Homogenization of Neumann boundary data with fully nonlinear operator

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Abstract

In this paper we study periodic homogenization problems for solutions of fully nonlinear PDEs, in half-spaces with oscillatory Neumann boundary data. We show the existence, uniqueness of the homogenized Neumann data for a given half-space. Moreover, we show that there exists a continuous extension of the homogenized slope as the normal of the half-space varies over "irrational" directions.

1 Introduction

In this paper, we consider the averaging phenomena for solutions of uniformly elliptic nonlinear PDEs, in half-spaces coupled with oscillatory Neumann boundary data. To be precise, let $\mathcal{M}^{n-1}$ be the normed space of symmetric $n \times n$ matrices and consider the function $F(M) : \mathcal{M}^{n-1} \to I\!R$ which satisfies

(F1) F is uniformly elliptic, i.e., there exists constants $0 < \lambda < \Lambda$ such that

$$\lambda \|N\| \leq F(M) - F(M+N) \leq \Lambda \|N\| \text{ for any } N \geq 0;$$

(F2) (homogeneity) $F(tM) = tF(M)$ for any $M \in \mathcal{M}^{n-1}$ and $t > 0$. In particular $F(0) = 0$;

The homogeneity condition (F2) can be relaxed (e.g. see (F4) of [3]). Typical examples of nonlinear operators which satisfy (F1)-(F3) are the Pucci extremal operators:

$$\mathcal{P}^+(D^2u(x)) := \lambda \sum_{\mu_i<0} \mu_i + \Lambda \sum_{\mu_i \geq 0} \mu_i; \quad \mathcal{P}^-(D^2u(x)) := \Lambda \sum_{\mu_i<0} \mu_i + \lambda \sum_{\mu_i \geq 0} \mu_i$$

where $\mu_1, \cdots, \mu_n$ are eigenvalues of $D^2u(x)$.

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Let $\{(e_1, ..., e_n)\}$ be an orthonormal basis of $I\!R^n$ and suppose $g(x) : I\!R^n \to I\!R$ satisfies

(a) $g \in C^\beta(I\!R^n)$, for some $0 < \beta \leq 1$;

(b) $g(x + e_k) = g(x)$ for $x \in I\!R^n$ for $k = 1, ..., n$.

Next, for given $p \in I\!R^n$ let $\Pi_\nu(p)$ be a strip domain in $I\!R^n$ with unit normal ν, that is

$$\Pi_\nu(p) = \{x : -1 \leq (x-p)\cdot\nu \leq 0\}, \text{ where } |\nu| = 1. \tag{1}$$

With F, g and Π_ν be as given above, our goal is to describe the limiting behavior of u_ϵ as $\epsilon \to 0$, where u_ϵ satisfies

$$(P_\epsilon) \qquad \begin{cases} F(D^2 u_\epsilon) = 0 & \text{in} \quad \Pi_\nu(p) \\ \nu \cdot Du_\epsilon = g(\frac{x}{\epsilon}) & \text{on} \quad \Gamma_0 := \{(x-p)\cdot\nu = 0\}. \\ u = 1 & \text{on} \quad \Gamma_I := \{(x-p)\cdot\nu = -1\}. \end{cases}$$

The fixed boundary data on Γ_I is introduced to avoid discussion of the compatibility condition on g and to ensure the existence of u^ϵ.

Homogenization of elliptic, divergence-form equations, with oscillatory coefficients and co-normal boundary data, is a classical subject. Let Ω be an open and bounded subset of $I\!R^n$. Consider $u^\epsilon : \bar{\Omega} \to I\!R$ solving

$$\nabla \cdot (A(\frac{x}{\epsilon})\nabla u_\epsilon) = 0, \tag{2}$$

with the Neumann (co-normal) condition

$$\nu \cdot (A(x/\epsilon)\nabla u)(x) = g(x/\epsilon), x \in \partial\Omega. \tag{3}$$

The problem (2)-(3) has been widely studied and by now has been well-understood (see [4] for an overview): first let us consider the case when Ω is a half-space: i.e. let

$$\Omega = \Sigma_\nu := \{x : (x-p)\cdot\nu \leq 0\}.$$

We define the averaged Neumann data

$$\mu(\nu, \epsilon) := \int_{(x-p)\cdot\nu=0, |x-p|\leq 1} g(\frac{x}{\epsilon})dx. \tag{4}$$

By integration by parts one can show that u^ϵ locally uniformly converges to a continuous function $u^0 : \bar{\Omega} \to I\!R$ as $\epsilon \to 0$ if and only if $\mu(\nu) := \lim_{\epsilon\to 0}\mu(\nu, \epsilon)$ exists, and that u^0 solves the averaged equation

$$(\bar{P}_{div}) \qquad \begin{cases} -\nabla \cdot (A^0 \nabla u^0)(x) = 0 & \text{for } x \in \Omega, \\ \nu \cdot (A^0 \nabla u^0) = \mu(\nu) & \text{for } x \in \partial\Omega. \end{cases}$$

Therefore different results hold depending on the choice of p and ν:

(a) If ν is parallel to a vector in $\mathbb{Z}^n$ (i.e. ν is a "rational" vector), then $\mu(\nu)$ exists if $p = 0$, and

$$\mu(\nu) = \text{ the average of } g(y) \text{ on the hyperplane } \{x \cdot \nu = 0\}.$$

(b) If ν is a rational vector and $p \neq 0$, then there may be no limit of $\mu(\nu, \epsilon)$ and u^ϵ can have different subsequential limits.

(c) If ν is not a rational vector, then due to Weyl's equi-distribution theorem (Lemma 2.5) $\mu(\nu, \epsilon)$ converges to

$$\mu(\nu) = < g > := \int_{[0,1]^n} g(y)dy,$$

independent of the choice of p. In particular, the homogenized slope $\mu(\nu)$ is discontinuous at every rational direction ν, but otherwise continuous.

From above results, the divergence form of the operator, and the fact that rational directions are of zero measure in $\mathcal{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$, the following results hold for the general domain Ω: if $\partial\Omega$ does not contain flat pieces whose normal vectors belong to $\mathbb{R}\mathbb{Z}^n$, then u^ϵ converges locally uniformly to the solution u^0 of $(\bar{P}_{div})$ with $\mu(\nu)$ replaced by $< g >$. We refer to [4] for detailed analysis. Note that u^0 is smooth up to the boundary due to the fact that $< g >$ is continuous (constant in particular).

For nonlinear or non-divergence type operators, or for linear operators with oscillatory nonlinear boundary data, little is known for the homogenization of the oscillating Neumann boundary data. Most available results concern half-space domains going through the origin with its normal pointing to a rational direction. In [14], Tanaka considered some model problems in half-space whose boundary is parallel to the axes of the periodicity by purely probabilistic methods. In [2], Arisawa studied special cases of problems in oscillatory domains near half spaces going through the origin, using viscosity solutions as well as stochastic control theory. Generalizing the results of [2], Barles, Da Lio and Souganidis [3] studied the problem for operators with oscillating coefficients, in half-space type domains whose boundary is parallel to the axes of periodicity, with a series of assumptions which guarantee the existence of approximate corrector.

In this paper we extend above results to the setting of general half-spaces Π_ν, defined in (1), where p is not necessarily zero and ν ranges over all directions in $\mathbb{R}^n$. In particular we show the continuity properties of the homogenized slope $\mu(\nu)$ over the normal directions ν (see Theorem 1.2 (ii)), with the hope that such results will lead to better understanding of homogenization phenomena in domains with general geometry (work in progress). Note that, as observed in the linear case, homogenized slope may not exist if ν is parallel to a vector in $\mathbb{Z}^n$ and if $p \neq 0$, therefore the best result we can hope for is the existence of the continuous function $\bar{\mu}(\nu) : S^{n-1} \to I\!R$ such that $\bar{\mu}(\nu) = \mu(\nu)$ for $\nu \in \mathcal{S}^{n-1} - I\!R\mathbb{Z}^n$. This is precisely what we will show.

Before stating the main theorem, let us introduce some notations.

