \int_{B(0,r)} (\rho(\cdot,t) * \Delta V)(x)dx. \tag{5.3}$$

Definition 5.1. *Let ρ_1 and ρ_2 be two non-negative radially symmetric functions in $L^1(\mathbb{R}^d)$. We say ρ_1 is* less concentrated than *ρ_2, or $\rho_1 \prec \rho_2$, if*

$$\int_{B(0,r)} \rho_1(x)dx \le \int_{B(0,r)} \rho_2(x)dx \quad \textit{for all } r \ge 0.$$

Definition 5.2. *Let $\rho_1(x,t)$ be a non-negative, radially symmetric function in $L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, which is C^1 in its positive set. We say ρ_1 is a* supersolution *of* (5.2) *if $M_1(r,t) := M(r,t;\rho_1)$ and $\tilde{M}_1 := \tilde{M}(r,t;\rho_1)$ satisfy*

$$\frac{\partial M_1}{\partial t} \ge \sigma_d r^{d-1}\frac{\partial}{\partial r}((\frac{\partial M_1}{\partial r}\frac{1}{\sigma_d r^{d-1}})^m) + (\frac{\partial M_1}{\partial r}\frac{1}{\sigma_d r^{d-1}})\tilde{M}_1 \tag{5.4}$$

in the positive set of ρ_1.

Similarly we define a subsolution of (5.2).

Proposition 5.3 (mass comparison)**.** *Suppose $m > 1$. Let V be given by (A) or (B), and let $\rho_1(x,t)$ be a supersolution and $\rho_2(x,t)$ be a subsolution of* (5.2)*. Further assume that ρ_i's preserve its mass over time, i.e., $\int \rho_1(\cdot,t)dx$ and $\int \rho_2(\cdot,t)dx$ stays constant for all $t \ge 0$. Then the mass functions are ordered for all times: i.e., if $\rho_1(x,0) \succ \rho_2(x,0)$, then we have $\rho_1(x,t) \succ \rho_2(x,t)$ for all $t > 0$.*

Proof. We claim that $M_1(r,t) \ge M_2(r,t)$ for all $r \ge 0, t > 0$, which proves the lemma. For the boundary conditions of M_i, note that

$$\begin{cases} M_1(0,t) = M_2(0,t) = 0 \quad \text{for all } t, \\ \lim_{r\to\infty}(M_1(r,t) - M_2(r,t)) = \int_{\mathbb{R}^d}(\rho_1(x,0) - \rho_2(x,0))dx \ge 0 \quad \text{for all } t. \end{cases}$$

As for initial data, we have $M_1(r,0) \ge M_2(r,0)$ for all $r \ge 0$.

To prove the claim, for given $\lambda > 0$, we define

$$w(r,t) := \big(M_2(r,t) - M_1(r,t)\big)e^{-\lambda t},$$

If the claim is false at $t = T$, then w attains a positive maximum at some point (r_1, t_1) in the domain $(0, \infty) \times (0, T]$. Moreover, since the total mass of ρ_1 and ρ_2 is preserved over time and thus is ordered, we know that (r_1, t_1) must lie inside the positive set for both ρ_1 and ρ_2, where M_i's are $C^{2,1}_{x,t}$.

At (r_1, t_1), since w attains a maximum, we have

$$w_t = 0 \implies \frac{\partial(M_2 - M_1)}{\partial t} = \lambda(M_2 - M_1) \tag{5.5}$$

$$w_r = 0 \implies \frac{\partial M_1}{\partial r} = \frac{\partial M_2}{\partial r} \tag{5.6}$$

$$w_{rr} \le 0 \implies \frac{\partial^2 M_1}{\partial r^2} \ge \frac{\partial^2 M_2}{\partial r^2} \tag{5.7}$$

Now consider the first term on the right hand side of (5.4), and the corresponding inequality for M_2. Using (5.6) and (5.7), we have

$$\frac{\partial}{\partial r}\Big(\big(\frac{\partial M_1}{\partial r}\frac{1}{\sigma_d r^{d-1}}\big)^m\Big) \ge \frac{\partial}{\partial r}\Big(\big(\frac{\partial M_2}{\partial r}\frac{1}{\sigma_d r^{d-1}}\big)^m\Big) \quad \text{at } (r_1, t_1). \tag{5.8}$$

Subtracting the inequality (5.4) with the corresponding inequality for M_2, and using (5.8), we obtain

$$\frac{\partial(M_2 - M_1)}{\partial t} \le \big(\frac{\partial M_1}{\partial r}\frac{1}{\sigma_d r^{d-1}}\big)(\tilde{M}_2 - \tilde{M}_1) \quad \text{at } (r_1, t_1). \tag{5.9}$$

We next claim

$$(\tilde{M}_2 - \tilde{M}_1)(r_1, t_1) \le C(M_2 - M_1)(r_1, t_1),$$

where C only depend on V. For Newtonian potential this is obvious, since $\tilde{M} \equiv M$. For mollified Newtonian potential, we estimate $(\tilde{M}_2 - \tilde{M}_1)(r_1, t_1)$ as following:

$$\begin{aligned}\tilde{M}_2(r_1, t_1) - \tilde{M}_1(r_1, t_1) &= \int_{\mathbb{R}^d} ((\rho_2 - \rho_1) * \Delta V)\ \chi_{B(0,r_1)} dx \\ &= \int_{\mathbb{R}^d} (\rho_2 - \rho_1)(\chi_{B(0,r_1)} * \Delta V) dx\end{aligned}$$

By our assumption, ΔV is radially decreasing, thus $\chi_{B(0,r_1)} * \Delta V$ is radially decreasing and has maximum less than or equal to $||\Delta V||_1$. Therefore we can use a sum of bump function to approximate $\chi_{B(0,r_1)} * \Delta V$, where the sum of the height is less than $||\Delta V||_1$. Hence

$$\tilde{M}_2(r_1, t_1) - \tilde{M}_1(r_1, t_1) \le ||\Delta V||_1 \sup_x (M_2 - M_1)(x, t_1) = ||\Delta V||_1 (M_2 - M_1)(r_1, t_1).$$

Once we get this estimate, plug it into (5.9), which then becomes

$$\frac{\partial(M_2 - M_1)}{\partial t} \le \rho_1(M_2 - M_1) \quad \text{at } (r_1, t_1). \tag{5.10}$$

Now choose $\lambda > \sup_{\mathbb{R}^n \times (0,T]} \rho_1$, then (5.10) will contradict (5.5). $\square$

Observe that (1.1) can be written as a transport equation

$$\rho_t + \nabla \cdot (\rho \vec{v}) = 0,$$

where the *velocity field* $\vec{v}$ is given by

$$\vec{v}(x,t;\rho) := -\frac{m}{m-1}\nabla(\rho^{m-1}) - \nabla(\rho * V). \tag{5.11}$$

Then the mass function for a radial solution of (1.1) satisfies

$$\frac{\partial}{\partial t}M(r,t) = -\rho(r,t)\int_{\partial B(0,r)} \vec{v}\cdot\vec{n}ds. \tag{5.12}$$

Above observation along with Proposition 5.3 immediately yields the following corollary:

Corollary 5.4. *Suppose $m > 1$. Let V be given by (A) or (B). Let $\rho_0(x)$ be a continuous radially symmetric function, which is differentiable in its positive set. We assume the velocity field of ρ_0 is pointing inside everywhere, i.e., for $\vec{v}$ as defined in* (5.11)*,*

$$\vec{v}(x;\rho_0)\cdot\frac{-x}{|x|} \geq 0 \quad \text{in } \{\rho_0 > 0\}. \tag{5.13}$$

Let ρ be the weak solution of (1.1) *with initial data $\rho(x,0)$, where $\rho(\cdot,0) \succ \rho_0$. Then $\rho(\cdot,t) \succ \rho_0$ for all $t \geq 0$.*

Proof. Let us define

$$\rho_1(x,t) := \rho_0(x), \text{ for } x \in \mathbb{R}^d, t \in \mathbb{R}^+.$$

Then (5.12) and (5.13) yield that ρ_1 is a subsolution of (5.2). Therefore, Proposition 5.3 applies to ρ and ρ_1 and we can conclude. □

As an application of Corollary 5.4, we will show that when the initial data is radially symmetric and compactly supported, the support of the solution will stay in a large ball for all times.

