Locating the first nodal set in higher dimensions

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Abstract

We extend the two dimensional results of Jerison [J1] on the location of the nodal set of the first Neumann eigenfunction of a convex domain to higher dimensions. If a convex domain Ω in $\mathbb{R}^n$ is contained in a long and thin cylinder $[0, N] \times B_\epsilon(0)$ with nonempty intersections with $\{x_1 = 0\}$ and $\{x_1 = N\}$, then the first nonzero eigenvalue is well approximated by the eigenvalue of an ordinary differential equation, by a bound proportional to ϵ, whose coefficients are expressed in terms of the volume of the cross sections of the domain. Also, the first nodal set is located within a distance comparable to ϵ near the zero of the corresponding ordinary differential equation.

1 Introduction

Let Ω be a bounded convex domain in $\mathbb{R}^n$. Let u be an eigenfunction for Ω associated with the smallest nonzero eigenvalue λ of the Neumann problem for Ω, that is,

$$\Delta u = -\lambda u \text{ in } \Omega, \;\; u_\nu = 0 \text{ on } \partial\Omega \tag{1.1}$$

where $u_\nu = \nu\cdot\nabla u$ and ν denotes the outer normal unit vector at each point on $\partial\Omega$. The purpose of this paper is to locate the first nodal set $\Lambda = \{u = 0\}$ and to estimate the first nonzero eigenvalue λ. We show that Λ is near the unique zero of the first nonconstant eigenfunction of a certain ordinary differential equation, and we estimate the difference between λ and the first nonzero eigenvalue of the corresponding ordinary differential equation.

Assume

$$\Omega \subset \{(x, y_1, ..., y_{n-1}) : x \in [0, N], (y_1, ..., y_{n-1}) \in B_\epsilon(0)\} \subset \mathbb{R}^n \tag{1.2}$$

and suppose further that

$$\Omega(s) := \Omega \cap \{x = s\} \text{ is nonempty for } 0 < s < N. \tag{1.3}$$

Let u be the first nonconstant eigenfunction for Ω. Denote by ϕ_1 the first nonconstant eigenfunction with the smallest nonzero eigenvalue μ_1 for the Neumann problem on $[0, N]$ given by

$$-(w\phi_1')' = \mu_1 w\phi_1 \text{ on } (0, N)\,;\; w(s)\phi_1'(s) \to 0 \text{ as } s \to 0^+ \text{ or } s \to N^- \quad (1.4)$$

where $w(s)$ is the $(n-1)$-dimensional volume of $\Omega(s)$. Let $s_1 \in (0, N)$ be the unique zero of ϕ_1, i.e., $\phi_1(s_1) = 0$. The main results of this paper are as follows.

Theorem 1.1. *If u is the first nonconstant Neumann eigenfunction of Ω and Ω satisfies (1.2) and (1.3), then there is a dimensional constant C such that*

(a) $u(x, y_1, ..., y_{n-1}) = 0$ *implies* $|x - s_1| < C\epsilon$

(b) $(1 - C\epsilon/N)\mu_1 \leq \lambda \leq \mu_1$.

where s_1 and μ_1 are given in (1.4).

Theorem 1.1 was proven by Jerison ([J1]) for $n = 2$. By taking a new coordinate system, he bounds the first eigenvalue λ from below by a formula, whose coefficients are expressed in terms of the width of the cross section $\Omega(x) = \{y : (x, y) \in \Omega\}$. (Here y-axis is chosen so that the projection of Ω onto y-axis has the shortest possible length.) Using ODE eigenvalue estimates, he first locate the nodal set, and then using the location of the first nodal set, he estimates the first eigenvalue. However for $n > 2$, a parallel approach leads to a weaker result mainly due to the fact that we do not have a proper coordinate system $(s, t_1, ..., t_{n-1}) \to (x, y_1, ..., y_{n-1})$ satisfying

$$\int_0^N |\partial_s y_j| ds \leq C\epsilon, \quad j = 1, ..., n-1. \qquad (1.5)$$

(see the Remarks at the end of this section.)

In this paper we extend results in [J1] to higher dimensions, with a different approach to the problem. We first estimate the difference between the eigenvalues of the original PDE and the corresponding ODE, by taking an one-dimensional test function and using a sharp result of Kröger on the upper bound of a gradient of a Neumann eigenfunction, and also using a new coordinate system which will be constructed in section 4. (we will only need a weaker pointwise estimate (see (II′) in section 4) than (1.5).) Based on these estimates, it turns out that we can find a bound on the width of the first nodal set. Once we prove that the nodal set is thin, i.e., the diameter of the nodal set is comparable to ϵ, then we are able to locate the nodal set near the zero of the eigenfunction of the corresponding ODE.

First, we derive the second inequality of (b). Recall that u minimizes the Dirichlet integral

$$J(v) = \int_\Omega |\nabla v|^2 \tag{1.6}$$

among all functions v on Ω satisfying

$$\int_\Omega v^2 = 1, \int_\Omega v = 0. \tag{1.7}$$

The minimum value of J is the eigenvalue λ. If we consider functions of x alone, i.e., $v(x, y_1, ..., y_{n-1}) = \phi(x)$, then

$$J(v) = I(\phi) := \int_0^N \phi'(s)^2 w(s) ds \tag{1.8}$$

and the constraints (1.7) become

$$\int_0^N \phi(s)^2 w(s) ds = 1, \int_0^N \phi(s) w(s) ds = 0. \tag{1.9}$$

As in [J1] we observe that the minimizer of (1.8) under the constraints (1.9) is the first nonzero eigenfunction ϕ_1 given in (1.4) and $I(\phi_1) = \mu_1$. Hence

$$\lambda \leq \mu_1. \tag{1.10}$$

Remark 1. If we normalize $N = 1$, then by [L] and [ZY], $C_1 \leq \lambda$ for some absolute constant $C_1 > 0$, and by plugging in the test function $\phi(x) = \sin \pi x_1$, we get $\lambda \leq C_2$ for some dimensional constant C_2.

Remark 2. In the case of Dirichlet problem on a planar convex domain, Jerison [J2] obtained results corresponding to Theorem 1.1. Later Grieser and Jerison ([GJ1], [GJ2]) showed that the nodal line is in an x-interval of much shorter length $C\epsilon/N$ (possibly at distance $C\epsilon$ from s_1). We expect that there is an analogous bound in the Neumann problem.

Remark 3. An analogous approach to the method in [J1] for higher dimensions, by modifying the methods in [J1]-[J2], leads to a result weaker than Theorem 1.1, i.e., with $\epsilon \log(N/\epsilon)$ instead of ϵ.

2 Preliminary results and corollaries

Throughout the paper we normalize Ω such that $N = 1$. As mentioned in the introduction, a key ingredient in the proof of Theorem 1.1 is Kröger's comparison theorems on the gradient of eigenfunction u, which we state below.

Theorem 2.1 (Kröger, [K1], [BQ])**.** *Let Ω be a convex domain in $I\!R^n$ with smooth boundary. Let u be the first eigenfunction for the Laplace operator on Ω with the associated eigenvalue $\lambda > 0$, under Neumann boundary conditions. Let v be a solution on some interval (a, b) of the differential equation*

$$v''(x) + \frac{n-1}{x}v'(x) = -\lambda v(x) \text{ on } (a, b) \;\; v'(a) = v'(b) = 0 \tag{2.1}$$

such that $v' \neq 0$ on (a, b) and $[\min u, \max u] \subset [\min v, \max v]$. Then

$$|\nabla(v^{-1} \circ u)| \leq 1.$$

Theorem 2.2 (Kröger, [K2])**.** *Let Ω, u and λ be given as in Theorem 2.1. Suppose $\max u \geq -\min u$. Let b be a positive number such that λ is the first nonzero eigenvalue of*

$$\psi''(x) + \frac{n-1}{x}\psi'(x) = -\lambda\psi(x) \text{ on } [0, b], \;\; \psi'(0) = \psi'(b) = 0. \tag{2.2}$$

If ψ is the corresponding eigenfunction with $\psi(0) > 0$, then

$$\frac{\max u}{-\min u} \leq \frac{\max \psi}{-\min \psi}.$$

Corollary 2.3. *Let Ω and u be given as in Theorem 2.1. If $N = 1$ and $\max |u| = 1$, then $|\nabla u| \leq C$ for some dimensional constant C.*

Proof. First, we claim that if $0 \leq a < b$ then the solution v to (2.1) satisfies

$$\max |v'| \leq \sqrt{\lambda}|v(a)|. \tag{2.3}$$

To see this, multiply v' to both sides of (2.1) to obtain

$$v'v'' + \lambda v'v = \frac{1}{2}((v')^2 + \lambda v^2)' = -\frac{n-1}{x}(v')^2 \leq 0 \tag{2.4}$$

for $x > 0$. Hence,

$$(v')^2(x) + \lambda v^2(x) \leq (v')^2(a) + \lambda v^2(a) = \lambda v^2(a)$$

and the claim follows.

Now, for sufficiently large $M > 0$, consider an interval $(a, b) \subset (M, \infty)$ such that λ is the first nonzero eigenvalue of the Neumann problem (2.1). Let v be the corresponding eigenfunction. If M is large enough, then v is close to a constant multiple of $\cos \sqrt{\lambda} x$, and thus we can normalize v so that $1 \leq -\min v \leq 2$ and $1 \leq \max v \leq 2$.

Then by Theorem 2.1, we get

$$\sup |\nabla u| \leq \sup |v'| \leq C$$

where the second inequality follows from Remark 1 and the above claim.

□

Corollary 2.4. *Let Ω and u be given as in Theorem 2.1. Suppose that $N = 1$ and $1 = \max u \geq -\min u$. Let $k(\epsilon)$ be the smallest integer such that $2^{-k(\epsilon)} \leq \epsilon$. For integers $1 \leq k < k(\epsilon)$, let*

$$I_k = [2^{-k-1}, 2^{-k}] \cup [1 - 2^{-k}, 1 - 2^{-k-1}]$$

and let $I_{k(\epsilon)} = [0, 2^{-k(\epsilon)}] \cup [1 - 2^{-k(\epsilon)}, 1]$. Then there exists a dimensional constant $C > 0$ such that

$$\sup_{x \in I_k} |\nabla u| \leq C 2^{-k} \text{ for } 1 \leq k \leq k(\epsilon). \tag{2.5}$$

Proof. Suppose $u < 0$ on $\Omega(0)$ and $u > 0$ on $\Omega(1)$. Denote $m = -\min u$. We claim that

$$\max_{x \in \Omega(0)} u \leq -m + 3M_0 \epsilon. \tag{2.6}$$

where M_0 is the upper bound for $|\nabla u|$. To see this, let

$$s_0 = \min\{x : \min_{\Omega(x)} u = -m\}.$$

By Corollary 2.3,

$$|\nabla u| \leq M_0 \tag{2.7}$$

with a dimensional constant M_0, and thus $\max_{\Omega(s_0)} u \leq -m + 2M_0\epsilon$. Assume that $\max_{\Omega(0)} u > -m + 3M_0\epsilon$, then $\{u > -m + 2M_0\epsilon\}$ has a component A such that $|A| > 0$ and $A \subset \{x \leq s_0\}$. Let

$$\tilde{u} = \begin{cases} u & \text{for} \quad x \in \Omega - A \\ -m + 2M_0\epsilon & \text{for} \quad x \in A \end{cases}$$

then

$$\frac{\int_{\Omega_-} |\nabla u|^2}{\int_{\Omega_-} u^2} > \frac{\int_{\Omega_-} |\nabla \tilde{u}|^2}{\int_{\Omega_-} \tilde{u}^2}$$

and we get a contradiction. A parallel argument yields

$$\min_{x\in\Omega(1)} u \geq 1 - 3M_0\epsilon. \tag{2.8}$$

Now we show (2.5) using Theorems 2.1 and 2.2. First, consider the case $k = k(\epsilon)$, on the interval $[0, 2^{-k(\epsilon)}] \subset [0, \epsilon]$. Recall that $m := -\min u \leq \max u = 1$ and $u \leq -m + 3M_0\epsilon$ on $\Omega(0)$. Let ψ be the first eigenfunction of (2.2) with $\psi' \neq 0$ on $(0, b)$, $\psi(0) > 0$ and $-\min\psi = m$. Observe that ψ is decreasing and thus $\psi(b) = -m$.