Definition 1.1.

1. *$\nu \in \mathcal{S}^{n-1}$ is a rational direction if $\nu \in I\!R\mathbb{Z}^n$.*
2. *$\nu \in \mathcal{S}^{n-1}$ is an irrational direction if ν is not a rational direction.*

Theorem 1.2 (Main Theorem)**.** *For a given $p \in I\!R^n$, let u_ϵ solve (P_ϵ). Then the following holds:*

(i) Let ν be an irrational direction. Then there is a unique constant $\mu(\nu) \in [\min g, \max g]$ such that u^ϵ locally uniformly converges to the solution of

$$(\bar{P}) \qquad \begin{cases} F(D^2u) = 0 & \text{in} \quad \Pi_\nu \\ \nu \cdot Du = \mu(\nu) & \text{on} \quad \Gamma_0 \\ u = 1 & \text{on} \quad \Gamma_I. \end{cases}$$

(ii) $\mu(\nu) : (\mathcal{S}^{n-1} - I\!R\mathbb{Z}^n) \to I\!R$ has a continuous extension $\bar{\mu}(\nu) : \mathcal{S}^{n-1} \to I\!R$.

(iii) For rational directions ν, if Γ_0 goes through the origin (that is if $p = 0$), then the statement in (i) holds for ν as well.

(iv) [Error estimate] Let ν be an irrational direction. Then for u^ϵ and u solving (P_ϵ) and $(\bar{P})$, we have the following estimate: for any $0 < \alpha < 1$, there exists a constant $C_\alpha > 0$ such that

$$|u^\epsilon - u| \leq C_\alpha \omega(\epsilon)^\alpha \text{ in } \Pi_\nu. \tag{5}$$

Here $\omega(\epsilon)$ depends on the "discrepancy" associated to ν as defined in (7).

Remark 1.3. *Our method can be applied to the operators of the form $F(D^2u, x) = f(x)$ with F and f continuous in x, but we will restrict ourselves to the simple case discussed in (P_ϵ) for the clarity of exposition. On the other hand, our proof for the continuity of $\mu(\nu)$ (Theorem 1.2 (ii)), presented in section 4.2, cannot handle the case where the operator F depends on the oscillatory variable x/ϵ (see Remark 4.9).*

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2 Preliminary results

Let Ω be an open, bounded domain. Let Γ_I be a part of its boundary, and define $\Gamma_0 := \partial\Omega - \Gamma_I$. For a continuous function $f(x,\nu) : I\!R^n \times \mathcal{S}^{n-1} \to I\!R$, let us recall the definition of viscosity solutions for the following problem:

$$(P)_f \qquad \begin{cases} F(D^2u) = 0 & \text{in} \quad \Omega; \\ \nu \cdot Du = f(x,\nu) & \text{on} \quad \partial\Omega - \Gamma_I; \\ u = 1 & \text{on} \quad \Gamma_I, \end{cases}$$

where $\nu = \nu_x$ denotes the outward normal at $x \in \partial\Omega$ with respect to Ω.

The following definition is equivalent to the ones given in [8]:

Definition 2.1. *(a) An upper semi-continuous function $u : \bar{\Omega} \to I\!R$ is a* viscosity subsolution *of $(P)_f$ if*

(i) $u \leq 1$ on Γ_I;

(ii) For given domain $\Sigma \subset I\!R^n$, u cannot cross from below any C^2 function ϕ in Σ which satisfies

$$\begin{cases} F(D^2\phi) > 0 & in \quad \Omega \cap \Sigma; \\ \nu \cdot D\phi > f(x,\nu) & on \quad \Gamma_0 \cap \Sigma; \\ \quad \phi > u & on \quad (\partial\Sigma \cup \Gamma_I) \cap \bar{\Sigma}. \end{cases}$$

(b) A lower semi-continuous function $u : \bar{\Omega} \to I\!R$ is a viscosity supersolution *of $(P)_f$ if*

(i) $u \geq 1$ on K,

(ii) For a given domain $\Sigma \subset I\!R^n$,u cannot cross from above any C^2 function φ which satisfies

$$\begin{cases} F(D^2\phi) < 0 & in \quad \Omega \cap \Sigma; \\ \nu \cdot D\phi < f(x,\nu) & on \quad \Gamma_0 \cap \Sigma; \\ \quad \phi < u & on \quad (\partial\Sigma \cup \Gamma_I) \cap \bar{\Sigma}. \end{cases}$$

(c) u is a viscosity solution *of $(P)_f$ if u is both a viscosity sub- and supersolution of $(P)_f$.*

Existence and uniqueness of viscosity solutions of $(P)_f$ is based on the comparison principle we state below:

Theorem 2.2 (Section V, [10])**.** *Suppose Ω,Γ_I, Γ_0, F and ν are as given above, and let $f : I\!R^n \times \mathcal{S}^{n-1} \to I\!R$ be continuous. Let u and v respectively be a viscosity sub- and supersolution of $(P)_f$ in a domain $\Sigma \subset I\!R^n$. If $u \leq v$ on $\partial\Sigma$, then $u \leq v$ in Ω.*

For details on the proof of the above theorem as well as well-posedness of the problem $(P)_f$, we refer to [8],[9], [10].

Next we state some regularity results that will be used in the paper.

Theorem 2.3. *[Chapter 8, [5], modified for our setting] Let u be a viscosity solution of $F(D^2u) = 0$ in a domain Ω. Then for any $0 < \alpha < 1$ and for any compact subset Ω' of Ω, we have*

$$\|u\|_{C^\alpha(\Omega')} \leq Cd^{-\alpha}\|u\|_{L^\infty(\Omega)}$$

where $C > 0$ depends on n, λ, Λ and $d = d(\Omega', \partial\Omega)$.

Theorem 2.4 (Theorem 8.1 and Theorem 8.2 in [13])**.** *Let*

$$B_r^+ := \{|x| < r\} \cap \{x \cdot e_n \geq 0\} \text{ and } \quad \Gamma := \{x \cdot e_n = 0\} \cap B_1.$$

Let u be a viscosity solution of

$$\begin{cases} F(D^2u) = 0 & in \quad B_1^+ \\ \nu \cdot Du = g & in \quad \Gamma. \end{cases}$$

(a) *Suppose g is bounded. Then u is in $C^\alpha(\overline{B_{1/2}^+})$ for some $\alpha = \alpha(n, \lambda, \Lambda)$. Moreover, we have the estimate*

$$\|u\|_{C^\alpha(\overline{B_{1/2}^+})} \leq C(\|u\|_{L^\infty(\overline{B_1^+})} + \max \|g\|).$$

(b) *Suppose $g \in C^\beta(\mathbb{R}^n)$ where $0 < \beta \leq 1$. Then u is in $C^{1,\gamma}(\overline{B_{1/2}^+})$ where $\gamma = \min(\alpha_0, \beta)$ and $\alpha_0 = \alpha_0(n, \lambda, \Lambda)$. Moreover, we have the estimate*

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}^+})} \leq C(\|u\|_{L^\infty(\overline{B_1^+})} + \|g\|_{C^\beta})$$

In (a)-(b), C denotes a positive constant depending only on n, λ, Λ and α.

Let us next discuss the averaging property of the sequence $(nx)_n$ mod 1, where x is an irrational number, and its applications to multi-dimensions which will serve useful in our analysis in section 3. Since we obtain estimates on the convergence rate of solutions for (P_ϵ) in our result, we are particularly interested in the estimates on the rate of convergence of the sequence $(nx)_n$ to the uniform distribution (Definition 2.6). We begin with recalling the notion of equi-distribution.

• A bounded sequence $(x_1, x_2, x_3...)$ of real numbers is said to be *equi-distributed* on an interval $[a, b]$ if for any $[c, d] \subset [a, b]$ we have

$$\lim_{n\to\infty} \frac{|\{x_1, ..., x_n\} \cap [c, d]|}{n} = \frac{d-c}{b-a}.$$

Here $|\{x_1, ..., x_n\} \cap [c, d]|$ denotes the number of elements.