Corollary 5.5 (compact solution stays compact)**.** *Let V be given by (A) or (B), and suppose $m > 2-\frac{2}{d}$. Let ρ be the solution to* (1.1)*, where the initial data $\rho(x,0) \in L^1(R^d;(1+|x|^2)dx)\cap L^\infty(\mathbb{R}^d)$ is continuous, radially symmetric and compactly supported. Then there exists $R > 0$ depending on $m, d, \|\Delta V\|_1$ and $\rho(\cdot,0)$, such that $\{\rho(\cdot,t) > 0\} \subset \{|x| \leq R\}$ for all $t > 0$.*

Proof. 1. We will first assume that $\rho(x,0) > 0$.

2. Let $A := \int_{\mathbb{R}^d} \rho(x,0)dx$, and let $\rho_A(x)$ be a radial stationary solution with mass A. For any continuous radial initial data with $\rho(0,0) > 0$, we can choose $a > 0$ sufficiently small, such that

$$\rho_1(x,t) := a^d \rho_A(ax) \prec \rho(x,0).$$

Our aim is to show that the velocity field of $\rho_1(x,t)$ is pointing towards inside all the time, i.e.,

$$v(r,t;\rho_1) := \vec{v}(r,t;\rho_1)\cdot\frac{-x}{|x|} = \frac{\partial}{\partial r}\rho_1^{m-1}(r) + \frac{\partial}{\partial r}(\rho_1 * V) \geq 0. \tag{5.14}$$

Let us assume that V is given by (B); the argument for V given by (A) is parallel and easier. Recall that the stationary solution $\rho_A(x,t)$ satisfies the following equation in its positive set:

$$\frac{m}{m-1}\frac{\partial \rho_A^{m-1}}{\partial r} + \frac{\tilde{M}(r;\rho_A)}{\sigma_d r^{d-1}} = 0 \tag{5.15}$$

Therefore it follows that

$$\begin{aligned} \frac{m}{m-1}\frac{\partial}{\partial r}\rho_1^{m-1}(r) &= a^{(m-1)d+1}\frac{m}{m-1}\frac{\partial \rho_A^{m-1}}{\partial r}(ar) \\ &= -a^{(m-1)d+1}\frac{\tilde{M}(ar;\rho_A)}{\sigma_d (ar)^{d-1}}, \end{aligned}$$

Secondly observe that $\tilde{M}(r,t;\rho_1)$ satisfies

$$\begin{aligned} \tilde{M}(r,t;\rho_1) &= \int_{B(0,r)}\int_{\mathbb{R}^d} a^d \rho_A(ay)\Delta V(y-x)dydx \\ &= \int_{B(0,ar)}\int_{\mathbb{R}^d} \rho_A(y) a^{-d}\Delta V(a^{-1}(y-x))dydx \\ &\geq \int_{B(0,ar)} \rho_A * \Delta V dx \quad (\text{since } a^{-d}\Delta V(a^{-1}x) \succ \Delta V \text{ when } 0<a<1) \\ &= \tilde{M}(ar;\rho_A). \end{aligned}$$

(Note that when V is given by (A), direct computation yields $M(r,t;\rho_1) = M(ar;\rho_A)$.)

Due to (2.3) and above inequalities, it follows that

$$\begin{aligned} v(r,t;\rho_1) &= \frac{\partial}{\partial r}\rho_1^{m-1}(r) + \frac{\partial}{\partial r}(\rho_1 * V) \\ &\geq \frac{\partial}{\partial r}\rho_1^{m-1}(r) + \frac{\tilde{M}(r;\rho_1)}{\sigma_d r^{d-1}} \qquad (5.16) \\ &\geq (1-a^{d(m-2+2/d)})a^{d-1}\frac{\tilde{M}(ar;\rho_A)}{\sigma_d (ar)^{d-1}}. \qquad (5.17) \end{aligned}$$

Since $m > 2-2/d$, above inequality yields that the inward velocity field $v(r,\rho_1) \geq 0$ when $a<1$. Therefore Corollary 5.4 implies that $\rho(\cdot,t) \succ \rho_1$ for all $t \geq 0$. Since they have the same mass A, it follows that

$$\text{supp}\rho(\cdot,t) \subset \text{supp}\rho_1, \text{ for all } t>0,$$

and we can conclude.

3. The assumption $\rho(0,0)>0$ can indeed be removed, since $\rho(0,t)$ would still become positive in finite time even if $\rho(0,0)=0$. That is because, for (1.2), it is a well-known fact that the solution will have a positive center density after finite time: this can be verified, for example, by maximum-principle type arguments using translations of Barenblatt solutions.

Note that a solution of (1.2) is a subsolution in the mass comparison sense. Hence one can compare ρ with a solution ψ of (1.2) with initial data ρ_0 and apply Proposition 5.3 to conclude that $\psi \prec \rho$. Now our assertion follows due to the continuity of ψ and ρ at the origin. □

5.2 Exponential convergence towards stationary solution in subcritical regime

As an application of Proposition 5.3, we will prove asymptotic convergence of general radial solutions to the unique radial stationary solution when the potential is given by (A) or (B).

Theorem 5.6 (Exponential convergence for Newtonian potential)**.** *Let $m > 2 - \frac{2}{d}$ and let V be given by (A) or (B). For given $\rho_0 \geq 0$: a continuous, radially symmetric function with compact support, let $\rho(x,t)$ be the solution to* (1.1) *with initial data ρ_0. Next let ρ_A be a radial stationary solution with mass $A := \int \rho_0(x)dx$. Then $M(r,t) := M(r,t;\rho)$ satisfies*

$$|M(r,t) - M(r;\rho_A)| \leq C_1 e^{-\lambda t}, \text{ for all } r > 0,$$

where C_1 depends on ρ_0, A, m, d, V, and the rate λ only depends on A, m, d, V.

Proof. 1. We will only prove the case when V satisfies (B); the case for (A) can be proven with a parallel (and easier) argument. Also note that we may assume $\rho_0(0) > 0$ since otherwise $\rho(0,t)$ will be come positive in finite time as explained in step 3. of the proof of Corollary 5.5.

2. Let ρ_A be a stationary solution with same mass as ρ_0, given as in the proof for Corollary 5.5. Since ρ_0 is compactly supported, continuous and with $\rho_0(0) > 0$, we can choose $0 < a < 1$ to be sufficiently small, such that

$$a^d \rho_A(ax) \prec \rho_0 \quad \text{ and } a^{-d}\rho_A(a^{-1}x) \succ \rho_0.$$

3. With above choice of a, we next construct a self-similar subsolution $\phi(x,t)$ of (5.2) with initial data $\phi(x,0) = a^d\rho_A(ax)$ such that $M_\phi(\cdot,t) := M(\cdot,t;\phi)$ converges exponentially to $M(\cdot;\rho_A)$ as $t \to \infty$.

Here is the strategy on construction of $\phi(x,t)$. Due to (5.17), for all $0 < a < 1$, the inward velocity field $v(r) := v(r; a^d\rho_A(ax))$ given by (5.14) satisfies

$$v(r) \geq (1 - a^{d(m-2+2/d)})a^d r \frac{\tilde{M}(ar;\rho_A)}{\sigma_d(ar)^d} \geq 0.$$

Observe that $\dfrac{d\tilde{M}(ar;\rho_A)}{\sigma_d(ar)^d}$ equals the average of $\rho_A * \Delta V$ in the ball $\{|x| \leq ar\}$. By Proposition 2.1, ρ_A (hence $\rho_A * \Delta V$) is radially decreasing, and thus we have

$$C_1 \leq \frac{\tilde{M}(ar;\rho_A)}{\sigma_d(ar)^d} \leq C_2 \quad \text{in } \{\rho_A > 0\} \tag{5.18}$$

where C_1, C_2 only depend on A, d, m, V. That gives a lower bound for the inward velocity field v

$$v(r) \geq C_1 a^d (1 - a^{d(m-2+2/d)})r. \tag{5.19}$$

Now if we have a self-similar function $\phi(x,t)$ where every point is moving towards center with the speed exactly $C_1 a^d(1 - a^{d(m-2+2/d)})r$, then we would expect it to be a subsolution of (5.2).