By Theorem 2.2, $\max\psi \geq \max u$ and thus

$$[\min u, \max u] \subset [\min\psi, \max\psi].$$

Also, since ψ satisfies (2.4) with v replaced by ψ,

$$\begin{aligned}
\psi'(x)^2 &= \int_x^b \frac{2(n-1)}{t}\psi'(t)^2 + \lambda(\psi(t)^2)' dt \\
&= \int_x^b \frac{2(n-1)}{t}\psi'(t)^2 dt + \lambda(\psi(b)^2 - \psi(x)^2)
\end{aligned}$$

Hence for $x \in J := \{\psi \leq -m + C\epsilon\}$

$$\psi'(x)^2 \leq \int_x^b \frac{2(n-1)}{t}\psi'(t)^2 dt + 2Cm\lambda\epsilon$$

Assume that $\psi'(x)^2 = M\epsilon$ for the first time in $[x, b] \subset J$. Then by above inequality

$$M\epsilon \leq M\epsilon \cdot 2(n-1)\log\frac{b}{x} + 2Cm\lambda\epsilon,$$

which yields $\log\frac{b}{x} \geq \frac{1 - M^{-1}C(2m\lambda)}{2(n-1)}$. Therefore if we let $M := 4Cm\lambda$, then we obtain $x \leq e^{-1/4(n-1)}b$.

Choose a sufficiently small dimensional constant $c_0 > 0$ such that if $\epsilon \leq c_0$ then

$$J = \{\psi \leq -m + C\epsilon\} \subset [e^{-1/4(n-1)}b, b].$$

It follows that if $\epsilon \leq c_0$, then

$$|\psi'| \leq C_1\sqrt{\epsilon},\ C_1 = \sqrt{4Cm\lambda} \text{ on } J.$$

By Theorem 2.1 and above argument with $C = 3M_0$, we obtain that if $\epsilon \leq c_0$ then

$$|Du| \leq M_1\sqrt{\epsilon} \text{ in } \{u \leq -m + 3M_0\epsilon\} \tag{2.9}$$

where $M_1 = 2\sqrt{3M_0 m\lambda}$.

Using the improved gradient bound (2.9) instead of (2.7), (2.6) improves to

$$u \leq -m + 3M_1\epsilon^{3/2} \text{ on } \{0 \leq x \leq \epsilon\}. \tag{2.10}$$

Next we repeat the argument with the improved bound (2.10), i.e., with $\{\psi \leq -m + C\epsilon\}$ replaced by $\{\psi \leq -m + 3M_1\epsilon^{3/2}\}$. It follows that for $\epsilon \leq c_0$

$$|Du| \leq M_2\epsilon^{1/2+1/4}, \;\; u \leq -m + 3M_2\epsilon^{1+1/2+1/4} \text{ on } \{0 \leq x \leq \epsilon\}$$

where

$$3M_2 := 6(6(3M_0 m\lambda)^{1/2} m\lambda)^{1/2} = 6^{1+1/2} 3{M_0}^{1/4} (m\lambda)^{1/2+1/4}.$$

Iteration of above argument will yield that, if $\epsilon < c_0$ then

$$|Du| \leq 36 m\lambda\epsilon \leq C_0\epsilon \text{ on } \{0 \leq x \leq \epsilon\}$$

where C_0 is a dimensional constant.

For other intervals $[2^{-k-1}, 2^{-k}]$ ($k < k(\epsilon)$), a similar iteration can be applied with the first step

$$\max_{x\in[0,2^{-k}]} u \leq -m + 4M_0 2^{-k}, \tag{2.11}$$

which follows from (2.6) and (2.7). Then a parallel argument with ϵ replaced by 2^{-k} proves (2.5) on $[2^{-k-1}, 2^{-k}]$ for k such that $2^{-k} \leq c_0$. Note that other than to derive (2.6), we do not use the fact that ϵ is the specific constant depending on Ω.

For intervals near $x = 1$, (i.e., for $[1-2^{-k}, 1-2^{-k-1}]$) the proof is divided into two cases. First if $\max u = 1 = -\min u$, then for
$w := -u(1-x, y_1, ..., y_{n-1})$,

$$-\min w = \max w = 1, \quad w < 0 \text{ on } \{x = 0\}, \text{ and } w > 0 \text{ on } \{x = 1\}.$$

Therefore the argument for w on intervals near $x = 0$ gives the result for u on intervals near $x = 1$.

Secondly if $\max u = 1 > -\min u$, then as in the proof of Corollary 2.3, choose sufficiently large constants a and b such that λ is the first nonzero eigenvalue of (2.1) and the first eigenfunction v satisfies

$$\max v = 1 \text{ and } -\min v \geq -\min u.$$

Since $[\min u, \max u] \subset [\min v, \max v]$ and

$$\min_{\Omega(1)} u \geq 1 - 3M_0\epsilon = \max v - 3M_0\epsilon,$$

a similar reasoning as in the interval $[2^{-k-1}, 2^{-k}]$, yields the result for the interval $[1 - 2^{-k}, 1 - 2^{-k-1}]$ near $x = 1$. □

¿From Lemma 2.3, we obtain Corollary 2.5, which states that the first nodal set is located in the middle part of Ω. Later in section 4, Corollary 2.5 and Theorem 2.1 will be used along with a new coordinate system to estimate the first nonzero eigenvalue λ. Based on the bound on λ, the width and location of the nodal set are derived, again by using Theorem 2.1.

Corollary 2.5. *Let Ω and u be given as in Theorem 2.1. Suppose $N = 1$ and $\sup |u| = 1$. Then there exist dimensional constants $c_1 > 0$ and $c_2 > 0$ such that*

$$c_2 \leq |u| \leq 1 \text{ for } x \in [0, c_1] \cup [1 - c_1, 1].$$

Proof. Without loss of generality, we may assume $\max |u| = u(\tilde{x}, \tilde{y}) = 1$ ($\tilde{y} \in I\!R^{n-1}$) and $u > 0$ on $\{x = 1\}$. By Corollary 2.3, there exists a dimensional constant $c_1 > 0$ such that $u(x, y) > 1/2$ if $|x - \tilde{x}| < c_1$. Since the Courant nodal domain theorem [CH, p.452] implies that Ω_+ and Ω_- are connected, $u > 1/2$ in $\{x > \tilde{x} - c_1\}$.

On the other hand, since $\int_{\Omega_-} u^- = \int_{\Omega_+} u^+$, Lemma 2.3 implies $-\min u \geq c_2$ for some dimensional constant $c_2 > 0$. Hence we obtain Corollary 2.5 by a similar reasoning for u_- as in u_+. □

3 ODE eigenvalue estimates

In this section we prove several lemmas on ODE eigenvalue estimates, which will be applied to the one-dimensional eigenfunction ϕ_1 in section 4. In particular Lemma 3.5 will be used to locate the nodal set in section 4, and Lemma 3.6 yields the bound on the width of nodal set. The proof for Lemmas 3.1, 3.2, 3.3 and 3.4 are parallel to those of the corresponding lemmas in [J1]. The only difference in the proof is that, instead of the concavity of the height of the cross-section $h(x)$ for $n = 2$, we have the concavity of $w^{1/n-1}(x)$ by the Brunn-Minkowski inequality for the volume of cross-section $w(x)$.

Lemma 3.1. *For $a \le 1/2$,*

$$\inf_{\{\phi:\phi(a)=0\}} \frac{\int_0^a \phi'(x)^2 w(x)dx}{\int_0^a \phi(x)^2 w(x)dx} \ge \frac{1}{2^{n-2}a^2}.$$

Proof. The proof is the same as that of Lemma 4.2 of [J1], using the fact that $w(x) \le 2^{n-1} w(t)$ for $0 \le x \le t \le a$. □

Lemma 3.2. *There exists a constant $C > 0$ depending on n such that*

$$\inf_{\{\phi:\phi(a)=0\}} \frac{\int_a^b \phi'(x)^2 w(x)dx}{\int_a^b \phi(x)^2 w(x)dx} \le \frac{C}{(b-a)^2}.$$

Proof. Take a test function $\phi(x) = x$. We need to show that

$$\int_0^1 w(x)dx \le C \int_0^1 x^2 w(x)dx,$$

where C depends on n. Multiply w by a constant so that $\int_0^1 w^{1/(n-1)}(t)dt = 1$. Due to the normalization and the concavity of $w^{1/(n-1)}$, the arguments in the proof of Lemma 4.3 in [J1] yield that

$$\int_x^1 w^{1/(n-1)}(t)dt \ge (1-x)^2 \text{ for } 0 \le x \le 1.$$

By Hölder inequality,

$$(\int_x^1 w(t)dt)^{1/(n-1)}(1-x)^{(n-2)/(n-1)} \ge \int_x^1 w^{1/(n-1)}(t)dt \ge (1-x)^2,$$

and thus $W(x) := \int_x^1 w(t)dt \ge (1-x)^n$. Therefore by integration by parts,

$$\begin{aligned} \int_0^1 x^2 w(x)dx = -\int_0^1 x^2 W'(x)dx &= \int_0^1 2xW(x)dx \\ &\ge \int_0^1 2x(1-x)^n dx = 2/(n+1)(n+2), \end{aligned}$$

where the second inequality holds because $x^2 W(x) = 0$ for $x = 0, 1$.

On the other hand, since $w^{1/(n-1)}(t)$ is concave with $w^{1/n-1}(0), w^{1/n-1}(1) \ge 0$, the graph of $w^{1/(n-1)}(t), 0 \le t \le 1$ is above the triangle generated by $(0,0), (1,0)$ and $(t_0, w^{1/n-1}(t_0))$ where $w(t_0) = \max w$. It follows that

$$\int_0^1 w(t)dt \le w(t_0) \le (2\int_0^1 w^{1/(n-1)}(t)dt)^{n-1} = 2^{n-1}$$

and our lemma holds with $C = C(n) = \dfrac{2^n}{(n+1)(n+2)}$.