• The sequence $(x_1, x_2, x_3, ...)$ is said to be equidistributed modulo 1 if $(x_1 - [x_1], x_2 - [x_2], ...)$ is equidistributed in the interval $[0, 1]$.

Lemma 2.5 ([15], Weyl's equidistribution theorem)**.** *If a is an irrational number, $(a, 2a, 3a, ...)$ is equidistributed modulo 1.*

To discuss quantitative versions of Lemma 2.5, we introduce the notion of discrepancy. The following definition is from the book [12].

Definition 2.6. *Let $(x_k), k = 1, 2, ...,$ be a sequence in $\mathbb{R}$. For a subset $E \subset [0, 1]$, let $A(E; N)$ denote the number of points $\{x_n\}$, $1 \le n \le N$, that lie in E.*

(a) The sequence (x_n), $n = 1, 2, ...$ is said to be uniformly distributed mode 1 in $\mathbb{R}$ if

$$\lim_{N \to \infty} \frac{A(E; N)}{N} = \mu(E)$$

for all $E = [a, b)$. Here μ denotes the Lebesgue measure.

(b) For $x \in [0, 1]$, let us define the discrepancy

$$D_N(x) := \sup_{E=[a,b)} \Big| \frac{A(E; N)}{N} - \mu(E) \Big|,$$

where $A(E; N)$ is defined with the sequence $(kx), k \in \mathbb{N}$, modulo 1.

It easily follows from Lemma 2.5 that the sequence $(x_k) = (kx)_{k \in \mathbb{N}}$ is uniformly distributed modulo 1 for any irrational number $x \in \mathbb{R}$. In particular $D_N(x)$ converges to zero as $N \to \infty$.

Next, let $\mathcal{S}^{n-1} = \{\nu \in \mathbb{R}^n : |\nu| = 1\}$. For a direction $\nu = (\nu_1, ..., \nu_n) \in \mathcal{S}^{n-1}$, let ν_i be the component with the biggest size, i.e.,

$$|\nu_i| = \max\{|\nu_j| : 1 \le j \le n\}$$

(If there are multiple components then we choose the one with largest index).

Let H_ν be the hyperplane in $\mathbb{R}^n$, which passes through 0 and is normal to ν, i.e.,

$$H_\nu = \{x \in \mathbb{R}^n : x \cdot \nu = 0\}.$$

Since $\nu_i \neq 0$, there exists $m(\nu)$ such that $(1, ..., 1, m(\nu), 1, ..., 1) \in H_\nu$, i.e.,

$$(1, ..., 1, m(\nu), 1, ..., 1) \cdot \nu = 0 \tag{6}$$

where $m(\nu)$ is the i-th component of $(1, ..., 1, m(\nu), 1, ..., 1)$. Then we define

$$\omega_\nu(\epsilon) := D_N(m(\nu)), \text{ where } N = \epsilon^{-9/10}. \tag{7}$$

Note that, if $m(\nu)$ is irrational, then $\omega_\nu(\epsilon) \to 0$ as $\epsilon \to 0$.

Now we are ready to state our quantitative estimate on the averaging properties of the vector sequence $(n\nu)$ with an irrational direction ν, which will be used in the rest of the paper. Recall that for $\nu \in \mathcal{S}^{n-1}$,
$\Pi_\nu(p) = \{x : -1 \le (x - p) \cdot \nu \le 0\}$. Denote $\Gamma_0 = \{x : (x - p) \cdot \nu = 0\}$ and define

$$H_\nu = \{x : x \cdot \nu = 0\}.$$

Lemma 2.7. *For $\nu \in \mathbb{R}^n$ and $x_0 \in \Pi_\nu$, let $H(x_0) := H_\nu + x_0$. Let $0 < \epsilon < \text{dist}(x_0, \Gamma_0)$.*

(i) Suppose that ν is a rational direction. Then for any $x \in H(x_0)$, there is $y \in H(x_0)$ such that

$$|x - y| \leq M_\nu \epsilon; \;\; y - x_0 \in \epsilon \mathbb{Z}^n$$

where $M_\nu > 0$ is a constant depending on ν.

(ii) Suppose that ν is an irrational direction, and let $w : [0, 1) \to \mathbb{R}^+$ defined as in (7). Then there exists a dimensional constant $M > 0$ such that such that the following is true: for any $x \in H(x_0)$, there is $y \in \mathbb{R}^n$ such that

$$|x - y| \leq M \epsilon^{1/10}; \;\; y - x_0 \in \epsilon \mathbb{Z}^n$$

and

$$\text{dist}(y, H(x_0)) < \epsilon \omega_\nu(\epsilon), \tag{8}$$

where ω_ν is as given in (7).

(iii) If ν is an irrational direction, then for any $z \in \mathbb{R}^n$ and $\delta > 0$, there is $w \in H(x_0)$ such that

$$|z - w| \leq \delta \;\; \textit{mod} \;\; \epsilon \mathbb{Z}^n.$$

Proof. The proof of (i) is immediate from the fact that for any rational direction ν, there exists an integer $M > 0$ depending on ν such that $M\nu \in \mathbb{Z}^n$.

Next, we prove (ii). Let ν be an irrational direction in $\mathbb{R}^n$. Without loss of generality, we may assume

$$|\nu_n| = \max\{|\nu_j| : 1 \leq j \leq n\}.$$

Let x be any point on $H(x_0)$: after a translation we may assume that $x = 0$. Choose m such that

$$\epsilon(1, 1, .., 1, m) \in H(x_0).$$

Note that $M = |m| \leq n^2$. Also note that m is irrational since ν is an irrational direction. Since $H(x_0)$ contains $x = 0$,

$$k\epsilon(1, 1, .., 1, m) \in H(x_0) \text{ for any integer } k.$$

Consider the sequence (km), $k \in \mathbb{N}$. From the definition of $\omega(\epsilon)$ and the discrepancy function $D_N(m)$, it follows that any interval $[a, b] \subset [0, 1]$ of length $\omega(\epsilon)$ contains at least one point km (mod 1), for some $k \leq N = \epsilon^{-9/10}$.

Hence for any

$$z = (0, 0, ..., 0, x_n) \in [0, \epsilon]^n,$$

there exists

$$w = k\epsilon(1, 1, .., 1, m) \in H(x_0), \;\; 0 \leq k \leq \epsilon^{-9/10}$$

such that

$$|z - w| \leq \epsilon\omega(\epsilon) \mod \epsilon\mathbb{Z}^n.$$

Similarly, for any $z \in [0, \epsilon]^n$, there exists $w \in H(x_0) \cap (k\epsilon(1, 1, ..., 1, m) + [0, \epsilon]^n)$ such that

$$|z - w| \leq \epsilon\omega(\epsilon) \text{ mod } \epsilon\mathbb{Z}^n; \quad 0 \leq k \leq \epsilon^{-9/10}. \tag{9}$$

We continue with the proof of (ii). Recall that the coordinates are shifted so that $x = 0$. Thus it suffices to find $y \in \mathbb{R}^n$ such that

$$|x - y| = |y| \leq M\epsilon^{1/10}; \ \ |y - x_0| = 0 \text{ mod } \epsilon\mathbb{Z}^n$$

and

$$\text{dist}(y, H(x_0))| < \epsilon\omega_\nu(\epsilon).$$

By (9), there exists $w \in H(x_0)$ such that

$$|x - w| = |w| \leq Mk\epsilon \leq M\epsilon^{1/10} \tag{10}$$

and

$$|x_0 - w| \leq \epsilon\omega_\nu(\epsilon) \ \text{ mod } \ \epsilon\mathbb{Z}^n. \tag{11}$$

Given w satisfying (11), we can take $y \in \mathbb{R}^n$ such that

$$|x_0 - y| = 0 \text{ mod } \epsilon\mathbb{Z}^n, \text{ and } |y - w| \leq \epsilon\omega_\nu(\epsilon).$$

Then by (10)

$$|y| \leq |y - w| + |w| \leq M\epsilon^{1/10} + \epsilon\omega_\nu(\epsilon) \leq M\epsilon^{1/10}.$$

Also since w is contained in $H(x_0)$,

$$\text{dist}(y, H(x_0)) \leq |y - w| \leq \epsilon\omega_\nu(\epsilon)$$

(iii) is a direct consequence of (9). □

3 In the strip domain

Fix $p \in \mathbb{R}^n$ and $\nu \in \mathcal{S}^{n-1}$ such that $p \cdot \nu \neq 0$. Let

$$\Pi = \Pi_\nu = \{x \in \mathbb{R}^n : -1 \leq (x - p) \cdot \nu \leq 0\}$$

We consider a bounded viscosity solution u_ϵ of

$$(P_\epsilon) \qquad \begin{cases} F(D^2 u_\epsilon) = 0 & \text{in} \quad \Pi \\ \partial u_\epsilon / \partial\nu = g(x/\epsilon) & \text{on} \quad \Gamma_0 := \{x : (x - p) \cdot \nu = 0\} \\ u_\epsilon = 1 & \text{on} \quad \Gamma_I := \{x : (x - p) \cdot \nu = -1\}. \end{cases}$$

Below we prove existence and uniqueness of u_ϵ.