Let us define

$$\phi(r,t) = k^d(t)\ \rho_A\big(k(t)r\big), \tag{5.20}$$

where the scaling factor $k(t)$ solves the following ODE with initial data $k(0) = a$:

$$k'(t) = C_1(k(t))^{d+1}(1-(k(t))^{d(m-2+2/d)}). \tag{5.21}$$

Since $m > 2-2/d$, $k'(t) > 0$ when $0 < k < 1$, and since $k = 1$ is the only non-zero stationary point for the ODE (5.21), for $0 < k(0) < 1$ we have $\lim_{t\to\infty} k(t) = 1$. Since

$$C_1k^d(1-k^{d(m-2+2/d)}) = -C_1d(m-2+2/d)(1-k) + o(1-k),$$

it follows that

$$0 \le 1-k(t) \lesssim e^{-C_1d(m-2+2/d)t}, \tag{5.22}$$

which implies

$$0 \le M_s(r,t) - M_\phi(r,t) \lesssim e^{-C_1d(m-2+2/d)t}, \text{ for all } x. \tag{5.23}$$

Next we claim that ϕ is a subsolution of (5.2), i.e.,

$$\frac{\partial M_\phi}{\partial t} \le \sigma_d r^{d-1}\frac{\partial}{\partial r}((\frac{\partial M_\phi}{\partial r}\frac{1}{\sigma_d r^{d-1}})^m) + (\frac{\partial M_\phi}{\partial r}\frac{1}{\sigma_d r^{d-1}})\tilde{M}_\phi \text{ in } \{\phi > 0\} \tag{5.24}$$

To prove the claim, first note that by definition of $\phi(r,t)$ we have $M_\phi(r,t) = M(k(t)r;\rho_A)$. Hence the left hand side of (5.24) is

$$\begin{aligned}
\frac{\partial M_\phi}{\partial t}(r,t) &= \partial_r M_s(k(t)r)\ k'(t)r \\
&= \sigma_d r^d \rho_A(k(t)r)k^{d-1}(t)k'(t) \\
&= \sigma_d r^d \phi(r,t) C_1 k^d(1-k^{d(m-2+2/d)}) \quad \text{(due to (5.21) and definition of } \phi)
\end{aligned}$$

On the other hand, we can proceed in the same way as (5.19), replacing a by k, to obtain

$$\frac{m}{m-1}\frac{\partial}{\partial r}\phi^{m-1} + \frac{\tilde{M}_\phi}{\sigma_d r^{d-1}} \ge C_1k^d(1-k^{d(m-2+2/d)})r.$$

Therefore

$$\begin{aligned}
\text{RHS of (5.24)} &= \sigma_d r^{d-1}\frac{\partial}{\partial r}\phi^m + \phi\tilde{M}_\phi \\
&= \sigma_d r^{d-1}\phi\Big(\frac{m}{m-1}\frac{\partial}{\partial r}\phi^{m-1} + \frac{\tilde{M}_\phi}{\sigma_d r^{d-1}}\Big) \\
&\ge \sigma_d r^d \phi C_1 k^d(1-k^{d(m-2+2/d)}),
\end{aligned}$$

thus M_ϕ indeed satisfies (5.24), and the claim is proved.

4. Similarly one can construct a supersolution of (5.2). Let us define

$$\eta(r,t) := k^d(t)\ \rho_A\big(k(t)r\big),$$

where $k(t)$ solves the following ODE with initial data $k(0) = \frac{1}{a}$:

$$k'(t) = C_2k^{d+1}(1-k^{d(m-2+2/d)}).$$

Parallel arguments as in step 2 yields that η is a supersolution of (5.2) and

$$0 \le M_\eta(r,t) - M(r;\rho_A) \lesssim e^{-C_2 d(m-2+2/d)t}, \text{ for all } r > 0. \tag{5.25}$$

5. Lastly we compare ϕ, η with the weak solution ρ of (1.1). Since

$$\phi(\cdot,0) \prec \rho(\cdot,0) \prec \eta(\cdot,0) \quad \text{(see Figure 4)},$$

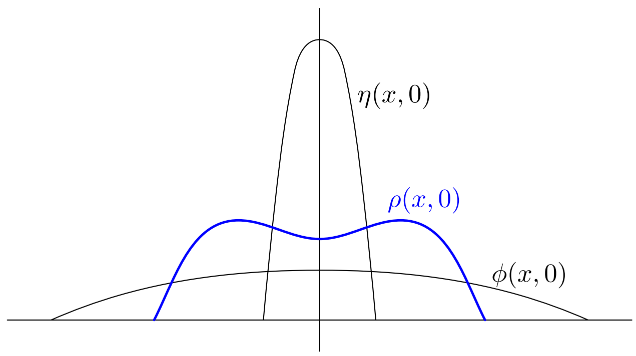

Figure 4: Initial data for ϕ, ρ and η

Proposition 5.3 yields that

$$M_\phi(\cdot,t) \le M(\cdot,t) \le M_\eta(\cdot,t),$$

By (5.23) and (5.25), we obtain

$$|M(r,t) - M(r;\rho_A)| \lesssim e^{-C_1 d(m-2+2/d)t}, \ \forall r.$$

□

Using the explicit subsolution and supersolution constructed in the proof of Theorem 5.6, we get exponential convergence of ρ/A towards the ρ_A/A in the p-Wasserstein metric, which is defined below. Note that the Wasserstein metric is natural for this problem, since as pointed out in [BD] and [BS], the equation (1.1) is a gradient flow of the energy (1.5) with respect to the 2-Wasserstein metric.

Definition 5.7. *Let μ_1 and μ_2 be two (Borel) probability measure on $\mathbb{R}^d$ with finite pth moment. Then the p*th Wasserstein distance *between μ_1 and μ_2 is defined as*

$$W_p(\mu_1,\mu_2) := \Big(\inf_{p\in\mathcal{P}(\mu_1,\mu_2)} \Big\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p p(dxdy) \Big\} \Big)^{\frac{1}{p}},$$

where $\mathcal{P}(\mu_1,\mu_2)$ is the set of all probability measures on $\mathbb{R}^d\times\mathbb{R}^d$ with first marginal μ_1 and second marginal μ_2.

Corollary 5.8. *Let $\rho, \rho_A, A, C_1, \lambda$ as given in Theorem 5.6. Then for all $p > 1$, we have*

$$W_p\Big(\frac{\rho(\cdot,t)}{A}, \frac{\rho_A}{A}\Big) \le C_1 e^{-\lambda t}.$$

In fact one can also obtain uniform convergence of ρ to ρ_A in sup-norm, however the convergence rate would depend on the modulus of continuity of ρ. Theorem 5.6 and the uniform continuity of ρ and ρ_A, as well as the fact that ρ_A is compactly supported, yields the following:

Corollary 5.9. *Let ρ,ρ_A, C_1 and λ as given in Theorem 5.6. Then we have*

$$\lim_{t\to\infty} \|\rho(x,t) - \rho_A(x)\|_{L^\infty(\mathbb{R}^d)} = 0,$$

Note that uniqueness of ρ_A is not required in the proof of Theorem 5.6. Indeed, uniqueness of ρ_A can be obtained as a consequence of asymptotic convergence of ρ: if there are two radial stationary solutions ρ_A^1 and ρ_A^2 with the same mass, Corollary 5.9 implies $\rho(\cdot,t) \to \rho_A^i$ in L^∞ norm for $i = 1, 2$ when ρ is given as in Theorem 5.6, which immediately establish the uniqueness of radial stationary solution:

Corollary 5.10. *Let V be given by (A) or (B), and let $m > 2 - \frac{2}{d}$. Then for all $A > 0$, the radial stationary solution ρ_A for* (1.1) *with $\int \rho_A(x)dx = A$ is unique.*

5.3 Algebraic convergence towards Barenblatt profile in supercritical regime

In this section, we consider the asymptotic behavior of radial solutions in the supercritical regime, i.e. for $1 < m < 2 - \frac{2}{d}$. In this case the diffusion overrides the aggregation and thus the solution is expected to behave similar to that of Porous Medium Equation (PME) in the long run. In fact recently it is shown in [B1] (and also in [S1]), by making use of entropy method as well as functional inequalities, that the solution of (1.1) with a general class of V and with small mass and small $L^{(2-m)d/2}$ norm converges to the self-similar Barenblatt solution $\mathcal{U}(x,t)$ with algebraic rate,

$$\mathcal{U}(x,t) = t^{-\beta d}(C - \frac{(m-1)\beta}{2m}|x|^2 t^{-2\beta})_+^{\frac{1}{m-1}}, \tag{5.26}$$

where C is some constant such that $\|\mathcal{U}(\cdot,0)\|_1 = \|\rho(\cdot,0)\|_1$.