□

Lemma 3.3. *Let s_1 be the zero point of ϕ_1 given in (1.4). Then there exist constants $c_1 > 0$ and $c_2 > 0$ depending on n such that*

$$c_1 < s_1 < c_2.$$

Proof. The lemma follows from Lemmas 3.1, 3.2 and the proof of Lemma 4.4 in [J1]. □

For $a \in [0, 1]$, define

$$E[a, 1] = \inf_{\{\phi:\phi(a)=0\}} \frac{\int_a^1 \phi'(x)^2 w(x)dx}{\int_a^1 \phi(x)^2 w(x)dx}.$$

Lemma 3.4. *Suppose that $c_0 \leq a \leq 1 - c_0$ for some $0 < c_0 < 1$. Then there exists a constant $C > 0$ depending on n and c_0 such that*

$$(\partial/\partial a)E[a, 1] \geq C.$$

Proof. Normalize w so that

$$\max_{0 \leq x \leq N} w(x) = 1. \tag{3.1}$$

By concavity of $w^{1/(n-1)}$ and (3.1), $w^{1/(n-1)}(c_0) \geq c_0$. Let ϕ be the unique nonnegative minimizer for $E[a, 1]$ with the normalization

$$\int_a^1 \phi(x)^2 w(x)dx = 1. \tag{3.2}$$

Following the proof of Lemma 3.4 in [J1], we only need to prove that

$$|\phi(x)| \leq CE^n \text{ for } a \leq x \leq 1, \tag{3.3}$$

where $E = E[a, 1]$ and C is a constant depending on n and c_0.

Observe that, since ϕ satisfies $-(w\phi')' = Ew\phi$ and $\phi'(1) = 0$,

$$\begin{aligned} |w\phi'|(x) &\leq |\int_x^1 Ew(t)\phi(t)dt| \\ &\leq E(\int_x^1 \phi^2(t)w(t)dt)^{1/2}(\int_x^1 w(t)dt)^{1/2} \end{aligned} \tag{3.4}$$

In particular, (3.1) and (3.2) imply that $|w\phi'| \le E$ for $a \le x < 1$. Since $\phi(a) = 0$, we have

$$\phi(t) \le E \int_a^t \frac{ds}{w(s)}. \tag{3.5}$$

On the other hand, by concavity of $w^{1/n-1}(t)$, for $a \le s \le t \le 1$, $w^{1/n-1}(s)$ is above the line $l(s) = \alpha s + \beta$ connecting (c_0, c_0) and $(t, w^{1/n-1}(t))$. Without loss of generality, we may assume that $c_0 > w^{1/n-1}(t)$ and $\alpha < 0$ (Other cases are better.) For $n \ge 3$

$$\int_a^t \frac{ds}{w(s)} \le \int_{c_0}^t \frac{ds}{(\alpha s + \beta)^{n-1}} \le \frac{C}{(\alpha t + \beta)^{n-2}} = \frac{C}{w^{(n-2)/n-1}(t)}. \tag{3.6}$$

Hence by (3.4), (3.5) and (3.6)

$$\begin{aligned} |\phi'(x)| &\le \frac{E}{w(x)} \int_x^1 w(t)\phi(t)dt \\ &\le \frac{E^2}{w(x)} \int_x^1 w(t)(\int_a^t \frac{1}{w(s)}ds)dt \\ &\le \frac{CE^2}{w(x)} \int_x^1 w^{1/(n-1)}(t)dt \\ &\le CE^2 \frac{(1-x)}{w^{(n-2)/(n-1)}(x)} \\ &\le CE^2 \frac{1}{w^{(n-3)/(n-1)}(x)} \end{aligned} \tag{3.7}$$

where the fourth and fifth inequalities follow respectively from

$$w^{1/(n-1)}(t) \le C(n) w^{1/(n-1)}(x) \text{ for } a \le x \le t \le 1 \tag{3.8}$$

and

$$w^{1/(n-1)}(x) \ge \min\{1-x, c_0\} \text{ for } a \le x \le 1. \tag{3.9}$$

(3.8) and (3.9) are due to the concavity of $w^{1/(n-1)}$ and the normalization (3.1). (For (3.8), see Remark 4.1 (b) in [J1].)

(3.7) and (3.8) yield that

$$|\phi(x)| \le \int_a^x |\phi'(t)|dt \le \frac{C_1 E^2}{w^{(n-3)/(n-1)}(x)} \text{ for } a \le x \le 1, \tag{3.10}$$

where C_1 is a constant depending on n and c_0. We go back to the first inequality of (3.7) and apply (3.10) and then (3.8) and (3.9) to obtain

$$|\phi'(x)| \le \frac{C_1 E^2}{w(x)} \int_x^1 w(t)^{2/(n-1)} dt \le \frac{C_2 E^3}{w(x)^{(n-4)/(n-1)}}$$

where C_2 is a dimensional constant. Now by similar reasoning as in (3.10), the improved estimate on $|\phi|$ holds:

$$|\phi(x)| \leq \frac{C_3 E^n}{w(x)^{(n-4)/(n-1)}}.$$

We repeat the above process $(n-4)$ more times to obtain $|\phi(x)| \leq C(n)E^n$. □

Lemma 3.5. *Let ϕ_1 and s_1 be given as in (1.4) and suppose $N = 1$. If ϕ is a function on $(0,1)$ such that $\phi(s_1') = 0$ and*

$$\frac{\int_{s_1'}^1 (\phi')^2 w ds}{\int_{s_1'}^1 \phi^2 w ds} \leq (1 + M\epsilon) \frac{\int_{s_1}^1 (\phi_1')^2 w ds}{\int_{s_1}^1 \phi_1^2 w ds},$$

then $s_1' \leq s_1 + C\epsilon$ for some constant C depending on n and M.

Proof. The lemma follows from Lemmas 3.3, 3.4 and from the fact

$$E[s_1, 1] = \mu_1 = \frac{\int_{s_1}^1 (\phi_1')^2 w ds}{\int_{s_1}^1 \phi_1^2 w ds}.$$

□

Next we show that if the energy associated with ϕ is bounded by $(1 + M\epsilon)\mu_1$, then $\sup |\phi|$ is bigger than ϵ on any interval of length $C\epsilon$.

Lemma 3.6. *Let $N = 1$ and μ_1 be given in (1.4). Suppose $\phi(s)$ is a function on $(0,1)$ such that $\int_0^1 \phi w ds = 0$, $\sup |\phi| = 1$, $\sup |\phi'| \leq C_1$ and*

$$\mu_1 \leq \frac{\int_0^1 (\phi')^2 w ds}{\int_0^1 \phi^2 w ds} \leq (1 + M\epsilon)\mu_1. \tag{3.11}$$

If for some $0 < a < b < 1$ and $C_2 > 0$

$$\sup_{[a,b]} |\phi| \leq 2C_1\epsilon \text{ and } |\int_a^1 \phi w ds| \geq C_2 \int_0^1 w ds,$$

then $b - a < C\epsilon$ for some $C > 0$ depending on C_1, C_2 and M.

Proof. Let ϕ_1 and s_1 be as given in (1.4). Suppose that

$$\sup_{[a,b]} |\phi| \leq 2C_1\epsilon \text{ and } b = a + C\epsilon,$$

for some a and sufficiently large $C > 0$. Without loss of generality we may assume

$$a \geq s_1 + C\epsilon/4 \text{ or } b \leq s_1 - C\epsilon/4. \tag{3.12}$$

(If $a < s_1 < b$, then $s_1 - a \geq C\epsilon/2$ or $b - s_1 \geq C\epsilon/2$. If $b - s_1 \geq C\epsilon/2$, replace a with $(s_1 + b)/2$. If $s_1 - a \geq C\epsilon/2$, replace b with $(s_1 + a)/2$. Lastly if $a < b < s_1$ or $s_1 < a < b$, by replacing a or b with $(a+b)/2$, we get (3.12).)

Changing the sign of ϕ if needed, we also set

$$|\int_a^1 \phi w ds| = \int_a^1 \phi w ds.$$

Define

$$A_\phi = \frac{\int_0^a \phi^2 w ds}{\int_0^1 \phi^2 w ds}, \quad B_\phi = \frac{\int_a^1 \phi^2 w ds}{\int_0^1 \phi^2 w ds}, \quad A_\phi + B_\phi = 1. \tag{3.13}$$

By our hypothesis,

$$\frac{\int_0^1 (\phi')^2 w ds}{\int_0^1 \phi^2 w ds} = A_\phi \frac{\int_0^a (\phi')^2 w ds}{\int_0^a \phi^2 w ds} + B_\phi \frac{\int_a^1 (\phi')^2 w ds}{\int_a^1 \phi^2 w ds} \leq (1 + M\epsilon)\mu_1.$$

If C is sufficiently large, then Lemma 3.4 and above inequality imply that

$$\frac{\int_0^a (\phi')^2 w ds}{\int_0^a \phi^2 w ds} \leq \mu_1 \leq \frac{\int_a^1 (\phi')^2 w ds}{\int_a^1 \phi^2 w ds}. \tag{3.14}$$

We will construct a test function ψ such that $\int_0^1 \psi w ds = 0$ and

$$(1 + M\epsilon)\frac{\int_0^1 (\psi')^2 w ds}{\int_0^1 \psi^2 w ds} < \frac{\int_0^1 (\phi')^2 w ds}{\int_0^1 \phi^2 w ds}, \tag{3.15}$$

which contradicts our hypothesis.

First, construct a continuous function $\tilde{\psi}$ such that $\tilde{\psi} = \phi$ on the left interval $[0, a]$ and $\tilde{\psi} = \phi + C_2(b - a)/10$ on the left interval $[b, 1]$. For $\alpha = C_2/10$, let

$$\tilde{\psi}(s) = \begin{cases} \phi(s) & \text{for} \quad 0 \leq s \leq a \\ \phi(s) + \alpha(s - a) & \text{for} \quad a \leq s \leq b = a + C\epsilon \\ \phi(s) + \alpha(b - a) & \text{for} \quad b \leq s \leq 1. \end{cases}$$

¿From a straightforward calculation, it follows that the energy of $\tilde{\psi}$ on $[a,1]$ gets smaller than the energy of ϕ on $[a,1]$ by some amount. More precisely

$$\frac{\int_a^1 \tilde{\psi}'^2 wds}{\int_a^1 \tilde{\psi}^2 wds} \leq (1 - C_0 C\epsilon)\frac{\int_a^1 \phi'^2 wds}{\int_a^1 \phi^2 wds} \tag{3.16}$$

for some $C_0 > 0$ depending on C_1 and C_2.