Lemma 3.1. *Let $f(x) : I\!R^n \to I\!R$ be continuous and bounded. Let Π_ν be as given above and define $B_R(p) := \{|x-p| \leq R\}$. Suppose w_1 and w_2 solve, in the viscosity sense,*

(a) $F(D^2 w_1) \leq 0, \quad F(D^2 w_2) \geq 0 \quad \text{in} \quad \Sigma_R := \Pi_\nu(p) \cap B_R(p);$

(b) $\partial w_1/\partial\nu \leq f(x) \leq \partial w_2/\partial\nu \quad \text{on} \quad \Gamma_0;$

(c) $w_1 = w_2 \quad \text{on} \quad \Gamma_I;$

(d) $w_1 = -M,\ w_2 = M \quad \text{on} \quad \Pi \cap B_R(p).$

Then, for $R > 2$ and $C = \frac{n\Lambda}{\lambda}$ we have

$$w_1 \leq w_2 \leq w_1 + \frac{3CM}{R^2} \quad \text{in} \quad \Pi \cap B_1(p).$$

Proof. Without loss of generality, let us set $\nu = e_n$ and $p = 0$. The first inequality, $w_1 \leq w_2$, directly follows from Theorem 2.2 To show the second inequality, consider $\tilde{\omega} := w_1 + M(h_1 + h_2)$, where

$$h_1 = \frac{1}{R^2}((x_1)^2 + ... + (x_n)^2) \text{ and } h_2 = \frac{C}{R^2}(1 - (x_n)^2)$$

with $C = \dfrac{n\Lambda}{\lambda}$. We claim $w_2 \leq \tilde{\omega}$. To see this, note that

$$\begin{aligned} F(D^2\tilde{\omega}) &= F(D^2 w_1 + D^2 h_1 + D^2 h_2) \\ &\geq F(D^2 w_1) + \frac{2}{R^2}(C\lambda - n\Lambda) \geq F(D^2 w_1) \text{ in } \Sigma_R. \end{aligned}$$

On the boundary of Σ_R, $\tilde{\omega}$ satisfies

$$\partial_{x_n}\tilde{\omega} = \partial_{x_n}\omega_1 = \partial_{x_n}\omega_2 \text{ on } \Sigma_R \cap \{x_n = 0\}$$

and

$$w_2 \leq \tilde{\omega} \quad \text{on } \Gamma_0 \cap B_R(0) \text{ and on } \partial B_R(0) \cap \Pi_\nu(p).$$

It follows from Theorem 2.2 that $w_2 \leq \tilde{\omega}$ in Σ_R, and we are done. □

Lemma 3.2. *There exists a unique bounded solution u of (P_ϵ).*

Proof. 1. Let Σ_R be as given in Lemma 3.1, and consider the viscosity solution $\omega_R(x)$ of (P_ϵ) in Σ_R with with the lateral boundary data $M = 1$ on $\partial B_R(p) \cap \Pi$. The existence and uniqueness of the viscosity solution ω_R is shown, for example, in [8], [9], [10].

Note that, by the maximum principle, $\omega_R \leq 1 + \max(g)$ in Σ_R. Due to Theorem 2.4 and Arzela-Ascoli Theorem, ω_R locally uniformly converges to a continuous function $u_\epsilon(x)$. Then the stability property of viscosity solutions it follows that $u_\epsilon(x)$ is a viscosity solution of (P_ϵ).

2. To show uniqueness, suppose u_1 and u_2 are both viscosity solutions of (P_ϵ) with $|u_1|, |u_2| \leq M$. Then Lemma 3.1 yields that, for any point $q \in \Gamma_0$ and any $R > 2$,

$$|u_1 - u_2| \leq O(1/R^2) \quad \text{in} \quad B_1(q) \cap \Pi.$$

Hence $u_1 = u_2$. □

The following is immediate from Theorem 2.2 and the construction of u_ϵ in above lemma.

Corollary 3.3. *Suppose u and v are bounded and continuous in $\bar{\Pi}_\nu(p)$ and solve*

(a) $F(D^2u) \leq 0 \leq F(D^2v)$ *in* $\Pi_\nu(p)$*;*

(b) $u \leq v$ *on* Γ_I*;*

(c) $\frac{\partial u}{\partial \nu} \leq f(x) \leq \frac{\partial v}{\partial \nu}$ *on* Γ_0*,*

where $f(x) : \mathbb{R}^n \to \mathbb{R}$ is continuous. Then $u \leq v$ in $\Pi_\nu(p)$.

In the rest of this section we will repeatedly use the fact that linear profiles as well as constants solve $F(D^2u) = 0$.

Lemma 3.4. *Let $\Pi_\nu(p)$ as given in (P_ϵ) and let $0 < \epsilon < 1$. Suppose that w_1 and w_2 are bounded and solve, in the viscosity sense,*

$$\begin{cases} F\left(D^2 w_i\right) = 0 & \text{in} \quad \Pi_\nu(p); \\ |w_1 - w_2| \leq \epsilon & \text{on} \quad \Gamma_I; \\ \partial w_1/\partial\nu - \partial w_2/\partial\nu = A & \text{on} \quad \Gamma_0. \end{cases}$$

Then there exists a positive constant $C = C(A)$ such that

$$|w_1 - w_2| \geq C - \epsilon \text{ in } \Pi_\nu(p) \cap B_{1/2}(p).$$

Proof. Let $\tilde{w} := w_2 + h$, where $h(x) = A(x-p)\cdot\nu + A - \epsilon$. Then $\partial\tilde{\omega}/\partial\nu = \partial\omega_1/\partial\nu$ on Γ_0. Also, $\tilde{\omega} \leq w_1$ on Γ_I. Therefore Corollary 3.3 yields that $w_2 + h \leq w_1$. Since $h \geq A/2 - \epsilon$ in $B_{1/2}(p)$, we are done. □

Lemma 3.5. *Let let $\tilde{\Pi} = \Pi + a\nu$ for some $0 \leq a \leq A\epsilon$ where $0 < A < 1$. Suppose u_ϵ and $\tilde{u}_\epsilon$ are bounded and solve (P_ϵ) respectively in the domains Π and $\tilde{\Pi}$. Then we have*

$$|u_\epsilon - \tilde{u}_\epsilon| \leq C(A^\beta + \epsilon^\alpha) \text{ in } \Pi \cap \tilde{\Pi},$$

where α is as given in Theorem 2.4 and β is the Hölder exponent of g.

Proof. 1. Let $v_\epsilon(x) = \tilde{u}_\epsilon(x + a\nu)$ so that v_ϵ and u_ϵ are defined in the same domain Π. Since $g(x) \in C^\beta(I\!R^n)$, $|\partial v_\epsilon/\partial\nu - \partial u_\epsilon/\partial\nu| \leq w^\beta(\epsilon)$ on Γ_0.