As another application of mass comparison, we will give an alternative, simpler proof for above result in the case of radial solutions, by using mass comparison (Proposition 5.3). We point out that in our result the mass does not need to be small, and provides an explicit description of solutions which are "sufficiently scattered" so that it does not blow up in finite time. We also point out that, of course, the method presented in [B1] is much more delicate and yields optimal convergence results for general solutions with small mass in the supercritical regime.

Let ρ be the weak solution to (1.1). Following [V], we re-scale ρ as follows:

$$\mu(\lambda,\tau) = (t+1)^\alpha \rho(x,t); \quad \lambda = x(t+1)^{-\beta}; \quad \tau = \ln(t+1). \tag{5.27}$$

where

$$\alpha = \frac{d}{d(m-1)+2}, \quad \beta = \alpha/d.$$

Then $\mu(\lambda,0) = \rho(x,0)$, and $\mu(\lambda,\tau)$ solves

$$\mu_\tau = \Delta\mu^m + \beta\nabla\cdot(\mu\nabla\frac{|\lambda|^2}{2}) + e^{(1-\alpha)\tau}\nabla\cdot(\mu\nabla(\mu * (\mathcal{N} * \tilde{h}(\lambda,\tau))), \tag{5.28}$$

where

$$\tilde{h}(\lambda, \tau) := e^{d\beta\tau} \Delta V(\lambda e^{\beta\tau}) \tag{5.29}$$

(when $V = \mathcal{N}$ one should replace the last term by $e^{(1-\alpha)\tau}\nabla \cdot (\mu \nabla(\mu * \mathcal{N}))$.

In the absense of the last term, equation (5.28) becomes a Fokker-Planck equation

$$\mu_\tau = \Delta \mu^m + \beta \nabla \cdot (\mu \nabla \frac{|\lambda|^2}{2}), \tag{5.30}$$

which is known to converge to the stationary solution μ_A exponentially, where μ_A has mass $A := \|\mu(\cdot, 0)\|_1$ and satisfies

$$\frac{m}{m-1}\mu_A^{m-1} = (C - \beta \frac{|\lambda|^2}{2})_+ \text{ for some } C > 0. \tag{5.31}$$

In Therorem 5.13, we will prove for $m < 2 - 2/d$, if the initial data is sufficiently less concentrated than μ_A, then $\mu(\cdot, \tau)$ also converges to the same limit μ_A exponentially as $\tau \to \infty$.

The proof of Therorem 5.13 is obtained by mass comparison. We define the following mass function for μ: $M^\mu(r, \tau) := M(r, \tau; \mu)$, where M is as given in (5.1). Also, for any function f, we define $\tilde{\mathcal{M}}(r, \tau; f) := \int_{B(0,r)} f * \tilde{h}(\cdot, \tau) d\lambda$, where $\tilde{h}$ is as given in (5.29). (For $V = \mathcal{N}$, $\tilde{h}(\cdot, \tau)$ is the delta function for all τ, hence $\tilde{\mathcal{M}} \equiv M$.)

Then M^μ satisfies the following PDE in the positive set of μ:

$$M^\mu_\tau = \sigma_d r^{d-1} (\frac{\partial M^\mu}{\partial r} \frac{1}{\sigma_d r^{d-1}}) \Big[\frac{m}{m-1} \frac{\partial}{\partial r} ((\frac{\partial M^\mu}{\partial r} \frac{1}{\sigma_d r^{d-1}})^{m-1}) + \beta r + e^{(1-\alpha)\tau} \frac{\tilde{\mathcal{M}}(r, \tau; \mu)}{\sigma_d r^{d-1}} \Big] \tag{5.32}$$

First we prove that mass comparison also holds for the rescaled equation (5.28).

Proposition 5.11. *Let $V(x)$ be given by (A) or (B), and let $m < 2 - \frac{2}{d}$. Assume $\mu_1(\lambda, \tau)$ is a subsolution and $\mu_2(\lambda, \tau)$ is a supersolution of (5.32). Further assume that μ_i's preserve its mass over time, i.e., $\int \mu_1(\cdot, \tau) d\lambda$ and $\int \mu_2(\cdot, \tau) d\lambda$ stays constant for all $t \geq 0$.*

Then the mass is ordered for all times: i.e., if $\mu_1(\lambda, 0) \prec \mu_2(\lambda, 0)$, then we have $\mu_1(\lambda, \tau) \prec \mu_2(\lambda, \tau)$ for all $\tau > 0$.

Proof. Let $\rho_i(x, t)$ be the corresponding re-scaled versions of μ_i, then ρ_1 (ρ_2) would be subsolution (supersolution) of (5.2). The proof is straightforward from Proposition 5.3 and from the fact that

$$M(r, \tau; \mu_i) = e^{(\alpha - \beta)\tau} M(re^{\beta\tau}, e^\tau; \rho_i).$$

$\square$

Now we state a technical lemma which is used later in the proof of the convergence theorem.

Lemma 5.12. *Let $k(t)$ solve the ODE*

$$k'(t) = C_1 k(1 - k^\alpha) + C_2 k^{d+1} e^{-\beta t}, \tag{5.33}$$

where C_1, C_2, α, β are positive constants. Then there exists a constant $\delta > 0$ such that if $0 < k(0) < \delta$, then $k(t) \to 1$ exponentially as $t \to \infty$.

Proof. When $0 < k < 1$, the right hand side of (5.33) is bounded above by $(C_1 + C_2)k$. Hence if the initial data satisfies $0 < k(0) < 1$, the inequality $k(t) \le k(0)e^{(C_1+C_2)t}$ will hold until k reaches 1. In other words, $k(t)$ is guaranteed to be smaller than 1 until time $t_1 := -\frac{\ln k(0)}{C_1+C_2}$.

Now if we choose $k(0)$ to be sufficiently small such that $0 < k(0) < \delta$, where

$$\delta := (\alpha C_1 C_2^{-1} 2^{-d-2})^{\frac{C_1+C_2}{\beta}},$$

then t_1 would be sufficiently large such that

$$C_2 2^{d+1} e^{-\beta t_1} \le \frac{C_1 \alpha}{2}.$$

We claim $g(t) := 1 + e^{-\epsilon(t-t_1)}$ is a supersolution of (5.33) for $t \ge t_1$, where $\epsilon := \min\{\beta, \frac{1}{2}C_1\alpha\}$. It is obvious that $g(t_1) > 1 \ge k(t_1)$, thus it suffices to show

$$g'(t) \ge C_1 g(1 - g^\alpha) + C_2 g^{d+1} e^{-\beta t} \quad \text{for } t \ge t_1. \tag{5.34}$$

By definition of g, we have

$$\begin{aligned} \text{RHS of (5.34)} &\le -C_1 \alpha e^{-\epsilon(t-t_1)} + C_2 2^{d+1} e^{-\beta t_1} e^{-\beta(t-t_1)} && (5.35)\\ &\le -\frac{1}{2} C_1 \alpha e^{-\epsilon(t-t_1)} && (5.36)\\ &\le -\epsilon e^{-\epsilon(t-t_1)} = \text{LHS of (5.34)}. && (5.37) \end{aligned}$$

Therefore $k(t) \le 1 + e^{-\epsilon(t-t_1)}$ for all $t \ge t_1$. Similarly we can show that $k(t) \ge 1 - e^{-\epsilon(t-t_1)}$ for all $t \ge t_1$, thus

$$|k(t) - 1| \le Ce^{-\epsilon t},$$

where $C := e^{-\epsilon t_1}$ depends on C_1, C_2, α, β.

□

Now we are ready to prove the main theorem. We will first prove it for radially decreasing solutions.