Next we perturb $\tilde{\psi}$ and get ψ such that $\int_0^1 \psi wds = 0$ and will show that ψ satisfies (3.15). Set

$$\psi(s) = \begin{cases} \tilde{\psi}(s) = \phi(s) & \text{for} \quad 0 \leq s \leq a \\ \\ \beta(\tilde{\psi}(s) - \phi(a)) + \phi(a) & \text{for} \quad a \leq s \leq 1 \end{cases}$$

where $\beta > 0$ is chosen to satisfy $\int_0^1 \psi wds = 0$, i.e., $\int_a^1 \psi wds = \int_a^1 \phi wds$. Then

$$\beta = 1 - \alpha|b-a|\frac{\int_a^1 wds}{\int_a^1 \phi wds} + O(\epsilon^2).$$

Since

$$\int_a^1 \phi wds \leq \sqrt{\int_a^1 \phi^2 wds}\sqrt{\int_a^1 wds}, \tag{3.17}$$

it follows that

$$\int_a^1 \tilde{\psi}^2 wds \leq (1 + 2\alpha|b-a|\frac{\int_a^1 wds}{\int_a^1 \phi wds} + C_3\epsilon^2)\int_a^1 \phi^2 wds.$$

Therefore

$$\int_a^1 \psi^2 wds \leq (1 + C_3\epsilon^2)\int_a^1 \phi^2 wds$$

and

$$B_\psi \leq (1 + C_3\epsilon^2) B_\phi \tag{3.18}$$

where B_ψ is defined as in (3.13) (Recall that $A_\psi = 1 - B_\psi$). Therefore,

$$
\begin{aligned}
\frac{\int_0^1 (\psi')^2 wds}{\int_0^1 \psi^2 wds} &= A_\psi \frac{\int_0^a (\psi')^2 wds}{\int_0^a \psi^2 wds} + B_\psi \frac{\int_a^1 (\psi')^2 wds}{\int_a^1 \psi^2 wds} \\
&= A_\psi \frac{\int_0^a (\phi')^2 wds}{\int_0^a \phi^2 wds} + B_\psi \frac{\int_a^1 (\psi')^2 wds}{\int_a^1 \psi^2 wds} \\
&\le (1 + C_3\epsilon^2)(A_\phi \frac{\int_0^a (\phi')^2 wds}{\int_0^a \phi^2 wds} + B_\phi \frac{\int_a^1 (\psi')^2 wds}{\int_a^1 \psi^2 wds}) \\
&\le (1 + C_3\epsilon^2)(A_\phi \frac{\int_0^a (\phi')^2 wds}{\int_0^a \phi^2 wds} + B_\phi (1 - C_0 C\epsilon) \frac{\int_a^1 (\phi')^2 wds}{\int_a^1 \phi^2 wds}) \\
&\le A_\phi \frac{\int_0^a (\phi')^2 wds}{\int_0^a \phi^2 wds} + B_\phi (1 - C_0 C\epsilon) \frac{\int_a^1 (\phi')^2 wds}{\int_a^1 \phi^2 wds} + C_3\epsilon^2 (1 + M\epsilon)\mu_1 \\
&\le \frac{\int_0^1 (\phi')^2 wds}{\int_0^1 \phi^2 wds} - B_\phi C_0 C\epsilon \cdot \frac{\int_a^1 (\phi')^2 wds}{\int_a^1 \phi^2 wds} + C_3\epsilon^2 (1 + M\epsilon)\mu_1
\end{aligned}
$$

where we obtain the first inequality from $A_\phi + B_\phi = A_\psi + B_\psi = 1$, (3.18) and (3.14), the second inequality from (3.16), the third inequality from (3.11). From the hypothesis $\int_a^1 \phi wds \ge C_2 \int_0^1 wds$ and (3.17), one can observe that B_ϕ is bounded below by a constant depending on C_2. Hence if we choose a sufficiently large C depending on C_1, C_2 and M, and if ϵ is sufficiently small compared to C, then (3.15) holds. □

In the next lemma, we show the nondegeneracy of ϕ_1 near the zero s_1.

Lemma 3.7. *Let ϕ_1 be the first nonzero eigenfunction of (1.4) with $N = 1$ and let s_1 be the zero of ϕ_1 - Note that $c_1 \le s_1 \le c_2$ by Lemma 3.3. Normalize ϕ_1 such that $\phi_1 > 0$ on $(s_1, 1]$ and* $\max \phi_1 = 1$. *Then $\phi_1' \ge 0$ on $[0, 1]$. Moreover there exists a dimensional constant $c = c(n)$ such that*

$$\phi_1' \ge c(n) \text{ on } [s_1 - c_1/2, s_1 + (1 - c_2)/2].$$

Proof. By the normalization $\phi_1 > 0$ on $(s_1, 1]$ and $\phi_1 < 0$ on $[0, s_1)$. Hence from (1.4) we observe that $w\phi_1'$ has its maximum at $x = s_1$ and $w\phi_1'$ is increasing on $[0, s_1]$ and decreasing on $[s_1, 1]$. From the boundary condition it follows that $\phi_1' \ge 0$ on $[0, 1]$ and thus $\phi_1(1) = 1$. We will only prove that $\phi_1' \ge c(n)$ on $[s_1, s_1 + (1 - c_2)/2]$: a parallel argument leads to the rest of the claim.

Observe that from (1.4) we have

$$\phi_1'(x) = \frac{\int_x^1 \mu_1 w(t)\phi_1(t)dt}{w(x)}. \tag{3.19}$$

We will use (3.19) to find lower bounds on ϕ' respectively near $t = 1$ and then near $t = s_1$, using that $w\phi'$ has its maximum at s_1. Note that for $c_1 \le x \le t \le 1$, the concavity of $w^{1/(n-1)}(x)$ implies

$$(1-t)w^{1/(n-1)}(x) \le w^{1/(n-1)}(t). \tag{3.20}$$

(See Remark 4.1 (a) in [J1].) Thus (3.8) and (3.20) imply that

$$C_1(1-x)^n w(x) \le \int_x^1 w(t)dt \le C_2(1-x)w(x) \tag{3.21}$$

where C_1 and C_2 are positive dimensional constants. Let $\phi_1(s_2) = 1/2$ for $s_2 \in [s_1, 1]$. Then by (3.19) and (3.21), it follows that for $x \in [s_2, 1]$,

$$C_1\mu_1(1-x)^{n+1} \le \phi_1'(x) \le C_2\mu_1(1-x). \tag{3.22}$$

Since $\phi_1(1) = 1$, (3.22) implies that $\phi_1(x) \ge 1 - C_3\mu_1(1-x) \ge 1/2$ if $x \in [A, 1]$ where C_3 and A are dimensional constants. Therefore $s_2 \le A$ and

$$\phi_1'(s_2) \ge C_1\mu_1(1-s_2)^{n+1} \ge C_4, \tag{3.23}$$

where C_4 is a dimensional constant. Now we have, for $s_1 \le t \le s_2$,

$$\phi_1'(t) \ge \frac{w(s_2)}{w(t)}\phi_1'(s_2) \ge C_5\phi_1'(s_2) \ge C_6$$

where the second inequality is due to $t \le s_2 < A$ and the last from (3.23). For $t \in [s_1, s_1 + (1-c_2)/2]$, (3.22) implies $\phi_1'(t) \ge C_7$ and our claim is proved. □

4 A new coordinate system

In this section, we define a new coordinate system for Ω satisfying certain properties, which will enable us to use Fubini's Theorem when we construct a one-dimensional test function with small energy. For the proof of the main lemma (Lemma 4.1), first we will prove Lemma 4.2 for general dimensions. Then we construct in detail coordinate systems for $n = 3$ and $n = 4$ satisfying the properties given in the main lemma. We will then discuss the general dimension based on the three and four-dimensional cases.

Lemma 4.1. *Let Ω be a bounded convex domain in $I\!R^n$ satisfying (1.2) and (1.3). Suppose $N = 1$. Then there exists a coordinate system $(s, t_1, ..., t_{n-1}) \in [0, 1] \times B_1(0)$ to $(x, y_1, ..., y_{n-1}) \in \Omega$ such that the following statements hold.*

(I) $s = x$

(II) $|\partial_s y_j| \leq C\epsilon \max\{1/s, 1/(1-s)\}$ *for a dimensional constant* C.

(III) the mapping $f : (s, t_1, ..., t_{n-1}) \to (s, y_1, ..., y_{n-1})$*, has a constant Jacobian* $a_n w(s)$*, where* $w(s)$ *is the volume of cross-section* $\Omega(s)$ *and* a_n *is a dimensional constant.*

Here we denote by $\partial_s y_j = \partial y_j/\partial s$, the partial derivative of y_j with t_1,..., t_{n-1} held fixed. For the proof of Lemma 4.1, we need the following lemma.

Lemma 4.2. *Let $\mathcal{D}$ be a planar bounded convex domain. Suppose $(0, \epsilon) \in \partial\mathcal{D}$, $(0, -C_1\epsilon) \in \partial\mathcal{D}$ ($C_1 > 0$), and $(\pm\epsilon, 0) \in \mathcal{D}$. Further suppose that the length of projection of $\mathcal{D}$ on the y-axis is less than $C_2\epsilon$ for some $C_2 > 0$. If*

$$(r\cos\theta, r\sin\theta) \in \partial\mathcal{D} \text{ and } (s\cos(\theta - \Delta\theta), s\sin(\theta - \Delta\theta)) \in \partial\mathcal{D},$$

then there exists C depending on C_1 and C_2 such that

$$|r - s| \leq Cr^2\Delta\theta/\epsilon$$

for sufficiently small $\Delta\theta$.

Proof. Suppose $0 \leq \theta \leq \pi/4$. Consider a line l passing through $(0, \epsilon)$ and $(r\cos\theta, r\sin\theta)$. Let

$$(\tilde{s}\cos(\theta - \Delta\theta), \tilde{s}\sin(\theta - \Delta\theta)) \in l$$

and

$$(s\cos(\theta - \Delta\theta), s\sin(\theta - \Delta\theta)) \in \partial\mathcal{D}.$$

Since Ω is convex, $s \leq \tilde{s}$. The equation for l is $y = \dfrac{r\sin\theta - \epsilon}{r\cos\theta}x + \epsilon$ and thus

$$\tilde{s}\sin(\theta - \Delta\theta) = \frac{r\sin\theta - \epsilon}{r\cos\theta}\tilde{s}\cos(\theta - \Delta\theta) + \epsilon.$$

If $\Delta\theta$ is sufficiently small,

$$s \leq \tilde{s} \leq r(1 + 2(\frac{r\cos\theta}{\epsilon} + \frac{r\sin\theta\tan\theta}{\epsilon})\Delta\theta) \leq r(1 + C\frac{r\Delta\theta}{\epsilon}).$$

On the other hand, if $\tilde{l}$ is a line passing through $(0, -C_1\epsilon)$ and $(r\cos\theta, r\sin\theta)$, by a similar reasoning

$$r \leq s(1 + C\frac{s\Delta\theta}{\epsilon})$$

for C depending on C_1 and the proof is done for $0 \leq \theta \leq \pi/4$.

Next, suppose $\pi/4 \leq \theta \leq \pi/2$. From the hypothesis on Ω, the angle between the tangent line to $\partial\Omega$ at $(r\cos\theta, r\sin\theta)$ and the line connecting the origin to $(r\cos\theta, r\sin\theta)$ is bounded below by an angle depending on C_2. Hence,

$$|s - r| \leq Cr\Delta\theta \leq C\frac{r^2\Delta\theta}{\epsilon}$$

for some C depending on C_2, where the second inequality follows from $r \leq C\epsilon$. $\square$

Using Lemma 4.2, we prove Lemma 4.1.