2. On Γ_I, $u_\epsilon = v_\epsilon = 1$. Hence one can compare $u_\epsilon \pm w(\epsilon)(1 + (x-p)\cdot\nu)$ with v_ϵ and apply Theorem 2.2 to obtain

$$|u_\epsilon - v_\epsilon| \leq w^\beta(\epsilon) \text{ in } \Pi.$$

Due to the Hölder continuity of u^ϵ given by Theorem 2.4, $|v_\epsilon - \tilde{u}_\epsilon| \leq CA^\beta + \epsilon^\alpha$ in $\Pi \cap \tilde{\Pi}$. This finishes the proof. □

The next lemma follows from Theorem 2.4 (b).

Lemma 3.6. *Let v_j be a bounded solution of (P_ϵ) with a constant Neumann condition $g(x) = \mu_j$. If $\mu_j \to \mu$, then v_j converges to v such that $\partial v/\partial\nu = \mu$ on Γ_0.*

4 Proof of the Main Theorem

In this section, we prove the main theorem of the paper, Theorem 1.2.

4.1 Proof of Theorem 1.2 (i), (iii) and (iv)

Recall

$$\Gamma_0 = \{x : (x-p)\cdot\nu = 0\};\ \Gamma_I = \{x : (x-p)\cdot\nu = -1\}.$$

Due to the uniform Hölder regularity of $\{u_\epsilon\}$ (Theorem 2.4(a)), along subsequences $u_{\epsilon_j} \to u$ in $\bar{\Pi}_\nu$. Note that there could be different limits along different subsequences ϵ_j. Below we will show that if ν is an irrational direction then all subsequential limits of $\{u_\epsilon\}$ coincide.

Suppose

$$0 \in \Pi_\nu = \{-1 < (x-p)\cdot\nu < 0\}.$$

Let us choose one of the convergent subsequence u_{ϵ_j} and denote $u_{\epsilon_j} = u_j$. Then for each j, there exists a constant μ_j and a function v_j in $\Pi_\nu(p)$ such that

$$(P_{\mu_j}) \qquad \begin{cases} F(D^2 v_j) = 0 & \text{in} \quad \Pi_\nu(p) \\ \partial v_j/\partial\nu = \mu_j & \text{on} \quad \Gamma_0 \\ v_j = u_j = 1 & \text{on} \quad \Gamma_I \\ v_j = u_j & \text{at} \quad x = 0. \end{cases}$$

Lemma 4.1. *$\mu_j \to \mu$ for some μ as $j \to \infty$. (μ may be different for different subsequences $\{\epsilon_j\}$.)*

Proof. Suppose not, then there would be a constant $A > 0$ such that for any $N > 0$, $|\mu_m - \mu_n| \geq A$ for some $m, n > N$. Then by Lemma 3.4,

$$|v_m(0) - v_n(0)| \geq C_A.$$

This contradicts the fact that $v_j(0) = u_j(0)$, since $u_j(0) \to u(0)$ as $j \to \infty$. □

The next lemma states that u_ϵ looks like a linear profile with respect to the direction ν as $\epsilon \to 0$.

Lemma 4.2. *Away from the Neumann boundary Γ_0, u_ϵ is almost a constant on hyperplanes parallel to Γ_0. More precisely, let $x_0 \in \Pi_\nu(p)$ with $\mathrm{dist}(x_0, \Gamma_0) > \epsilon^{1/20}$. Then for $0 < \alpha < 1$ the following holds:*

(i) If ν is a rational direction, there exists a constant $C > 0$ depending on ν, α and n, such that for any $x \in H(x_0) := \{(x - x_0) \cdot \nu = 0\}$

$$|u_\epsilon(x) - u_\epsilon(x_0)| \leq C\epsilon^{\alpha/2}. \tag{12}$$

(ii) If ν is any irrational direction, there exists a constant $C > 0$ depending on α and n, such that for any $x \in H(x_0)$

$$|u_\epsilon(x) - u_\epsilon(x_0)| \leq C\epsilon^{\alpha/20} + Cw_\nu(\epsilon)^\alpha \tag{13}$$

where $w_\nu : [0, \infty) \to [0, \infty)$ is a mode of continuity given as in (ii) of Lemma 2.7.

Proof. First, let ν be a rational direction. Lemma 2.7 implies that for any $x \in H(x_0)$, there is $y \in H(x_0)$ such that $|x - y| \leq M_\nu \epsilon$ and $u_\epsilon(y) = u_\epsilon(x_0)$. Then by Theorem 2.3,

$$|u_\epsilon(x_0) - u_\epsilon(x)| \leq C\epsilon^{-\alpha/20}(M_\nu \epsilon)^\alpha \leq C\epsilon^{\alpha/2}.$$

Next, we assume that ν is an irrational direction and $x \in H(x_0)$. By (ii) of Lemma 2.7, there exists $y \in I\!R^n$ such that $|x - y| \leq M\epsilon^{1/10}$, $y - x_0 \in \epsilon Z\!\!Z^n$ and

$$\mathrm{dist}(y, H(x_0)) < \epsilon w(\epsilon). \tag{14}$$

Then we obtain

$$\begin{aligned} |u_\epsilon(x_0) - u_\epsilon(x)| &\leq |u_\epsilon(x_0) - u_\epsilon(y)| + |u_\epsilon(y) - u_\epsilon(x)| \\ &\leq Cw(\epsilon)^\beta + |u_\epsilon(y) - u_\epsilon(x)| \\ &\leq Cw(\epsilon)^\beta + C\epsilon^{-\alpha/20}(M\epsilon^{1/10})^\alpha \\ &\leq Cw(\epsilon)^\beta + C\epsilon^{\alpha/20}, \end{aligned} \tag{15}$$

where the second inequality follows from Lemma 3.5 with (14), and the third inequality follows from Theorem 2.3. □

By Lemma 4.2 and by the comparison principle (Theorem 2.2), we obtain the following estimate: For $x \in \Pi$,

$$|u_\epsilon(x) - v_\epsilon(x)| \leq \Lambda(\epsilon) \tag{16}$$

where

$$\Lambda(\epsilon) = \begin{cases} C\epsilon^{\alpha/2} & \text{if} \quad \nu \text{ is a rational direction} \\ \\ C\epsilon^{\alpha/20} + Cw_\nu(\epsilon)^\beta & \text{if} \quad \nu \text{ is any irrational direction.} \end{cases}$$

Lemma 4.3. $\lim v_j = \lim u_j$ *and hence* $\partial u/\partial\nu = \mu$ *on* Γ_0.

Proof. Observe that v_j solves (P_{ϵ_j}) with $g = \mu_j$: Note that v_j is then a linear profile, i.e., $v_j(x) = \mu_j((x-p)\cdot\nu + 1) + 1$. Let x_0 be a point between Γ_0 and $H(0)$. Then by Lemma 4.2, applied to u_j and v_j,

$$|(u_j(x) - v_j(x)) - (u_j(x_0) - v_j(x_0))| \leq C\tilde{w}(\epsilon_j). \tag{17}$$

for all $x \in H(x_0)$, if j is sufficiently large. Suppose now that

$$u_j(x_0) - v_j(x_0) > c > 0 \text{ for sufficiently large } j.$$

Then due to (16) $u_j - v_j \geq c/2$ on $H(x_0)$ if j is sufficiently large. Note that u_j can be constructed as the locally uniform liimt of $u_{j,R}$, where $u_{j,R}$ solves

$$F(D^2 u_{j,R}) = 0 \text{ in } B_R(x_0) \cap \Pi; \quad u_{j,R} = v_j \text{ on } \partial B_R(x_0) \cap \Pi$$

with

$$u_{j,R} = 1 \text{ on } \Gamma_I; \quad \frac{\partial}{\partial\nu} u_{j,R}(x) = g(\frac{x}{\epsilon_j}) \text{ on } \Gamma_0.$$

Comparing $u_{j,R}$ and $v_j + c((x-x_0)\cdot\nu + 1)$ on the domain

$$B_R(x_0) \cap \{x : -1 \leq (x-p)\cdot\nu \leq (x-x_0)\cdot\nu\}$$

for sufficiently large R then yields that $u_{j,R}(0) \geq v_j(0) + c_0$ for all sufficiently large R, which would contradict the fact that $v_j(0) = u_j(0)$. Similarly the case $\liminf_j (u_j(x_0) - v_j(x_0) < 0$ can be excluded. and it follows that

$$|u_j(x_0) - v_j(x_0)| \to 0 \text{ as } j \to \infty.$$

Hence we get $v_j \to u$ in each compact subset of Π. By Lemma 4.1 and Lemma 3.6, the limit $u = v$ of v_j satisfies $\partial u/\partial\nu = \mu$ on Γ_0.