Theorem 5.13. *Let $V(x)$ be given by (A) or (B), and let $1 < m < 2 - \frac{2}{d}$. Assume $\mu_0(\lambda)$ is radially decreasing, compactly supported and has mass A. Then there exists a sufficiently small constant $\delta > 0$ depending on d, m, μ_0 and V, such that if*

$$\mu_0(\lambda) \prec \delta^d \mu_A(\delta\lambda),$$

where $\mu_A(\lambda)$ is given in (5.31), then the weak solution $\mu(\lambda, \tau)$ to (5.28) with initial data μ_0 exists globally. Furthermore, the mass function $M(r, \tau; \mu)$ defined in (5.1) converges to $M(r, \tau; \mu_A)$ exponentially as $t \to \infty$ and uniformly in r.

Proof. The proof of theorem is analogous to that of Theorem 5.6: we will construct a self-similar subsolution $\phi(\lambda, \tau)$ and supersolution $\eta(\lambda, \tau)$ to (5.28), both of which converge to μ_A exponentially.

Observe that (5.28) can be written as a transport equation

$$\mu_t + \nabla \cdot (\mu \vec{v}) = 0,$$

where the *velocity field* $\vec{v}$ is given by

$$\vec{v} := \frac{m}{m-1}\nabla(\mu^{m-1}) + \beta\lambda + e^{(1-\alpha)\tau}\nabla(\mu * (\mathcal{N} * \tilde{h}(y,\tau)).$$

Hence that the inward velocity field $v(r,\tau;\mu) := -\vec{v}\cdot\frac{x}{|x|}$ for the rescaled PDE (5.28) is

$$v(r,\tau;\mu) = \frac{m}{m-1}\frac{\partial}{\partial r}(\mu^{m-1}) + \beta r + e^{(1-\alpha)\tau}\frac{\tilde{\mathcal{M}}(r,\tau;\mu)}{\sigma_d r^{d-1}}.$$

We first construct a self-similar subsolution $\phi(\lambda,\tau)$, which is a continuous scaling of μ_A with the scaling factor $k(\tau)$ to be determined later:

$$\phi(\lambda,\tau) := k^d(\tau)\mu_A\big(k(\tau)\lambda\big),$$

Since μ_A satisfies (5.31), the inward velocity field of ϕ is then given by

$$v(r,\tau;\phi) = (1-k^{d(m-1)+2})\beta r + e^{(1-\alpha)\tau}\frac{\tilde{\mathcal{M}}(r,\tau;\phi)}{\sigma_d r^{d-1}}.$$

Note that the last term of $v(r,\tau;\phi)$ is always non-negative, thus $v(r,\tau;\phi) \geq (1-k^{d(m-1)+2})\beta r$. That motivates us to choose $k(\tau)$ to be the solution of the following equation

$$k'(\tau) = \beta k(1-k^{d(m-1)+2}), \tag{5.38}$$

with initial data $k(0)$ sufficiently small such that $\phi(\cdot,0) \prec \mu_A$ and $\phi(\cdot,0) \prec \mu(\cdot,0)$.

One can proceed as in the proof of Theorem 5.6 to verify ϕ is indeed a subsolution. Moreover, it can be easily checked that $k(\tau) \to 1$ exponentially, hence $M(r,\tau;\phi)$ converges to $M(r;\mu_A)$ exponentially as $t \to \infty$ and uniformly in r.

In the construction of the supersolution

$$\eta(\lambda,\tau) := k^d(\tau)\mu_A\big(k(\tau)\lambda\big),$$

the main difficulty comes from the aggregation term, which might cause the solution to blow up in finite time. To find an upper bound of the inward velocity field, we first need to control $\tilde{\mathcal{M}}(r,\tau,k^d\mu_A(k\lambda))$:

$$\begin{aligned}
\tilde{\mathcal{M}}(r,\tau;k^d\mu_A(k\lambda)) &= \int_{B(0,r)} k^d\mu_A(k\cdot) * e^{d\beta\tau}\Delta V(e^{\beta\tau}\cdot)(\lambda)d\lambda \\
&\leq \|\Delta V\|_1 \int_{B(0,r)} k^d\mu_A(k\lambda)d\lambda \\
&= \|\Delta V\|_1 \int_{B(0,kr)} \mu_A(\lambda)d\lambda \ \leq C(kr)^d/\sigma_d,
\end{aligned}$$

where the first inequality is due to Riesz's rearrangement inequality and the fact that μ_A is radially decreasing; and C is some constant that does not depend on k, r, τ.

The above inequality gives the following upper bound for the inward velocity field of η:

$$v(r,\tau;\eta) \leq (1-k^{d(m-1)+2})\beta r + Ck^d e^{(1-\alpha)\tau} r.$$

Therefore we let $k(t)$ solve the following ODE

$$k'(\tau) = \beta k(1 - k^{d(m-1)+2}) + Ck^{d+1}e^{(1-\alpha)\tau}, \tag{5.39}$$

and choose the initial data $k(0)$ such that $\eta(\cdot, 0) = k^d(0)\mu_A\big(k(0)\lambda\big) \succ \mu(\cdot, 0)$, then η would be a supersolution to (5.28).

Now it suffices to show the solution to (5.39) exists globally and converges to 1 exponentially. From the assumption that $\mu(\lambda, 0) \prec \delta^d\mu_A(\delta\lambda)$, we may choose $k(0) = \delta$. Due to Lemma 5.12, $k(\tau) \to 1$ exponentially when $k(0) = \delta$ is sufficiently small, hence $M(r, \tau; \eta)$ converges to $M(r; \mu_A)$ exponentially.

Since the supersolution η exists globally, we claim the weak solution μ exists globally as well. Suppose not, then due to Theorem 4 of [BRB], μ has a maximal time interval of existence T^*, and $\lim_{\tau \nearrow T^*} \|\mu(\cdot, \tau)\|_\infty = \infty$. On the other hand, Proposition 5.11 yields that

$$\mu(\cdot, \tau) \prec \eta(\cdot, \tau) \text{ for all } \tau < T^*. \tag{5.40}$$

Note that Proposition 4.2 implies that μ is radially decreasing for all $\tau < T^*$, (Proposition 4.2 is proved for the solution to (1.1), however it works for the solution to (5.28) as well, since (5.28) is derived from a continuous scaling of (1.1)), which gives

$$\|\mu(\cdot, \tau)\|_\infty \le \|\eta(\cdot, \tau)\|_\infty \text{ for all } \tau < T^*. \tag{5.41}$$

The above inequality implies $\lim_{\tau \nearrow T^*} \|\eta(\cdot, \tau)\|_\infty = \infty$, which contradicts the fact that $\|\eta(\cdot, \tau)\|_\infty$ is uniformly bounded for all τ.

Once we have global existence of μ, Proposition 5.11 yields

$$\phi(\cdot, \tau) \prec \mu(\cdot, \tau) \prec \eta(\cdot, \tau) \text{ for all } \tau \ge 0.$$

Since both ϕ and η converges exponentially towards μ_A, we can conclude. $\square$

The following generalization of Theorem 5.13 will be proved in section 6. We mention that the conditions on the initial data given in this section does not restrict to solutions with small mass.

Corollary 5.14. *Let $V(x)$ be given by (A) or (B), and suppose $1 < m < 2 - \frac{2}{d}$. For a nonnegative fucntion μ_0 in $L^1(\mathbb{R}^d)$, define $A := \int \mu_0(\lambda)d\lambda$, and let $\mu_A(\lambda)$ be as given in* (5.31). *Then the followings are true:*

(a) there exists a sufficiently small constant $\delta > 0$ depending on d, m, μ_0 and V, such that if

$$\mu_0^*(\lambda) \prec \delta^d\mu_A(\delta\lambda),$$

then the weak solution $\mu(\lambda, \tau)$ to (5.28) *with initial data μ_0 exists globally.*

(b) If μ_0 is radially symmetric and compactly supported, the mass function $M(r, \tau; \mu)$ defined in (5.1) *converges to $M(r, \tau; \mu_A)$ exponentially as $\tau \to \infty$ and uniformly in r.*

Remark 5.15. *It is shown in [L] that when V is given by (B) and $m < 2 - \frac{2}{d}$, there exists a stationary solution to* (1.1) *for large mass. We suspect that in this case similar techniques as in [BKLN] may yield stability.*

If we rescale back to the original space and time variables, Theorem 5.13 immediately yields the algebraic convergence of mass function for the solution to (1.1).