Proof of Lemma 4.1. Without loss of generality, we may assume that

$$(0, 0, ..., 0), (1, 0, ..., 0) \in \partial\Omega,$$

since if $l = \{(x, l_1(x), ..., l_{n-1}(x)) : 0 \leq x \leq 1\}$ is a line segment connecting a point on $\bar{\Omega} \cap \{x = 0\}$ to a point on $\bar{\Omega} \cap \{x = 1\}$, then $1 \leq \text{length}(l) \leq 1 + C\epsilon$ and $|l_j'| \leq C\epsilon$ for $1 \leq j \leq n - 1$ and C depending on n.

Denote $a_i = 2^{-i}$, $b_i = 1 - 2^{-i}$ $(a_1 = b_1)$. For $i \geq 1$, set

$$I_i = (a_{i+1}, a_i), \quad I_{-i} = (b_i, b_{i+1}).$$

With this notation, the condition (II) of Lemma 4.1 becomes equivalent to the following statement:

$$(II') \qquad |\partial_s y_j| \leq C2^i\epsilon \text{ for } s \in I_{\pm i}$$

which will be verified in the proof instead of (II).

Recall that $\Omega(s) = \Omega \cap \{x = s\}$. Denote by $h_1(\Omega(x))$, the minimum length of projection of $\Omega(x) \subset I\!R^{n-1}$ on a line $l_1 \subset I\!R^{n-1}$, and denote by $h_j(\Omega(x)), j = 2, ..., n-1$, the minimum length of projection of $\Omega(x)$ on a line l_j which is perpendicular to l_k for $1 \leq k \leq j-1$ (See Figure 1). Since Ω is convex, each $\Omega_i := \Omega \cap \{x \in I_i\}$ has orthonormal basis

$$e_{i1}, e_{i2}, ..., e_{in-1} \in I\!R^{n-1}$$

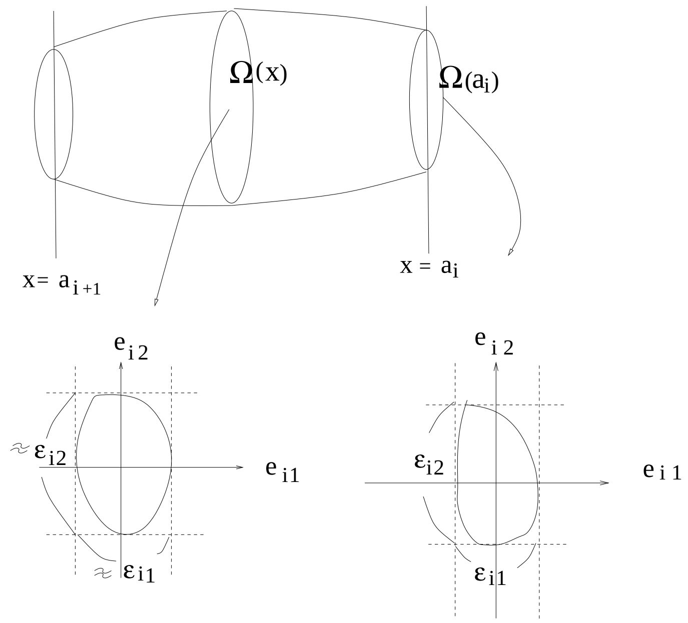

Figure 1

such that for $x \in I_i$,

$$\begin{cases} |p_{e_{i1}}(\Omega(x))| \approx h_1(\Omega(x)) \approx h_1(\Omega(a_i)) \\ . \\ . \\ |p_{e_{in-1}}(\Omega(x))| \approx h_{n-1}(\Omega(x)) \approx h_{n-1}(\Omega(a_i)) \end{cases} \tag{4.1}$$

where $|p_{e_{ij}}(\Omega(x))|$ denotes the length of projection of $\Omega(x)$ on a line parallel to e_{ij}. Denote

$$\epsilon_{i1} = h_1(\Omega(a_i)), \ \ ..., \ \ \epsilon_{in-1} = h_{n-1}(\Omega(a_i)),$$

so that $\Omega(x)$ has dimensions comparable to ϵ_{i1}, ..., ϵ_{in-1} for $x \in I_i$ where $\epsilon_{i1} \leq \epsilon_{i2} \leq ... \leq \epsilon_{in-1}$.

For a domain $D \in I\!R^n$, define *the center of mass* of D as

$$\frac{1}{|D|}\int_D x dx,$$

where $|D|$ denotes the volume of D. Let $(x, L(x))$ be the curve in Ω, that is linear on each interval $I_{\pm i}$ and equal to the center of mass of the cross-section at each endpoints of I_i. In other words, $L(a_i)$ is the center of mass of $\Omega(a_i)$, and similarly for b_i. On I_i, denote

$$L(x) = L_1(x)e_{i1} + ... + L_{n-1}(x)e_{in-1}, \quad x \in I_i.$$

Since the line segment connecting $(0, 0, ..., 0)$ and $(1, 0, ..., 0)$ is contained in Ω, $(a_i, 0, ..., 0) \in \Omega(a_i)$ and $(a_{i+1}, 0, ...0) \in \Omega(a_{i+1})$, which imply

$$|L_j(a_i) - L_j(a_{i+1})| \leq |L_j(a_i)| + |L_j(a_{i+1})| \leq C\epsilon_{ij}$$

and hence

$$|L_j'(x)| \leq C2^i\epsilon_{ij} \text{ for } x \in I_i. \tag{4.2}$$

4.1 $n = 3$

First, when $n = 3$, we define a new coordinate system $(s, t_1, t_2) \in [0, 1] \times B_1(0)$ to $(x, y, z) \in \Omega$ as follows.

(i) $x = s$, $(y, z) = f(s, t_1, t_2)$

(ii) $f(s, 0, 0) = L(s)$, $f(s, 1, 0) - f(s, 0, 0)$ is parallel to e_{i1} for every $s \in I_i$

(iii) the mapping $f : (t_1, t_2) \to (y, z)$, with s held fixed, is linear on every line segment from $(0, 0)$ to $\partial B_1(0)$

(iv) the mapping $f : (t_1, t_2) \to (y, z)$, with s held fixed, has a constant Jacobian $w(s)/\pi$, where $w(s)$ is the area of $\Omega(s)$.

Note that with s held fixed, $f(t_1, t_2)$ is defined so that $B_1(0) \cap \{0 \leq \theta \leq a\}$ is mapped onto a sector $\Omega(s) \cap \{0 \leq \angle(e_{i1}, (y_1, y_2) - L(s)) \leq b\}$ with volume V satisfying

$$\frac{a}{2\pi} = \frac{V}{|\Omega(s)|}.$$

(Here, $\angle(v_1, v_2)$ denotes the angle between v_1 and v_2, that is

$$\angle(v_1, v_2) = \arccos(\frac{(v_1, v_2)}{|v_1||v_2|}).)$$

Observe that if $g_1(s)$ is a function on I_i such that

$$\{(s, g_1(s)e_{i1}) : s \in I_i\} \subset \partial\Omega,$$

then by the convexity of Ω,

$$|g_1'(s)| \leq C|g_1(s)|/|s| \leq C2^i\epsilon_{i1} \text{ for } s \in I_i. \tag{4.3}$$

Similarly, if $\{(s, g_2(s)e_{i2}) : s \in I_i\}$ is a curve on $\partial\Omega$, then

$$|g_2'(s)| \leq C2^i \epsilon_{i2} \text{ for } s \in I_i. \tag{4.4}$$

(4.3) and (4.4) imply that if $x \in I_i$ and $h > 0$ is sufficiently small,

$$|\Omega(x) - \Omega(x+h)| \leq C2^i h \epsilon_{i1} \epsilon_{i2} \tag{4.5}$$

where

$$\Omega(x) - \Omega(x+h) := (\Omega(x) \cup \Omega(x+h)) - (\Omega(x) \cap \Omega(x+h)).$$

Now to prove (II′) for $x \in I_i$, fix $(t_1, t_2) \in \partial B_1(0)$. Since $\nabla L(x)$ satisfies (4.2) and $\epsilon_{ij} \leq \epsilon$, we may assume $L(x) = L(x+h)$ without loss of generality. Denote

$$\theta_x = \angle(e_{i1}, f(x, t_1, t_2) - L(x))$$

and

$$r_x = |f(x, t_1, t_2) - L(x)|.$$

Change the coordinates in $I\!R^2$ so that $L(x) = L(x+h) = 0$ and

$$f(x, t_1, t_2) = (r_x \cos\theta_x, r_x \sin\theta_x).$$

Fix a sufficiently small $h > 0$, and denote

$$f(x+h, t_1, t_2) = (r_{x+h} \cos\theta_{x+h}, r_{x+h} \sin\theta_{x+h}).$$

Since f has a constant Jacobian, (4.5) implies

$$r_x^2 |\theta_{x+h} - \theta_x| \leq C2^i h \epsilon_{i1} \epsilon_{i2} \tag{4.6}$$

since

$$\begin{aligned} |\Omega(x) \cap \{0 \leq \theta \leq \theta_x\} - \Omega(x+h) \cap \{0 \leq \theta \leq \theta_x\}| &\leq |\Omega(x) - \Omega(x+h)| \\ &\leq C2^i h \epsilon_{i1} \epsilon_{i2}. \end{aligned}$$

Let $s > 0$ be a number such that $(s\cos\theta_{x+h}, s\sin\theta_{x+h}) \in \partial\Omega(x)$. Then Lemma 4.2 implies

$$|s - r_x| \leq C r_x^2 |\theta_{x+h} - \theta_x| / \epsilon_{i1}, \tag{4.7}$$

which yields

$$\begin{aligned} |(r_x\cos\theta_x, r_x\sin\theta_x) - (s\cos\theta_{x+h}, s\sin\theta_{x+h})| &\leq C r_x |\theta_{x+h} - \theta_x| + |s - r_x| \\ &\leq C2^i h \epsilon_{i2} \end{aligned}$$

where the last inequality follows from (4.6), (4.7) and $\epsilon_{i1} \le Cr_x$.

On the other hand, since

$$A := (s\cos\theta_{x+h}, s\sin\theta_{x+h}) \text{ and } B := (r_{x+h}\cos\theta_{x+h}, r_{x+h}\sin\theta_{x+h})$$

are points on $\partial\Omega(x)$ and $\partial\Omega(x+h)$ with the same angle from e_{i1}, a similar argument as in (4.3) and (4.4) yields

$$|A - B| \le C2^i hs \le C2^i h\epsilon_{i2}.$$

Hence we conclude

$$\begin{aligned}
&|f(x,t_1,t_2) - f(x+h,t_1,t_2)| \\
&\qquad = |(r_x\cos\theta_x, r_x\sin\theta_x) - (r_{x+h}\cos\theta_{x+h}, r_{x+h}\sin\theta_{x+h})| \\
&\qquad \le C2^i h\epsilon_{i2} \le C2^i h\epsilon
\end{aligned}$$

and this implies (II') by sending h to 0.