□

Lemma 4.4. *If* ν *is an irrational direction,* $\partial u/\partial\nu = \mu_\nu$ *for a constant* μ_ν *which depends on* ν*, not on the subsequence* ϵ_j*.*

Proof. 1. Let $0 < \eta < \epsilon$ be sufficiently small. Let

$$w_\epsilon(x) = \frac{u_\epsilon(\epsilon x)}{\epsilon}, \quad w_\eta(x) = \frac{u_\eta(\eta x)}{\eta}$$

and denote by Γ_1 and Γ_2 as the corresponding Neumann boundary of w_ϵ and w_η, respectively. By (iii) of Lemma 2.7, for a point $p \in I\!R^n$, there exist $q_1 \in \Gamma_1$ and $q_2 \in \Gamma_2$ such that

$$|p - q_1| \leq \eta \mod Z\!\!Z^n, \text{ and } |p - q_2| \leq \eta \mod Z\!\!Z^n.$$

Hence after translations by $p - q_1$ and $p - q_2$, we may suppose that $w_\epsilon(x)$ and $w_\eta(x)$ are defined, respectively, on the extended strips

$$\Omega_\epsilon := \{x : -\frac{1}{\epsilon} \leq (x-p)\cdot\nu \leq 0\}$$

and

$$\Omega_\eta := \{x : -\frac{1}{\eta} \leq (x-p)\cdot\nu \leq 0\}.$$

Here $w_\epsilon = 1/\epsilon$ on $\{(x-p)\cdot\nu = -\frac{1}{\epsilon}\}$ and $w_\eta = 1/\eta$ on $\{(x-p)\cdot\nu = -\frac{1}{\eta}\}$. Moreover on $\Gamma_0 := \{(x-p)\cdot\nu = 0\}$ we have

$$\partial w_\epsilon/\partial\nu = g_1(x) := g(x - z_1) \text{ and } \partial w_\eta/\partial\nu = g_2(x) := g(x - z_2)$$

where $|z_1|, |z_2| \leq \eta$. Observe that since g has Hölder exponent $0 < \beta \leq 1$, $|g_1 - g_2| \leq \eta^\beta$.

Let v_ϵ be a solution of the problem (P_ϵ) with constant Neumann data $\partial v_\epsilon/\partial\nu = \mu_\epsilon$ on Γ_0 such that v_ϵ coincides with u_ϵ at $x = 0$ and on Γ_I. By (16)

$$|w_\epsilon(x) - \frac{v_\epsilon(\epsilon x)}{\epsilon}| \leq \frac{C\epsilon^{\alpha/20} + Cw(\epsilon)^\beta}{\epsilon}. \tag{18}$$

Note that v_ϵ is a linear profile: indeed

$$\frac{v_\epsilon(\epsilon x)}{\epsilon} = \mu_\epsilon((x-p)\cdot\nu + \frac{1}{\epsilon}) + \frac{1}{\epsilon}.$$

¿From (18) and the comparison principle, it follows that

$$(\mu_\epsilon - \Lambda(\epsilon))((x-p)\cdot\nu + \frac{1}{\epsilon}) \leq w_\epsilon(x) - \frac{1}{\epsilon} \leq (\mu_\epsilon + \Lambda(\epsilon))((x-p)\cdot\nu + \frac{1}{\epsilon}) \tag{19}$$

where $\Lambda(\epsilon) = C\epsilon^{\alpha/20} + Cw(\epsilon)^\beta$.

2. (19) means that the slope of w_ϵ in the direction of ν (i.e. $\nu \cdot Dw_\epsilon$) is between $\mu_\epsilon + \Lambda(\epsilon)$ and $\mu_\epsilon - \Lambda(\epsilon)$ on $\{x : (x-p)\cdot\nu = -\frac{1}{\epsilon}\}$. Now let us consider linear profiles

$$l_1(x) = a_1(x-p)\cdot\nu + b_1 \text{ and } l_2(x) = a_2(x-p)\cdot\nu + b_2,$$

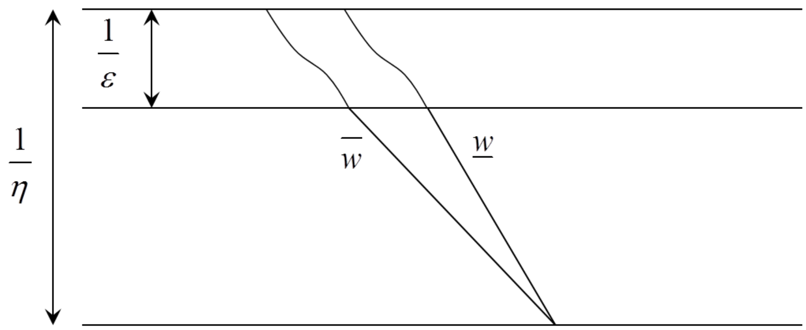

Figure 2

whose respective slopes are $a_1 = \mu_\epsilon + \Lambda(\epsilon)$ and $a_2 = \mu_\epsilon - \Lambda(\epsilon)$. Here b_1 and b_2 are chosen such that

$$l_1 = l_2 = \omega_\eta(x) \;\text{ on }\; \{x : (x-p)\cdot\nu = -\frac{1}{\eta}\}.$$

3. Now we define

$$\overline{w}(x) := \begin{cases} l_1(x) & \text{in} \quad \{-1/\eta \le (x-p)\cdot\nu \le -1/\epsilon\} \\ w_\epsilon(x) + c_1 & \text{in} \quad \{-1/\epsilon \le (x-p)\cdot\nu \le 0\} \end{cases}$$

and

$$\underline{w}(x) := \begin{cases} l_2(x) & \text{in} \quad \{-1/\eta \le (x-p)\cdot\nu \le -1/\epsilon\} \\ w_\epsilon(x) + c_2 & \text{in} \quad \{-1/\epsilon \le (x-p)\cdot\nu \le 0\} \end{cases}$$

where c_1 and c_2 are constants satisfying $l_1 = w_\epsilon + c_1$ and $l_2 = w_\epsilon + c_2$ on $\{(x-p)\cdot\nu = -1/\epsilon\}$. (See Figure 2.)

Note that, due to (19), in $\{-\frac{1}{\epsilon} \le (x-p)\cdot\nu \le 0\}$ we have

$$\overline{w}(x) = \min(l_1(x), w_\epsilon(x) + c_1) \text{ and } \underline{w}(x) = \max(l_2(x), w_\epsilon(x) + c_2),$$

and thus it follows that $\overline{w}$ and $\underline{w}$ are respectively viscosity super- and subsolution of (P).

4. Let us define

$$h_1(x) = \eta^\beta((x-p)\cdot\nu + 1/\eta).$$

Then $w^+ := \overline{w} + h_1$ and $w^- := \underline{w} - h_1$ respectively solves the following:

$$\begin{cases} F(Dw^+) \le 0 & \text{in} \quad \Omega_\eta; \\ \dfrac{\partial w^+}{\partial \nu} = g(x) + \eta^\beta & \text{on} \quad \Gamma_0 \end{cases}$$

and

$$\begin{cases} F(Dw^-) \geq 0 & \text{in} \quad \Omega_\eta \\ \dfrac{\partial w^-}{\partial \nu} = g(x) - \eta^\beta & \text{on} \quad \Gamma_0. \end{cases}$$

Since $|g - \tilde{g}| \leq \eta^\beta$ and $w^+ = w^- = w_\eta$ on $\{(x-p)\cdot\nu = -\frac{1}{\eta}\}$, from the comparison principle for (P_ϵ) it follows that

$$w^- \leq w_\eta \leq w^+ \text{ in } \Omega_\eta. \tag{20}$$

Hence we conclude

$$|\mu_\eta - \mu_\epsilon| \leq \Lambda(\epsilon) + \eta^\beta, \tag{21}$$

where μ_η is the slope of v_η, and $\Lambda(\epsilon) = C\epsilon^{\alpha/20} + Cw(\epsilon)^\beta \to 0$ as $\epsilon \to 0$.