Corollary 5.16. *Let V, m, μ and μ_0 be as given in Corollary 5.14, and let ρ be given by* (5.27). *Let $\mathcal{U}(x,t)$ denote the Barenblatt function defined in* (5.26). *Then ρ is a weak solution to* (1.1), *and ρ vanishes to zero as $t \to \infty$ with algebraic decay. In particular if ρ_0 is radially symmetric then*

$$|M(r,t) - M(r,t;\mathcal{U})| \leq Ct^{-\gamma}, \text{ for all } r \geq 0,$$

for some C, γ depending on $\rho(x,0), m, d$ and V.

Corollary 5.17. *Let V, m, μ, μ_0, A and ρ be as given in Corollary 5.16. If μ_0 is radially symmetric, then for all $p > 1$ we have*

$$W_p\left(\frac{\rho(\cdot,t)}{A}, \frac{\mathcal{U}(\cdot,t)}{A}\right) \leq Ct^{-\gamma},$$

where C, γ depend on $\rho(x,0), m, d$ and V.

6 A comparison principle for general solutions and Instant regularization in L^∞

In this last section we consider general (non-radial) solutions of (1.1). Our goal is to prove the following result:

Theorem 6.1. *Suppose $m > 1$. Let V be given by (A) or (B), and let ρ be the weak solution to* (1.1) *with initial data $\rho(x,0) = \rho_0(x)$. Let $\bar{\rho}$ be the weak solution to* (1.1) *with initial data $\bar{\rho}(x,0) = \rho_0^*(x)$. Assume $\bar{\rho}$ exists for $t \in [0,T)$, where T can be either finite or ∞. Then $\rho^*(\cdot,t) \prec \bar{\rho}(\cdot,t)$ for all $t \in [0,T)$.*

As an application of Theorem 6.1, we will show that solutions of (1.1) with its initial data in L^1 immediately regularizes in L^∞ (see Proposition 6.6.)

The proof of Theorem 6.1, which we divide into several subsections follows that of the corresponding theorem for solutions of (1.2) (see Chapter 10 of [V]). The theorem in [V] is proved by taking the semi-group approach and applying the Crandall-Liggett Theorem. The challenge lies in the fact that our operator in (1.1) is not a contraction, in either L^1 or L^∞. For this reason the proof requires an additional approximation of our equation with the one with fixed drift: see (6.5).

6.1 Implicit Time Discretization for PME with drift

Consider the following equation

$$\rho_t = \Delta\rho^m + \nabla \cdot (\rho\nabla\Phi), \tag{6.1}$$

where Φ is a priori given function such that $\Phi(x,t) \in C(\mathbb{R}^d \times [0,\infty))$, and $\Phi(\cdot,t) \in C^2(\mathbb{R}^d)$ for all t.

Following the proof in the case of (1.2) in [V], let us approximate (6.1) by an implicit time discrete scheme. For a small constant $h > 0$, U_i is recursively defined as the solution of the following elliptic equation:

$$\frac{U_i - U_{i-1}}{h} = \Delta U_i^m + \nabla \cdot (U_i \nabla \Phi_i), \quad i = 1, 2, \dots \tag{6.2}$$

where $U_0 = u(\cdot, 0), \Phi_i = \Phi(\cdot, ih)$. Now define

$$\rho_h(t) := U_i \quad \text{for } (i-1)h < t \leq ih \text{ where } i = 1, 2, .. \tag{6.3}$$

The following result, whose proof is given in the Appendix, states that our approximation scheme is valid.

Proposition 6.2. *Let $u_0 \in L^1(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$, and let ρ_h be defined by (6.3).Then we have*

$$\rho(x,t) := \lim_{h \to 0} \rho_h(x,t), \tag{6.4}$$

where the convergence is in $L^1(\mathbb{R}^d)$ in x-variable and uniform on $t \in [0,T]$ for all $T > 0$. Moreover, ρ coincides with the unique weak solution for (6.1).

6.2 Rearrangement comparison

For a given function $u(x) : \mathbb{R}^d \to \mathbb{R}$, let us define u^* as given in (1.6).

Consider the following equation, where $f(x,t) \in C([0,\infty); L^1(\mathbb{R}^d))$ be a given function:

$$\rho_t = \Delta \rho^m + \nabla \cdot (\rho \nabla (f * V)), \tag{6.5}$$

Let us first prove the rearrangement result for above equation.

Theorem 6.3. *Suppose $m > 1$. Let V be given by (A) or (B), and let ρ be the weak solution to (6.5) with initial data $\rho(x,0) = \rho_0(x)$. Let $\bar{\rho}$ be the weak solution to the symmetrized problem*

$$\rho_t = \Delta \rho^m + \nabla \cdot (\rho \nabla (f^* * V)), \tag{6.6}$$

with initial data $\bar{\rho}(x,0) = \rho_0^*(x)$. Then we have $\bar{\rho}$ is radially decreasing, and

$$\rho^*(\cdot,t) \prec \bar{\rho}(\cdot,t) \text{ for all } t > 0.$$

Due to Proposition 6.2, it suffices to show the following Proposition, whose proof will be given in the appendix.

Proposition 6.4. *Suppose $m > 1$, and let V be given by (B). Let $u \in D$ (the domain D is defined in (A.1)) be the weak solution of*

$$-h\Delta u^m - h\nabla \cdot (u \nabla (f * V)) + u = g, \tag{6.7}$$

where $f, g \in L^1(\mathbb{R}^d)$ is nonnegative. And let $\bar{u} \in D$ be the solution to the symmetrized problem, i.e. $\bar{u}$ solves (6.7) with f, g replaced by $\bar{f}$ and $\bar{g}$ respectively, where $\bar{f}$ and $\bar{g}$ are radially decreasing, have same mass as f and g respectively, and satisfy $f^* \prec \bar{f}$ and $g^* \prec \bar{g}$. Then $u^* \prec \bar{u}$.

Proof of Theorem 6.3:

We can prove $\bar{\rho}$ is radially decreasing by using a similar argument as in Theorem 4.2. In fact the argument is easier here since we already know $f^* * \Delta V$ is a radially decreasing function.

Now we prove $\rho^* \prec \bar{\rho}$ for all $t \geq 0$ when V is given by (B). Let U_i be the discrete solution for the original problem, and let V_i be the discrete solution for the symmetrized problem. Due to Proposition 6.2 it suffices to prove that $U_i^* \prec V_i$ for all $i \in \mathbb{N}$. Here U_i solves

$$\frac{U_i - U_{i-1}}{h} = \Delta U_i^m + \nabla \cdot (U_i \nabla (f_i * V)), \tag{6.8}$$

where $U_0 = u(\cdot, 0), f_i = f(\cdot, ih)$, and V_i solves

$$\frac{V_i - V_{i-1}}{h} = \Delta V_i^m + \nabla \cdot (V_i \nabla (f_i^* * V)), \tag{6.9}$$

where $V_0 = u^*(\cdot, 0)$.

Since $U_0^* \prec V_0$, by applying Prop 6.4 inductively, we can conclude.

When $V = \mathcal{N}$, we can use mollified Newtonian kernel to approximate $\mathcal{N}$, and the result follows via Proposition 3.3. $\square$

Now we are ready to prove our main result:

Proof for Theorem 6.1:

Let us define a sequence $\{\rho_i\}_{i \in \mathbb{N}}$ as follows:

Let us first prove the theorem when V is given by (B), where we have global existence of solutions. Let $\rho_1(\cdot, t) := \rho^*(\cdot, t)$ for all $t \geq 0$, where $\rho(x, t)$ is the weak solution of (1.1) with initial data $\rho(x, 0) = \rho_0(x)$. For $i > 1$, we let ρ_i be the weak solution to the following equation:

$$(\rho_i)_t = \Delta(\rho_i)^m + \nabla \cdot (\rho_i \cdot \nabla(\rho_{i-1} * V)), \tag{6.10}$$

with initial data $\rho_i(x, 0) = \rho^*(x, 0)$. Observe that $\rho_i(\cdot, t)$ is radially decreasing for all $i \in \mathbb{N}^+, t \geq 0$.