4.2 $n = 4$

Next we consider $n = 4$. Recall that for $x \in I_i$, $p_{e_{ij}}(\Omega(x)) \approx \epsilon_{ij}$ with $\epsilon_{i1} \le \epsilon_{i2} \le \epsilon_{i3}$, and

$$(x, L(x)) = (x, L_1(x)e_{i1} + L_2(x)e_{i2} + L_3(x)e_{i3})$$

is a line segment connecting the center of mass of $\Omega(a_i)$ to that of $\Omega(a_{i+1})$ on I_i. To define a new coordinate system, we first divide $\Omega(x)$ into two parts with the same volume as follows. Let $\tilde{L}_3(x)$ be a number such that

$$|\{y_1e_{i1} + y_2e_{i2} + y_3e_{i3} \in \Omega(x) : y_3 \ge \tilde{L}_3(x)\}| = |\Omega(x)|/2.$$

Let (r, θ_1, θ_2) be a polar coordinate in $\Omega(x)$ with $r = 0$ at

$$\tilde{L}(x) := L_1(x)e_{i1} + L_2(x)e_{i2} + \tilde{L}_3(x)e_{i3}. \tag{4.8}$$

Let g_x be a density function on $\partial\Omega(x)$ with total mass 1, i.e., for any $0 \le a_1, a_2, b_1, b_2 \le 2\pi$

$$\begin{aligned}
&|\{(r,\theta_1,\theta_2) \in \Omega(x) : a_1 \le \theta_1 \le b_1, a_2 \le \theta_2 \le b_2\}| \\
&\qquad = |\Omega(x)| \int_{\partial\Omega(x)\cap\{a_j\le\theta_j\le b_j,\, j=1,2\}} g_x d\theta_1 d\theta_2.
\end{aligned} \tag{4.9}$$

(Note that

$$g_x(y) = C\frac{r(y)^3}{|\Omega(x)|}, \quad r(y) = |y - \tilde{L}(x)|$$

where $C > 0$ is a dimensional constant.)

Now we construct a coordinate system $(s, t_1, t_2, t_3) \in [0,1] \times B_1(0)$ to $(x, y_1, y_2, y_3) \in \Omega$ as follows.

(i) $x = s$, $(y_1, y_2, y_3) = f(s, t_1, t_2, t_3)$

(ii) $f(s, 0, 0, 0) = \tilde{L}(s)$, $f(s, 1, 0, 0) - f(s, 0, 0, 0)$ is parallel to e_{i1} for every $s \in I_i$

(iii) the mapping $f : (t_1, t_2, t_3) \to (y_1, y_2, y_3)$, with s held fixed, is linear on every line segment from $(0, 0, 0)$ to $\partial B_1(0)$

(iv) the mapping $f : (t_1, t_2, t_3) \to (y_1, y_2, y_3)$ satisfies the following properties

(a) f maps the half sphere

$$B_1^{right}(0) := \{(t_1, t_2, t_3) \in B_1(0) : t_3 \geq 0\}$$

onto

$$\Omega^{right}(x) := \{y_1 e_{i1} + y_2 e_{i2} + y_3 e_{i3} \in \Omega(x) : y_3 \geq \tilde{L}_3(x)\}.$$

(b) f maps the half cone

$$B(\theta) := \{t \in B_1^{right}(0) : \angle(t, e_1) \leq \theta\}$$

onto

$$\Omega(x, \phi(x, \theta)) := \{y \in \Omega^{right}(x) : \angle(\overrightarrow{y}, e_{i1}) \leq \phi(x, \theta)\}$$

where $\overrightarrow{y}$ is a vector from $\tilde{L}(x)$ to y, and $\phi(x, \theta)$ is the angle such that

$$\frac{|B(\theta)|}{|B_1^{right}(0)|} = \frac{|\Omega(x, \phi(x, \theta))|}{|\Omega^{right}(x)|}.$$

(c) Let $\tilde{B}(\theta) = \{t \in \partial B_1^{right}(0) : \angle(t, e_1) = \theta\}$ and

$$\tilde{\Omega}(x, \phi(x, \theta)) = \{y \in \partial\Omega^{right}(x) : \angle(\overrightarrow{y}, e_{i1}) = \phi(x, \theta)\}.$$

Then the mapping $f : \tilde{B}(\theta) \to \tilde{\Omega}(x, \phi(x,\theta))$, with s and θ held fixed, satisfies

$$\frac{\displaystyle\int_{f(A)} g_x}{\displaystyle\int_{\tilde{\Omega}(x,\phi(x,\theta))} g_x} = \frac{|A|}{|\tilde{B}(\theta)|} \tag{4.10}$$

for any $A \subset \tilde{B}(\theta)$.

(d) f satisfies parallel properties for $B^{left}(0) := B_1(0) - B_1^{right}(0)$ and $\Omega^{left}(x) := \Omega(x) - \Omega^{right}(x)$.

(f might be discontinuous on the intersection of $B^{right}(0)$ and $B^{left}(0)$.)

The conditions (i)-(iii) are parallel to those when $n = 3$, and the only difference between $n = 3$ and $n = 4$ is the condition (iv). In fact, when $n = 3$, there is a unique map $f : (s, t_1, t_2) \to (y, z)$ with a constant Jacobian on each cross sections, if we fix one direction e_{i1} (See condition (ii)). However when $n = 4$, there are infinitely many maps $f : (s, t_1, t_2, t_3) \to (y_1, y_2, y_3)$ with a constant Jacobian, even if we fix any two directions. In other words when $n = 4$, the properties stated for three dimensional case do not suffice to construct a function f which satisfies (II$'$). Hence when $n = 4$, we construct a (unique) map f with a constant Jacobian, under the constraint (iv) that f maps a two dimensional surface in $B_1(0)$ with a fixed angle θ from e_1, to a two dimensional surface in $\Omega(x)$ with a fixed angle $\phi(x,\theta)$ from e_{i1}. Then the map f, with θ fixed, is two-dimensional and area-preserving with respect to the normalized density function $g_x/|\int g_x|$. Hence we can proceed as in the case $n = 3$. Here we divide Ω into two parts $\Omega^{right}(x)$ and $\Omega^{left}(x)$, so that the shorter arcs of

$$\{y \in \Omega(x) : \angle(y, e_{i1}) = \phi(x,\theta)\}$$

are mapped to the shorter arcs of

$$\{y \in \Omega(\tilde{x}) : \angle(y, e_{i1}) = \phi(\tilde{x},\theta)\}$$

by $f_{\tilde{x}} \circ f_x^{-1}$, where f_x and $f_{\tilde{x}}$ denote the map f with $s = x$ and $s = \tilde{x}$, respectively.

Now to prove (II$'$), let $x \in I_i$ and fix a sufficiently small $h > 0$. Translate Ω so that $(x, \tilde{L}(x)) = 0$. Let $z_1 = a_1 e_{i1} + a_2 e_{i2}$ and $z_2 = b_1 e_{i1} + b_2 e_{i3}$ be points on

$$\tilde{\Omega}(x, \phi(x,\theta))$$

(See Figure 2).

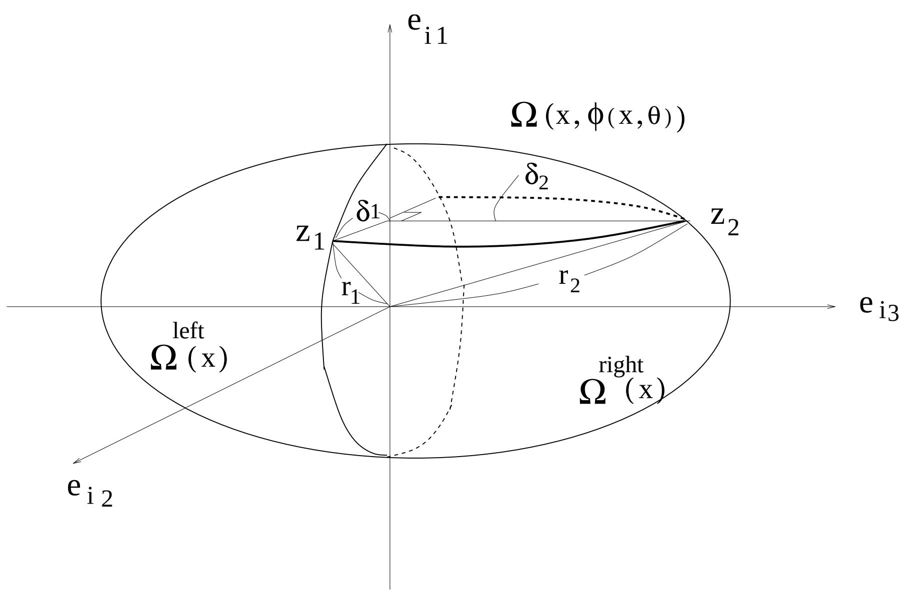

Figure 2

In other words, $z_1 \in \partial\Omega(x)$ is a point on $e_{i1}e_{i2}$ plane such that the angle between e_{i1} and z_1 equals $\phi(x,\theta)$. Similarly, $z_2 \in \partial\Omega(x)$ is the point on $e_{i1}e_{i3}$ plane such that the angle between e_{i1} and z_2 equals $\phi(x,\theta)$. Denote by δ_1, the distance from z_1 to e_{i1}-axis and by δ_2, the distance from z_2 to e_{i1}-axis. Since $\epsilon_{i1} \le \epsilon_{i2} \le \epsilon_{i3}$, one can observe that $\delta_1 \le C\delta_2$ for a dimensional constant C. Also denote $r_1 = |z_1|$ and $r_2 = |z_2|$, then $r_1 \le Cr_2$. Observe that since $\Omega(x,\phi(x,\theta))$ is convex,

$$\epsilon_{i1}\delta_1\delta_2/C \le |\Omega(x,\phi(x,\theta))| \le C\epsilon_{i1}\delta_1\delta_2 \tag{4.11}$$

for a dimensional constant C.