□

The proof of the following lemma is immediate from Lemma 4.4 and (21) .

Lemma 4.5. *[Error estimate: Theorem 1.2 (iv)] For any irrational direction ν, there is a unique homogenized slope $\mu(\nu) \in I\!R$ and $\epsilon_0 = \epsilon_0(\nu) > 0$ such that for $0 < \epsilon < \epsilon_0$ the following holds: for any $0 < \alpha < 1$, there exists a constant $C = C(\alpha, n, \lambda, \Lambda)$ such that*

$$|u_\epsilon(x) - (1 + \mu(\nu)((x-p)\cdot\nu + 1))| \leq \Lambda(\epsilon) := C\epsilon^{\alpha/20} + Cw_\nu(\epsilon)^\alpha \text{ in } \Pi_\nu(p), \tag{22}$$

where $\omega_\nu(\epsilon)$ is as given in (7).

Lemma 4.6. *Let ν be a rational direction. If the Neumann boundary Γ_0 passes through $p = 0$, then there is a unique homogenized slope $\mu(\nu)$ for which the result of Lemma 4.5 holds with $\Lambda(\epsilon) = C\epsilon^{\alpha/2}$.*

Proof. The proof is parallel to that of Lemma 4.4. Let ω_ϵ and ω_η be as given in the proof of Lemma 4.4. Note that since Ω_ϵ and Ω_η have their Neumann boundaries passing through the origin, $\partial w_\epsilon/\partial\nu = g(x) = \partial w_\eta/\partial\nu$ without translation of the x variable, and thus we do not need to use the properties of hyperplanes with an irrational normal (Lemma 2.7 (b)) to estimate the error between the shifted Neumann boundary datas.

□

Remark 4.7. *As mentioned in the introduction, if ν is a rational direction with $p \neq 0$, the values of $g(\cdot/\epsilon)$ on $\partial\Omega_\epsilon$ and $\partial\Omega_\eta$ may be very different under any translation, and thus the proof of Lemma 4.4 fails. In this case u_ϵ may converge to solutions of different Neumann boundary data depending on the subsequences.*

4.2 Proof of Theorem 1.2 (ii)

Proposition 4.8 (Theorem 1.2 (ii))**.** *The homogenized limit $\mu(\nu)$, defined in Lemma 4.5 for irrational directions in $\mathcal{S}^{n-1}$, has a continuous extension $\bar{\mu}(\nu)$: $\mathcal{S}^{n-1} \to I\!R$.*

Proof. Let us fix a unit vector $\nu \in \mathcal{S}^{n-1}$. Then we will show that there exists a positive constant $C > 0$ depending on ν such that the following holds: Given $\delta > 0$ there exists $\epsilon > 0$ such that for any two irrational directions $\nu_1, \nu_2 \in \mathcal{S}^{n-1}$

$$|\mu(\nu_1) - \mu(\nu_2)| < C\delta^{1/2} \text{ whenever } 0 < |\nu_1 - \nu|, |\nu_2 - \nu| < \epsilon. \tag{23}$$

1. To simplify the proof, we first present the case $n = 2$. For simplicity of notations, we may assume that $|\nu \cdot e_1| \leq |\nu \cdot e_2|$ and $p = 0$. First we introduce several notations. Again for notational simplicity and clarity in the proof, we assume that $\nu = e_2$: we will explain in the paragraph below how to modify the notations and the proof for $\nu \neq e_2$. Let us define

$$\Omega_0 := \Pi_\nu(0) = \{(x, y) \in I\!R^2 : -1 \leq y \leq 0\}$$

and for $i = 1, 2$

$$\Omega_i := \Pi_{\nu_i}(0) = \{(x, y) \in I\!R^2 : -1 \leq (x, y) \cdot \nu_i \leq 0\}.$$

Let us also define the family of functions

$$g_i(x_1, x_2) = g_i(x_1) = g(x_1, \delta(i-1)), \text{ where } i = 1, ..., m := [\frac{1}{\delta}] + 1.$$

(see Figure 3).

If ν is a rational direction different from e_2, take the smallest $K_\nu \in I\!N$ such that $K_\nu \nu = 0 \text{ mod } I\!N^2$. Then g can be considered as a K_ν-periodic function with the new direction of axis of ν. If ν is an irrational direction, take the smallest $K_\nu \in I\!N$ such that $|K_\nu \nu| \leq \delta \text{ mod } I\!N^2$. Then g is almost K_ν- periodic up to the order of δ with the new axis of ν. We point out that it does not make any difference in the proof if we replace the periodicity of g by the fact that g is almost periodic up to the order δ.

Before moving onto the next step, we briefly discuss the heuristics in the proof.

Proof by heuristics:

Since the domains Ω_1 and Ω_2 point toward different directions ν_1 and ν_2, we cannot directly compare their boundary data, even if $\partial\Omega_1$ and $\partial\Omega_2$ cover most part of the unit cell in $I\!R^n / Z\!\!Z^n$. To overcome this difficulty we perform a two-scale homogenization.

First we consider the functions g_i $(i = 1, .., m)$, whose profiles cover most values of g in $I\!R^2$ up to the order of δ^β, where β is the Hölder exponent of g. Note that most values of g in $I\!R^2$ are taken on $\partial\Omega_1$ and on $\partial\Omega_2$ since ν_1 and ν_2

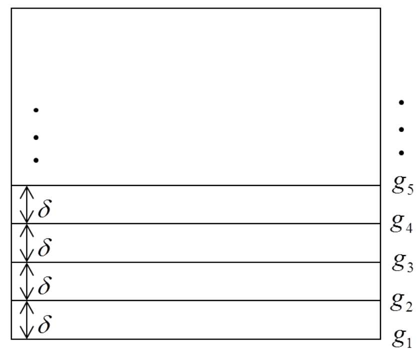

Figure 3

are both irrational directions. On the other hand, since ν_1 and ν_2 are very close to ν which may be a rational direction, the averaging behavior of a solution u_ϵ in Ω_1 (or Ω_2) would occur only if ϵ gets very small.

If $|\nu_1 - \nu| = |\nu_1 - e_2|$ is chosen much smaller than δ, we can say that the Neumann data $g_1(\cdot/\epsilon)$ is (almost) repeated $N := [\delta/|\nu_1 - \nu|]$ times on $\partial\Omega_1$ with period ϵ, up to the error $O(\delta^\beta)$. (See Figure 4.) Similarly, on the next piece of the boundary, $g_2(\cdot/\epsilon)$ is (almost) repeated N times and then $g_3(\cdot/\epsilon)$ is repeated N times: this pattern will repeat with g_k ($k \in I\!N$ mod m).

If N is sufficiently large, i.e., if $|\nu_1 - \nu|$ is sufficiently small compared to δ, the solution u_ϵ in Ω_1 will exhibit averaging behavior, $N\epsilon$-away from $\partial\Omega_1$. More precisely, on the $N\epsilon$-sized segments of hyperplane H located $N\epsilon$-away from $\partial\Omega_1$, u_ϵ would be homogenized by the repeating the profiles of g_i (for some fixed i) with an error of $O(\delta^\beta)$. This is the first homogenization of u_ϵ near the boundary of Ω_1: we denote by $\mu(g_i)$, the corresponding values of the homogenized slopes of u_ϵ on H.