By Theorem 6.3, we have $\rho_i \prec \rho_{i+1}$ for all $i \in \mathbb{N}$. Hence we have

$$\rho^*(\cdot, t) = \rho_1(\cdot, t) \prec \rho_2(\cdot, t) \prec \rho_3(\cdot, t) \prec \ldots, \quad \text{for all } t. \tag{6.11}$$

Due to Theorem 3.1, $\{\rho_i\}$ is locally uniformly continuous in space and time. Hence by Arzela-Ascoli Theorem any subsequence of $\{\rho_i\}$ locally uniformly converges to a function $\bar{\rho}$ along a subsequence. On the other hand $\bar{\rho}$ is the unique weak solution for (1.1) with initial data $\bar{\rho}(x, 0) = \rho_0^*(x)$. This means that the whole sequence $\{\rho_i\}$ locally uniformly converges to $\bar{\rho}$. Now we can conclude due to (6.11).

When $V = \mathcal{N}$, we can use mollified Newtonian kernel to approximate $\mathcal{N}$, and the result follows via Proposition 3.3. $\square$

Corollary 6.5. *Suppose $m > 1$. Let V be given by (A) or (B), and let ρ be the weak solution of* (1.1) *with initial data $\rho_0(x)$. Let $\bar{\rho}$ be the solution to the symmetrized problem, i.e. $\bar{\rho}$ is the weak solution to* (1.1) *with initial data $\rho_0^*(x)$. Assume $\bar{\rho}$ exists for $t \in [0, T)$, where T can be either finite or ∞. Then for any $p \in (1, \infty]$ we have*

$$\|\rho(\cdot, t)\|_{L^p(\mathbb{R}^d)} < \|\bar{\rho}(\cdot, t)\|_{L^p(\mathbb{R}^d)}, \quad \textit{for all } t \in [0, T).$$

We are now ready to generalize Theorem 5.13.

Proof of Corollary 5.14: Let $\bar{\mu}(\lambda,\tau)$ be the weak solution to (5.28) with initial data $\mu_0^*(\lambda)$. Then $\bar{\mu}(\cdot,0)$ meets the assumptions for Theorem 5.13, which implies the global existence of $\bar{\mu}$. Due to Corollary 6.5, $\|\mu(\cdot,\tau)\|_\infty \leq \|\bar{\mu}(\cdot,\tau)\|$ for all τ during the existence of μ; hence the uniform boundedness of $\bar{\mu}$ yields that μ cannot blow up and thus must exist globally.

When μ is radially symmetric and compactly supported, once we obtain global existence of μ, we can construct subsolution and supersolution as in the proof for Theorem 5.13 and conclude. □

6.3 Instant regularization in L^∞

Lastly, we present the following regularization result as a corollary of Theorem 6.1.

Proposition 6.6. *Let V be given by (A) or (B), and let $m > 2 - 2/d$. Let $\rho(x,t)$ be the weak solution for* (1.1)*, with initial data $\rho_0 \in L^1(\mathbb{R}^d;(1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$. Then for every $t > 0$ we have $\rho(\cdot,t) \in L^\infty(\mathbb{R}^d)$ with*

$$\|\rho(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq c(m,d,A,V)t^{-\alpha} \text{ for all } t < 1$$

where $A = \|\rho_0\|_1$ and $\alpha := \frac{d}{d(m-1)+2}$.

Proof. By Corollary 6.5, it suffices to prove the inequality when ρ_0 is radially symmetric. Also, in this proof we denote $c(m,d,A,V)$ by all constants which only depends on m,d,A,V.

Let ρ_A to be the radial stationary solution of (1.1) with mass A. Note that ρ_A is radially decreasing, and thus $\rho_A(0) > 0$. Since u_0 is a radial function in L^∞, we can scale ρ_A to make it more concentrated than u_0, i.e. we choose $0 < a < 1$ to be sufficiently small, such that

$$u_0 \prec a^{-d}\rho_A(a^{-1}x).$$

As in the proof of Theorem 5.6, let us define

$$\eta(r,t) := k^d(t)\ \rho_A\big(k(t)r\big),$$

where $k(t)$ solves the following ODE with initial data $k(0) = a^{-1}$:

$$k'(t) = c(m,d,A,V)k^{d+1}(1 - k^{d(m-2+2/d)}),$$

where $c(m,d,A,V)$ corresponds to C_2 in the proof for Theorem 5.6. It is shown in the proof that

$$\rho(\cdot,t) \prec \eta(\cdot,t) \quad \text{for all } t \geq 0,$$

which in particular yields that

$$\|\eta(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \geq \|\rho(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \quad \text{for all } t \geq 0.$$

Observe that, by definition,

$$h(t) := \|\eta(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} = k^d(t)\rho_A(0) = c(m,d,A,V)k^d(t).$$

Therefore to prove our proposition it is enough to show

$$h(t) \leq c(m,d,A,V)t^{-\alpha} \quad \text{for all } h(0) > 0,$$

which becomes an ODE problem: $h(t)$ solves

$$\begin{aligned} h'(t) &= c(m,d,A,V)k^{d-1}k' \\ &= c(m,d,A,V)h^2\big(1 - h^{m-2+2/d}\big). \end{aligned}$$

Hence when $h \geq 2$, $h(t)$ satisfies the following inequality

$$h'(t) \leq -c(m,d,A,V)h^{m+2/d}.$$

If we replace $h(t)$ with $c(m,d,A,V)t^{-\alpha}$ with α as given above, then it will achieve equality. Therefore $h(t) \leq \eta(t)$ for $t \geq 0$ and we can conclude. □

A Appendix

A.1 Proof for Proposition 6.2

The proof for Proposition 6.2, which is an application of Crandall-Liggett Theorem ([CL], also see Theorem 10.16 in [V]), are based on the following two lemmas.

Let us consider the following domain:

$$D := \Big\{ u \in L^1(\mathbb{R}^d) : u^m \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d), \Delta u^m \in L^1(\mathbb{R}^d), |\nabla u^m| \in M^{d/(d-1)}(\mathbb{R}^d) \Big\}. \tag{A.1}$$

Here, the Marcinkiewicz space $M^p(\mathbb{R}^d), 1 < p < \infty$, is defined as set of $f \in L^1_{loc}(\mathbb{R}^d)$ such that

$$\int_K |f(x)|dx \leq C|K|^{(p-1)/p},$$

for all subsets K of finite measure. The minimal C in the above inequality gives a norm in this space, i.e.

$$\|f\|_{M^p(\mathbb{R}^d)} = \sup \Big\{ \mathrm{meas}(K)^{-(p-1)/p} \int_K |f|dx : K \subset \mathbb{R}^d, \mathrm{meas}(K) > 0 \Big\}.$$

A parallel argument as in Theorem 2.1 of [BBC] yields existence for the discretized equation.

Lemma A.1 (existence)**.** *Let $d \geq 3$ and let $u_0 \in L^1(\mathbb{R}^d), \Phi \in C^2(\mathbb{R}^d)$. then there exists a unique weak solution $u \in D$ of the following equation:*

$$\frac{u - u_0}{h} = \Delta u^m + \nabla \cdot (u \nabla \Phi). \tag{A.2}$$

Next we state the L^1-contraction result. The proof is parallel to that of Prop 3.5 in [V] for (1.2).

Lemma A.2 (L^1 contraction)**.** *Let $\Phi \in C^2(\mathbb{R}^d)$, $u_{0i} \in L^1(\mathbb{R}^d)$ and let $u_1, u_2 \in D$ be the weak solution to the degenerate elliptic equation*

$$\frac{u_i - u_{0i}}{h} = \Delta(u_i)^m + \nabla \cdot (u_i \nabla \Phi), \quad i = 1, 2 \tag{A.3}$$

Then u_i satisfies

$$\|u_1 - u_2\|_{L^1(\mathbb{R}^d)} \leq \|u_{01} - u_{02}\|_{L^1(\mathbb{R}^d)}. \tag{A.4}$$

Proof for Proposition 6.2

Proof. Let us define the nonlinear operator $\mathcal{A} : D \to L^1(\mathbb{R}^d)$ by the formula

$$\mathcal{A}(u) = -\Delta u^m - \nabla(u \nabla \Phi),$$

in the domain D defined above.

Then Lemma A.1 and Lemma A.2 yields that for any $h > 0$, there is a unique solution u in D for the equation

$$h\mathcal{A}(u) + u = f,$$

and the map $f \mapsto u$ is a contraction in $L^1(\mathbb{R}^d)$. Now arguing as in [V], the Crandall-Liggett Theorem yields the conclusion. □

A.2 Proof for Proposition 6.4

The proof for Proposition 6.4 is parallel to that of Theorem 11.7 in [V] for (1.2). First we state a lemma which deals with the extra convolution term.