On the other hand, by similar arguments as in (4.3) and (4.4)

$$|\Omega(x) - \Omega(x+h)| \le C2^i h|\Omega(x)| \tag{4.12}$$

for $x \in I_i$ and a small $h > 0$. This implies

$$\epsilon_{i1}\epsilon_{i2}|\tilde{L}_3(x) - \tilde{L}_3(x+h)| \le C2^i h|\Omega(x)|$$

and thus

$$|\tilde{L}_3'(x)| \le C2^i\epsilon_{i3},\ \ |L_1'(x)| \le C2^i\epsilon_{i1},\ \ |L_2'(x)| \le C2^i\epsilon_{i2} \tag{4.13}$$

where the second and third inequalities follow by the same argument as in (4.2). Recall that by definition of $\phi(x,\theta)$ and $\phi(x+h,\theta)$,

$$\frac{|\Omega(x,\phi(x,\theta))|}{|\Omega(x)|} = \frac{|B(\theta)|}{|B_1(0)|} = \frac{|\Omega(x+h,\phi(x+h,\theta))|}{|\Omega(x+h)|}.$$

The above equality and (4.12) imply

$$||\Omega(x,\phi(x,\theta))| - |\Omega(x+h,\phi(x+h,\theta))|| \le C2^i h|\Omega(x,\phi(x,\theta))|. \tag{4.14}$$

To simplify notations, denote

$$\Omega_0 = \Omega(x,\phi(x,\theta)), \quad \Omega_2 = \Omega(x+h,\phi(x+h,\theta))$$

and let Ω_1 be an intermediate region such that

$$\Omega_1 = \Omega^{right}(x) \cap \{y : \angle(\overrightarrow{y}, e_{i1}) \le \phi(x+h,\theta)\}.$$

(Note that Ω_1 is an intermediate region in the sense that Ω_0 and Ω_1 are contained in $\Omega(x)$, but the angle from ϵ_{i1} to $\partial\Omega_1$ is the same as the angle from ϵ_{i1} to $\partial\Omega_2$.) Then by similar reasoning as in (4.3) and (4.4), and by (4.13)

$$|\Omega_1 - \Omega_2| \le C2^i h|\Omega_1| \le C2^i h|\Omega_0|$$

if h is sufficiently small so that $|\phi(x,\theta) - \phi(x+h,\theta)|$ is small enough. Hence

$$\begin{aligned} |\Omega_0 - \Omega_1| = ||\Omega_0| - |\Omega_1|| &\le ||\Omega_0| - |\Omega_2|| + ||\Omega_1| - |\Omega_2|| \\ &\le ||\Omega_0| - |\Omega_2|| + |\Omega_1 - \Omega_2| \\ &\le C2^i h|\Omega_0| \\ &\le C2^i h\epsilon_{i1}\delta_1\delta_2 \end{aligned}$$

where the first inequality is due to the fact that either $\Omega_0 \subset \Omega_1$ or $\Omega_1 \subset \Omega_0$, the third inequality follows from (4.14), and the last equality follows from (4.11). Denote

$$\Delta\phi = |\phi(x+h,\theta) - \phi(x,\theta)|$$

then by the above inequality,

$$|\Delta\phi| r_2^2 \delta_1 \le |\Omega_0 - \Omega_1| \le C2^i h\epsilon_{i1}\delta_1\delta_2$$

and which yields

$$|\Delta\phi| \le C2^i h\epsilon_{i1}\delta_2 / r_2^2. \tag{4.15}$$

Fix $t = (t_1, t_2, t_3) \in \partial B_1(0)$. Without loss of generality, we may assume $t_2, t_3 \geq 0$. Denote by z and w, the points on $\Omega(x)$ and $\Omega(x+h)$ such that

$$z = f(x, t), \quad w = f(x+h, t).$$

(Note that $\angle(e_{i1}, \overrightarrow{z}) = \phi(x, \theta)$ and $\angle(e_{i1}, \overrightarrow{w}) = \phi(x+h, \theta)$ for the angle θ between e_1 and t.) For the proof of (II$'$), it suffices to show

$$|z - w| \leq C2^i h\epsilon_{i3} \leq C2^i h\epsilon. \tag{4.16}$$

To prove (4.16), introduce intermediate points z_1 and z_2 on $\partial\Omega(x)$ such that

$$\angle(e_{i1}, \overrightarrow{z_1}) = \angle(e_{i1}, \overrightarrow{z}), \quad \angle(e_{i2}, \overrightarrow{p(z_1)}) = \angle(e_{i2}, \overrightarrow{p(w)})$$

and

$$\angle(e_{i2}, \overrightarrow{p(z_2)}) = \angle(e_{i2}, \overrightarrow{p(z_1)}), \quad \angle(e_{i1}, \overrightarrow{z_2}) = \angle(e_{i1}, \overrightarrow{w})$$

where p is a projection on the $e_{i2}e_{i3}$-plane. Using the bound (4.15) on $\Delta\phi$, Lemma 4.2 implies

$$\begin{aligned} |z_1 - z_2| &\leq C(r\Delta\phi + r^2\Delta\phi/\epsilon_{i1}) \\ &\leq C2^i h\delta_2 r^2/r_2^2 \\ &\leq C2^i h\epsilon_{i3} \end{aligned} \tag{4.17}$$

where $r = |z_1 - \tilde{L}(x)|$, the second inequality follows from $\epsilon_{i1} \leq Cr$ and the last inequality follows from $r \leq Cr_2$ and $\delta_2 \leq C\epsilon_{i3}$. Moreover, since z_2 and w are points on $\Omega(x)$ and $\Omega(x+h)$ with the same angles from e_{i1} and e_{i2}, (4.13) and a similar argument as in (4.3) yield that

$$|z_2 - w| \leq C2^i h\epsilon_{i3}.$$

Hence it suffices to prove $|z - \tilde{z}| \leq C2^i h\epsilon_{i3}$ for the proof of (4.16).

To simplify notations, denote

$$\theta_2(z) = \angle(e_{i2}, \overrightarrow{p(z)})$$

where p is a projection on $e_{i2}e_{i3}$-plane. Also let

$$\begin{aligned} S_0 &= \{y \in \partial\Omega(x) : \angle(e_{i1}, y) = \phi(x, \theta), 0 \leq \theta_2(y) \leq \pi\}, \\ S_1 &= \{y \in \partial\Omega(x) : \angle(e_{i1}, y) = \phi(x, \theta), 0 \leq \theta_2(y) \leq \theta_2(z)\}, \\ S_0^h &= \{y \in \partial\Omega(x+h) : \angle(e_{i1}, y) = \phi(x+h, \theta), 0 \leq \theta_2(y) \leq \pi\}, \\ S_1^h &= \{y \in \partial\Omega(x+h) : \angle(e_{i1}, y) = \phi(x+h, \theta), 0 \leq \theta_2(y) \leq \theta_2(w)\}. \end{aligned}$$

Then (4.10) implies

$$\frac{\displaystyle\int_{S_1} g_x}{\displaystyle\int_{S_0} g_x} = \frac{\displaystyle\int_{S_1^h} g_{x+h}}{\displaystyle\int_{S_0^h} g_{x+h}}. \tag{4.18}$$

We claim that the denominators in (4.18) are very close to each other, more precisely,

$$|\int_{S_0} g_x - \int_{S_0^h} g_{x+h}| \le C2^i h \int_{S_0^h} g_x.$$

To prove the claim, recall that

$$g_x(y) = Cr(y)^3/|\Omega(x)|$$

where $r(y) = |y - \tilde{L}(x)|$ and C is a dimensional constant. If $y \in S_0$ and $y' \in S_0^h$ are points satisfy $\theta_2(y) = \theta_2(y')$, then

$$\begin{aligned} |g_x(y) - g_{x+h}(y')| &\le |\frac{Cr(y)^3}{|\Omega(x)|} - \frac{Cr(\tilde{y})^3}{|\Omega(x)|}| + C2^i h g_x(y) \\ &= (|1 - \frac{r(\tilde{y})^3}{r(y)^3}| + C2^i h) g_x(y) \\ &\le (\frac{C2^i h \delta_2 r(y)^2}{r_2^2 r(y)} + C2^i h) g_x(y) \\ &\le C2^i h g_x(y) \end{aligned}$$

where the first inequality follows from (4.12), the second inequality follows from a similar argument as in the second inequality of (4.17), and the last inequality follows from $\delta_2 r(y) \le r_2^2$. Thus

$$|\int_{S_0} g_x - \int_{S_0^h} g_{x+h}| \le C2^i h \int_{S_0^h} g_x$$

and due to (4.18)

$$\begin{aligned} |\int_{S_1} g_x - \int_{S_1^h} g_{x+h}| &\le C2^i h \int_{S_1} g_x \\ &= C2^i h \int_{S_1} \frac{r(y)^3}{|\Omega(x)|} \\ &\le C2^i h r(z) \int_{S_1} \frac{r(y)^2}{|\Omega(x)|} \\ &\le \frac{C2^i h r(z)^2 r_1}{|\Omega(x)|} = \frac{C2^i h r^2 r_1}{|\Omega(x)|} \end{aligned} \tag{4.19}$$

where $r = r(z) = |z - \tilde{L}(x)|$. Here the second inequality is from the fact $r(y) \le Cr(z)$ on S_1 (See Figure 2). Thus

$$\frac{r^3|\theta_2(z) - \theta_2(w)|}{|\Omega(x)|} \le C|\int_{S_1} g_x - \int_{S_1^h} g_{x+h}| \le \frac{C2^i h r^2 r_1}{|\Omega(x)|} \tag{4.20}$$

and which yields

$$\Delta\theta_2 := |\theta_2(z) - \theta_2(w)| \le C2^i h r_1 / r. \tag{4.21}$$

By applying Lemma 4.2 on $\partial\Omega(x, \phi(x, \theta))$ with $\epsilon = r_1$, we get

$$\begin{aligned} |z - z_1| &\le C(r\Delta\theta_2 + r^2\Delta\theta_2/r_1) \\ &\le C2^i h r \\ &\le C2^i h \epsilon_{i3}. \end{aligned} \tag{4.22}$$

Combining (4.22) with the bounds on $|z_1 - z_2|$ and $|z_2 - w|$, (4.16) is proved.

Remark Observe that in the construction of f we divided $\Omega(x)$ into $\Omega^{right}(x)$ and $\Omega^{left}(x)$ in a way that any point y on

$$\partial\Omega(x) \cap \partial\Omega^{right}(x) \cap \partial\Omega^{left}(x)$$

has the shortest distance from the center $\tilde{L}(x)$ among the points on

$$\{z \in \partial\Omega(x) : \angle(e_{i1}, \overrightarrow{z}) = \angle(e_{i1}, \overrightarrow{y})\}.$$

In fact, if the shorter arcs were not fixed by f, then our bound in (4.21) would be $C2^i h\tilde{r}/r$ for some $\tilde{r} \ge r_1$, which is not strong enough to obtain the second inequality of (4.22).

4.3 $n > 4$

As in $n = 4$, in each I_i we fix orthonormal basis $e_{i1}, ..., e_{in-1}$ as before (with $\epsilon_{i1} \le ... \le \epsilon_{in-1}$) and construct one-to-one map $f(s, t_1, ..., t_{n-1})$ with a constant Jacobian, under the constraint

$$f(s, \cdot) : B_1(0) \cap \{t_{n-1} \ge 0\} \to \Omega(x) \cap \{y_{n-1} \ge \tilde{L}_{n-1}(x)\},$$

and $f(s, \cdot)$ maps the $(n-2)$-dimensional surface

$$\{t \in B_1(0) : \angle(t, e_k) = \phi_k(y, \theta_k)\}$$

to

$$\{y \in \Omega(x) : \angle(x, e_{ik}) = \phi_k(x, \theta_k)\}.$$

Then with $\theta_1, ..., \theta_{n-3}$ fixed, f is a one-to-one map between two-dimensional surfaces, with shorter arcs (of length r_1) being fixed, and area-preserving with respect to the normalized density function $\frac{g_x}{|\int g_x|}$, where g_x is a density function on $\partial\Omega(x) \subset R^{n-1}$ defined similarly as in (53).

Then parallel arguments as in $n = 4$ would yield the corresponding inequality to (61):

$$|\Delta\phi_k| \leq C2^i h\epsilon_{i1}\delta_{n-2}/r_{n-2}^2$$

where

$$\Delta\phi_k = \phi_k(x, \theta_k) - \phi(x+h, \theta_k) \text{ and } 1 \leq k \leq n-3.$$

Thus we get (4.17) with $\tilde{z}'$ replaced by $\tilde{z}_k'$ and ϵ_{i3} replaced by ϵ_{in-1}, where

$$\angle(\tilde{z}, e_{ik}) = \phi_k(x, \theta_k), \ \ \angle(\tilde{z_k}', e_{ik}) = \phi_k(x+h, \theta_k)$$

and $1 \leq k \leq n-3$.