Lemma A.3. *Let V be given by (B). Let $f \in L^1(\mathbb{R}^d)$ and $\phi \in W_0^{1,\infty}(\mathbb{R}^d)$ be non-negative functions. Then for any non-negative number a, b, we have*

$$\int_{\{a<\phi<b\}} \nabla(f * (-V)) \cdot \nabla\phi \leq \int_{\{\phi^*>a\}} (f^* * \Delta V)(\max\{\phi^*, b\} - a), \tag{A.5}$$

where the equality is achieved if f, ϕ are both radially decreasing.

Proof. Let

$$\eta(x) := \begin{cases} b & \text{if } \phi(x) \geq b, \\ \phi(x) - a & \text{if } a < \phi(x) < b, \\ 0 & \text{if } \phi(x) \leq a. \end{cases}$$

Then $\eta(x) \in W_0^{1,\infty}(\mathbb{R}^d)$, $\nabla\phi = \nabla\eta$ in $\{a < \phi(x) < b\}$, and $\nabla\eta = 0$ in $\mathbb{R}^d \backslash \{a < \phi(x) < b\}$. Therefore

$$\begin{aligned} \text{LHS of (A.5)} \quad &= \int_{\mathbb{R}^d} \nabla(f * (-V)) \cdot \nabla\eta \\ &\leq \int_{\mathbb{R}^d} (f^* * \Delta V)\eta^* = \int_{\{\phi^*>a\}} (f^* * \Delta V)(\max\{\phi^*, b\} - a), \end{aligned}$$

where the inequality comes from Riesz's rearrangement inqeuality. Note that it would be an equality if $f = f^*$ and $\eta = \eta^*$, hence the lemma is proved. □

The following lemma corresponds to the Theorem 17.5 in [V].

Lemma A.4. *Let V be given by (B). Let $f, \bar{f}$ and g be non-negative radially decreasing functions in $L^1(\mathbb{R}^d)$, where $f \prec \bar{f}$.*

Let $h > 0$, and let $v_1, v_2 \in D$ be two non-negative radial decreasing functions. Assume v_1 satisfies

$$-h\Delta(v_1)^m - h\nabla \cdot (v_1 \nabla(f * V)) + v_1 \prec g, \tag{A.6}$$

and v_2 solves

$$-h\Delta(v_2)^m - h\nabla \cdot (v_2 \nabla(\bar{f} * V)) + v_2 = g \tag{A.7}$$

Then we have $v_1 \prec v_2$.

Proof. Let $u_i := v_i^m$ and define $u := u_1 - u_2$, $v := v_1 - v_2$, $A(r) := \int_{B(0,r)} v(x)dx$. Our goal is to show $A(r) \leq 0$ for all $r \geq 0$.

For all $r \geq 0$, subtracting (A.6) from (A.7) and integrating it in $B(0,r)$ yields

$$\int_{B(0,r)} -h\Delta u dx - h\Big(v_1(r)\int_{B(0,r)} f * \Delta V dx - v_2(r)\int_{B(0,r)} \bar{f} * \Delta V dx\Big) + A(r) \leq 0, \tag{A.8}$$

which can be written as

$$-hc_d r^{d-1} u'(r) - hv(r)\int_{B(0,r)} f * \Delta V dx - hv_2(r)\int_{B(0,r)} (f - \bar{f}) * \Delta V dx + A(r) \leq 0. \tag{A.9}$$

(Here the existence of $u'(r)$ is guaranteed by the assumption that $v_i \in D$ for $i = 1, 2$, which implies that Δu is in L^1.) Due to our assumption $f \prec \bar{f}$ and V given by (B), we have $\int_{B(0,r)}((f-\bar{f})*\Delta V)dx \leq 0$ for all $r \geq 0$. Therefore

$$-hc_d r^{d-1} u'(r) - hv(r)\int_{B(0,r)} f * \Delta V + A(r) \leq 0 \quad \text{for all } r \geq 0. \tag{A.10}$$

Note that since u_i and v_i both vanishes at infinity, from (A.10) it follows that $\lim_{r\to\infty} A(r) \leq 0$. Hence suppose $A(r)$ is positive somewhere, it must achieve its positive maximum at some $r_0 > 0$. At $r = r_0$ we have $v(r_0) = A'(r_0) = 0$, and (A.10) becomes

$$u'(r_0) \geq \frac{A(r_0)}{hc_d r^{d-1}} > 0,$$

which means $u_2 - u_1$ is strictly increasing at r_0: hence $v_2 - v_1$ will also be strictly positive in $(r_0, r_0 + \epsilon)$ for some small ϵ, which implies $A(r_0 + \epsilon) > A(r_0)$, contradicting our assumption that $A(r)$ achieves its maximum at r_0. Therefore $A(r)$ must be ≤ 0 for all r, which means $v_2 \prec v_1$. □

Proof of Proposition 6.4: The proof is parallel to that of Theorem 11.7 as in [V]. For any test function $\phi \in W_0^{1,\infty}(\mathbb{R}^d)$, we have

$$h\int_{\mathbb{R}^d} \nabla u^m \cdot \nabla\phi + h\int_{\mathbb{R}^d} u\nabla(f * V) \cdot \nabla\phi + \int_{\mathbb{R}^d} u\phi = \int_{\mathbb{R}^d} g\phi, \tag{A.11}$$

where $\phi \in W_0^{1,\infty}(\mathbb{R}^d)$ is any test function. Now let us plug in $\phi(x) := (u^m(x)-t)_+$ for any real number $t>0$, and differentiate the equation with respect to t. Then we have:

$$\underbrace{-h(\frac{d}{dt}\int_{\{u^m>t\}}|\nabla u^m|^2)}_{I_1}\underbrace{-h(\frac{d}{dt}\int_{\{u^m>t\}}\frac{m}{m+1}\nabla(f*V)\cdot\nabla(u^{m+1}))}_{I_2}+\underbrace{\int_{\{u^m>t\}}u}_{I_3}=\underbrace{\int_{\{u^m>t\}}g}_{I_4} \tag{A.12}$$

Following the proof of Theorem 17.7 in [V], one can check that

$$\begin{aligned} I_1 &\le \int_{\{(u^*)^m>t\}} h\Delta((u^*)^m), \quad \text{(with equality if } u\equiv u^*) \\ I_3 &= \int_{\{(u^*)^m>t\}} u^*, \\ I_4 &\le \sup_{|\Omega|=|\{u^m>t\}|}\int_\Omega g^* = \int_{\{(u^*)^m>t\}} g^*. \end{aligned}$$

It remains to examine I_2. Using Lemma A.3, it follows that

$$\begin{aligned} I_2 &= h\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\{t<u^m<t+\epsilon\}}\frac{m}{m+1}\nabla(f*(-V))\cdot\nabla(u^{m+1}) \\ &\le h\liminf_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\{t<(u^*)^m<t+\epsilon\}}\frac{m}{m+1}(f^**\Delta V)(\max\{u^{m+1},(t+\epsilon)^{1+\frac{1}{m}}\}-t^{1+\frac{1}{m}})_+ \\ &= ht^{\frac{1}{m}}\int_{\{(u^*)^m>t\}} f^**\Delta V \end{aligned}$$

Plug in the four inequalities into (A.12), the following inequality holds for all $t\ge 0$:

$$-\int_{\{(u^*)^m>t\}} h\Delta((u^*)^m) - ht^{\frac{1}{m}}\int_{\{(u^*)^m>t\}} f^**\Delta V + \int_{\{(u^*)^m>t\}} u^* \le \int_{\{(u^*)^m>t\}} g^*, \tag{A.13}$$

since $t\ge 0$ is arbitrary, the above inequality implies

$$-h\Delta((u^*)^m) - h\nabla\cdot(u^*\nabla(f^**V)) + u^* \prec g^*. \tag{A.14}$$

On the other hand, by assumption, $\bar{u}$ solves

$$-h\Delta(\bar{u}^m) - h\nabla\cdot(\bar{u}\nabla(\bar{f}*V)) + \bar{u} = \bar{g}, \tag{A.15}$$

where $\bar{f}\succ f^*$ and $\bar{g}\succ g^*$. Note that $u\in D$ implies $u^*\in D$, hence we can apply Lemma A.4 and get $u^*\prec\bar{u}$. □

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