Similar arguments as in (4.20) would yield that

$$\frac{r^{n-1}|\theta_{n-2}(z) - \theta_{n-2}(w)|}{|\Omega(x)|} \leq C|\int_{S_1} g_x - \int_{S_1^h} g_{x+h}| \leq \frac{C2^i h r^{n-2} r_1}{|\Omega(x)|},$$

which would yield (4.21) and thus (4.22) with ϵ_{i3} replaced by ϵ_{in-1}. Combining these inequalities as in $n = 4$, we obtain (II').

5 Proof of Theorem 1.1

Throughout the proof, all dimensional constants will be denoted by C. We start with assuming that Ω has a smooth boundary - we will consider the general case at the end of the proof. Normalize u and Ω so that $0 < -\min u \leq \max u = 1$ and $N = 1$. In the first part of the proof, the difference between the first eigenvalues λ and μ of (1.1) and (1.4) will be estimated by a bound $C\epsilon$. From this bound, the nodal set will be located in an x- interval of length $C\epsilon$, which is also near the zero s_1 of the first eigenfunction ϕ_1 of the corresponding ordinary differential equation (1.4).

Let $(s, t_1, ..., t_{n-1})$ be the new coordinate system constructed in Lemma 4.1. Using Fubini's Theorem with this new coordinate system, we can choose $(\tilde{t_1}, ..., \tilde{t}_{n-1})$ such that

$$\begin{aligned} \int_\Omega |\nabla u|^2 &= \int_{[0,1]\times B_1(0)} |\nabla u|^2 a_n w(s) ds...dt_{n-1} \\ &\geq \int_0^1 |\nabla u(s, f(s, \tilde{t_1}, ..., \tilde{t}_{n-1}))|^2 w(s) ds. \end{aligned} \quad (5.1)$$

Based on (5.1), we will construct a one-dimensional test function ψ to compare with ϕ_1 as follows. Let k_0 be the smallest integer such that $2^{-k_0} \le \epsilon$ and let

$$J_i = [2^{-i-1}, 2^{-i}] \cup [1 - 2^{-i}, 1 - 2^{-i-1}] \text{ for } 1 \le i \le k_0.$$

Define a function $\tilde{\phi}$ in Ω such that

(i) $\tilde{\phi}(z) = \tilde{\phi}(w)$ for $z,\ w \in \Omega(x)$, $0 \le x \le 1$

(ii) $\tilde{\phi} = u$ on $\{(s, f(s, \tilde{t_1}, ..., \tilde{t}_{n-1})) : 2^{-k_0} \le s \le 1 - 2^{-k_0}\}$

(iii) $\tilde{\phi} = \tilde{\phi}(2^{-k_0}, f(2^{-k_0}, \tilde{t}_1, ..., \tilde{t}_{n-1}))$ for $0 \le x \le 2^{-k_0}$

(iv) $\tilde{\phi} = \tilde{\phi}(1 - 2^{-k_0}, f(1 - 2^{-k_0}, \tilde{t}_1, ..., \tilde{t}_{n-1}))$ for $1 - 2^{-k_0} \le x \le 1$.

Observe that $\tilde{\phi}$ is continuous on J_i and may have a jump discontinuity at endpoints of J_i. But since

$$|\nabla u| \le C2^{-i} \text{ on } J_i$$

due to Corollary 2.4, and since $\Omega(x)$ has a diameter less than 2ϵ,

$$u(z) - u(w) \le C2^{-i}\epsilon \text{ for } z, w \in \Omega(x). \tag{5.2}$$

(5.2) implies that there exists a continuous function ϕ such that for $1 \le i \le k_0$

$$\phi = \tilde{\phi} + d_i \text{ on } [2^{-i-1}, 2^{-i}], \quad d_i \text{ is a constant with } |d_i| \le C\epsilon$$

and similarly on $[1-2^{-i}, 1-2^{-i-1}]$. In other words, ϕ is constructed so that ϕ is continuous, $\nabla\phi = \nabla\tilde{\phi}$ on J_i and

$$1 - C\epsilon \le \frac{\displaystyle\int_\Omega |\phi|^2}{\displaystyle\int_\Omega |\tilde{\phi}|^2} \le 1 + C\epsilon. \tag{5.3}$$

Now we will obtain an estimate on the first eigenvalue λ ((b) of Theorem 1.1) using a function perturbed from ϕ. On J_i,

$$\begin{aligned} |\nabla\phi| = |\nabla\tilde{\phi}| &= |\partial\tilde{\phi}/\partial x| \\ &\le (1 + |\partial_s y_j|)|\nabla u(s, f(s, \tilde{t_1}, ..., \tilde{t}_{n-1}))| \\ &\le (1 + C2^i\epsilon)v(s) \end{aligned}$$

where $v(s)$ is a one-dimensional function in Ω such that

$$v(s) = |\nabla u(s, f(s, \tilde{t_1}, ..., \tilde{t}_{n-1})|.$$

Hence

$$\begin{aligned}
\int_\Omega |\nabla\phi|^2 &= \int_{\Omega(\cup_{1\le i<k_0} J_i)} |\nabla\phi|^2 \\
&\le \int_\Omega v^2 + \sum_{i=1}^{k_0-1} C\epsilon 2^i \int_{\Omega(J_i)} v^2 \\
&\le \int_\Omega v^2 + C\sum_{k=1}^{k_0-1} \epsilon 2^{-i} \int_{\Omega(J_i)} 1 \\
&\le (1+C\epsilon)\int_\Omega |\nabla u|^2 + C\epsilon|\Omega| \qquad (5.4) \\
&\le (1+C\epsilon)\int_\Omega |\nabla u|^2
\end{aligned}$$

where $\Omega(J_i) := \Omega\cap\{x \in J_i\}$, the second inequality follows from Corollary 2.4 and the third inequality follows from (5.1). On the other hand, by (5.2), Corollary 2.5 and (5.3)

$$1 - C\epsilon \le \frac{\int_\Omega \phi^2}{\int_\Omega u^2} \le 1 + C\epsilon. \tag{5.5}$$

Therefore (5.4) and (5.5) imply

$$\frac{\int_\Omega |\nabla\phi|^2}{\int_\Omega \phi^2} \le (1+C\epsilon)\frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} \tag{5.6}$$

Since ϕ does not satisfy the constraint $\int_\Omega \phi = 0$, we construct ψ by perturbing ϕ, which satisfies $\int_\Omega \psi = 0$ as well as (5.6). From (5.2) and $\int_\Omega u = 0$,

$$|\int_\Omega \phi| \le C\epsilon|\Omega|. \tag{5.7}$$

Let $z = (z_1, ..., z_n)$ be a point on

$$\{(s, f(s, f(s, \tilde{t_1}, ..., \tilde{t}_{n-1})) : 0 \le s \le 1\}$$

such that $u(z) = 0$. (Note that $z = (z_1, ..., z_n) \in \Lambda$.) Then by Corollary 2.5, $c_1 \le z_1 \le 1 - c_1$ for a dimensional constant $c_1 > 0$. This bound on z_1 and (5.7) imply that there exists c_0 such that $\psi(x) := \phi(x) + c_0\epsilon(x - z_1)^+$ satisfies $\int_\Omega \psi = 0$ with $|c_0| < C$ for a dimensional constant C. Observe that (5.6) also holds for ψ, and thus

$$\mu_1 \le \frac{\int_\Omega |\nabla\psi|^2}{\int_\Omega \psi^2} \le (1 + C\epsilon)\frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} = (1 + C\epsilon)\lambda \tag{5.8}$$

which proves part (b) of Theorem 1.1.

For the proof of part (a), we first show that the projection of the nodal set Λ of u onto x axis is contained in an interval of length $C\epsilon$ and then we locate Λ near the zero of ϕ_1. Let p be a projection on the x-axis. Then $p(\bar{\Omega}_+)$ and $p(\bar{\Omega}_-)$ are intervals because the Courant nodal domain theorem [CH, p.452] implies Ω_+ and Ω_- are connected. Hence $p(\Lambda) = [a, b]$ for some $a \le b$. By (5.2),

$$\max_{[a,b]} |u| < C\epsilon$$

and since $\max |\psi - u| \le C\epsilon$,

$$\sup_{[a,b]} |\psi| < C\epsilon. \tag{5.9}$$

By Corollary 2.5, $|\int_a^1 \phi w ds| > C\int_0^1 w ds$ for a dimensional constant $C > 0$. Hence Lemma 3.6 with (5.8) and (5.9) implies that $b - a < C\epsilon$ for a dimensional constant C. In other words, the nodal set Λ is contained in an x-interval of length $C\epsilon$.

Now, observe

$$
\begin{aligned}
\frac{\displaystyle\int_{z_1}^{1} \psi'^2 w ds}{\displaystyle\int_{z_1}^{1} \psi^2 w ds} &\le (1+C\epsilon)\frac{\displaystyle\int_{z_1}^{1} \phi'^2 w ds}{\displaystyle\int_{z_1}^{1} \phi^2 w ds} \le (1+C\epsilon)\frac{\displaystyle\int_{\Omega'} |\nabla u|^2}{\displaystyle\int_{\Omega'} u^2} \\
&\le (1+C\epsilon)\frac{\displaystyle\int_{\Omega_+} |\nabla u|^2}{\displaystyle\int_{\Omega_+} u^2} = (1+C\epsilon)\lambda \le (1+C\epsilon)\mu_1
\end{aligned}
$$

where $\Omega' = \Omega \cap \{z_1 \le x \le 1\}$, $\psi(z_1) = \phi(z_1) = 0$, the second inequality follows from a similar argument as in (5.6) and the third inequality follows from $b - a < C\epsilon$, $|u| \le 1$ and $|\nabla u| \le C$. By Lemma 3.5, $z_1 \le s_1 + C\epsilon$. By a similar argument on the interval $[0, z_1]$, we obtain $s_1 - C\epsilon \le z_1$. Since the length of the projection of the nodal set on the x-axis is less than $C\epsilon$, part (a) is proved.

Lastly we discuss the general case. For a general domain Ω, let $\{\Omega_k\}_k$ be an increasing sequence of smooth domains which converges to Ω uniformly on each cross sections $\Omega_k(x_1)$. Let u_k be the corresponding first nonzero eigenfunctions of Ω_k with $\sup |u_k| = 1$. Then by Kröger's theorem (Theorem 2.1), $\sup_{\Omega_k} |\nabla u_k|$ is uniformly bounded. Hence there exists a subsequence $\{u_{k_j}\}_j$ which converges uniformly to u and $\{\lambda_{k_j}\}_j$ converges to λ as $j \to \infty$. On the other hand, since the volume $w_k(x)$ of $\Omega_k(x)$ uniformly converges to $w(x)$, we may assume that $\{\phi_{k_j}\}_j$ converges uniformly to ϕ_1 and $\{\mu_{k_j}\}_j$ converge to μ as $j \to \infty$. Now by the nondegeneracy of ϕ_{k_j} (Lemma 3.7) and the nondegeneracy of u_{k_j} in the scale ϵ (Lemma 3.6), we obtain Theorem 1.1 for u and λ.

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