Now a unit distance away from $\partial\Omega_1$, we obtain the second homogenization of u_ϵ, whose slope is determined by $\mu(g_i)$, $i = 1, .., m$. Note that this estimate does not depend on the direction ν_1, but on the quantity $|\nu_1 - \nu|$. Hence applying the same argument for ν_2, we conclude that $|\mu(\nu_1) - \mu(\nu_2)|$ is small. Note that $\mu(\nu_1)$ and $\mu(\nu_2)$ are uniquely determined because ν_1 and ν_2 are irrational directions (Lemma 4.6).

A rigorous proof of above observation is rather lengthy: the main difficulty lies in the fact that, to perform the first homogenization $N\epsilon$-away from the boundary, one requires the solution u_ϵ to be sufficiently flat in tangential directions to ν, which we do not know a priori. We will go around this difficulty by constructing sub- and supersolutions by patching up solutions from near-boundary region and from the region away from the boundary. The proof is given in steps 2-8 below.

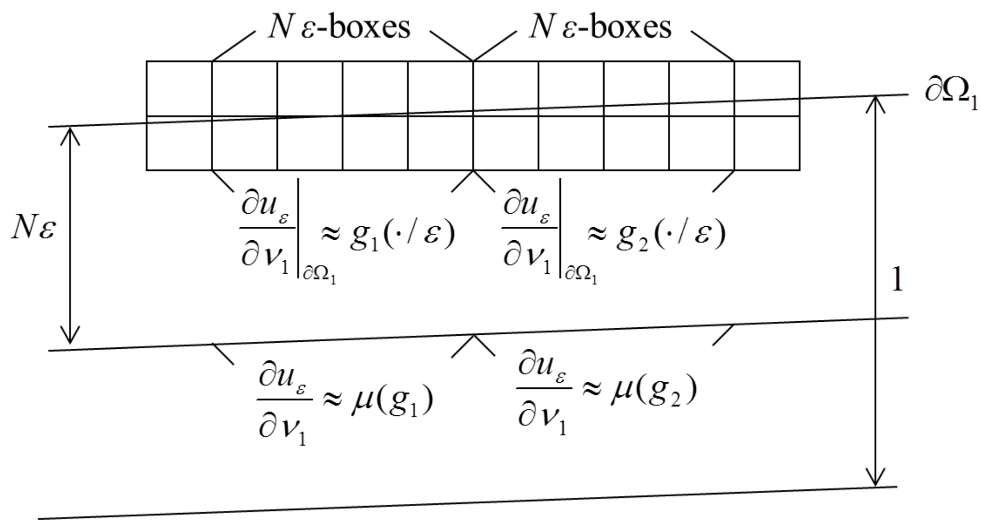

Figure 4

2. Given $\delta > 0$, let us choose irrational unit vectors $\nu_1, \nu_2 \in I\!R^2$ such that

$$0 < \bar{\epsilon}_0^{1/1000} \le \epsilon_0^{1/1000} = \delta,$$

where$\epsilon_0 = |\nu_1 - e_2|$, $\epsilon = \epsilon_0^{21/20}$ and $\bar{\epsilon} = |\nu_2 - e_2|$, $\bar{\epsilon} = \bar{\epsilon_0}^{21/20}$. Let us also define

$$N = [\frac{\delta}{|\nu_1 - e_n|}] = [\frac{\delta}{\epsilon_0}]. \tag{24}$$

Then $N\epsilon = \delta\epsilon_0^{1/20} := \delta_0$. Note that

$$\delta_0 \ge \epsilon^{1/20} \text{ and } \delta_0 \ge \delta^{100}.$$

With above definition of ϵ and N, consider the strip regions $I_0 = [-N\epsilon, 0] \times I\!R$, $I_1 = [0, N\epsilon] \times I\!R$, $I_{-1} = [-2N\epsilon, -N\epsilon] \times I\!R$, $I_2 = [N\epsilon, 2N\epsilon] \times I\!R$,..., i.e.,

$$I_k = [(k-1)N\epsilon, kN\epsilon] \times I\!R \quad \text{for} \quad k \in Z\!\!Z.$$

Let $\tilde{k} \in [1, m]$ denote k in modulo m, where $m = [\frac{1}{\delta}] + 1$. Note that, since $N|\nu_1 - e_n| = \delta$, $g_{\tilde{k}}(\cdot/\epsilon)$ is (almost) repeated N times on $I_k \cap \partial\Omega_1$. This fact and the Hölder continuity of g yield that

$$|g(\frac{x}{\epsilon}, \frac{y}{\epsilon}) - g_{\tilde{k}}(\frac{x}{\epsilon})| < C\delta^\beta \text{ on } \partial\Omega_1 \cap I_k \quad \text{for } k \in Z\!\!Z. \tag{25}$$

3. Let w_ϵ solve $(P): F(D^2 w_\epsilon) = 0$ in Ω_0 with

$$\begin{cases} \dfrac{\partial w_\epsilon}{\partial \nu}(x,0) = g_{\tilde{k}}(\dfrac{x}{\epsilon}) & \text{for} \quad (x,0) \in I_k \cap \{y=0\} \\ w_\epsilon = 1 & \text{on} \quad \{y=-1\}. \end{cases}$$

Next let u_ϵ solve (P) in Ω_1 with

$$\begin{cases} \dfrac{\partial u_\epsilon}{\partial \nu_1}(x,0) = g(\dfrac{x}{\epsilon}, \dfrac{y}{\epsilon}) & \text{on} \quad \{(x,y)\cdot\nu_1 = 0\}, \\ u_\epsilon = 1 & \text{on} \quad \{(x,y)\cdot\nu_1 = -1\}. \end{cases}$$

Let $\mu(w_\epsilon)$ ($\mu(u_\epsilon)$) be chosen as the slope μ_j in the linearized problem (P_{μ_j}) in section 4, where u_j is replaced by w_ϵ (u_ϵ) and the reference point $x=0$ is replaced by $x = -e_2/2 = (0,-1/2)$. (Recall that we assumed $0 \in \partial\Omega_1$, and $(0,-1/2) \in \Omega_i$ for $i=1,2$.) $\mu(w_\epsilon)$ and $\mu(u_\epsilon)$ then denote the slopes of a linear approximation of ω_ϵ and u_ϵ. From (25) it follows that

$$|\mu(w_\epsilon) - \mu(u_\epsilon)| < C\delta^\beta. \tag{26}$$

We point out that $\mu(w_\epsilon)$ and $\mu(u_\epsilon)$ respectively converge to a unique limit as $\epsilon \to 0$ since ν_1 is irrational.

4.

We begin with introducing $\mu_{1/N}(g_{\tilde{k}})$, which denotes the average of slope for our solution, $\delta_0 = N\epsilon$-away from the Neumann boundary $\{y=0\}$, in I_k.

Let us define

$$H := \partial\Omega_0 - N\epsilon\nu = \partial\Omega_0 - N\epsilon e_2 = \{(x,y) : y = -\delta_0\}.$$

Let $\eta = 1/N$ and let $w_{\eta,1}$ solve

$$\begin{cases} F(D^2 w_{\eta,1}) = 0 & \text{in} \quad \{-\delta_0 \le y \le 0\} \\ w_{\eta,1} = w_\epsilon(0,-\delta) & \text{on} \quad H = \{y = -\delta_0\} \\ \dfrac{\partial w_{\eta,1}}{\partial y}(x,0) = g_1(\frac{x}{\epsilon}, 0) & \text{on} \quad \partial\Omega_0 = \{y=0\} \end{cases}$$

where $g_1(x,0) = g_1(x+k,0)$ for $k \in \mathbb{Z}$. Let $\mu_{\frac{1}{N}}(g_1)$ be the slope of the linear approximation of $w_{\eta,1}$, defined as below: choose a linear solution $v_{\eta,1}(\cdot)$ such that

$$\begin{cases} F(D^2 v_{\eta,1}) = 0 & \text{in} \quad \{-\delta_0 \le y \le 0\} \\ v_{\eta,1} = w_{\eta,1}(0,-\delta_0) & \text{on} \quad H = \{y=-\delta_0\} \\ v_{\eta,1}(0, -\frac{\delta_0}{2}) = w_{\eta,1}(0,-\frac{\delta_0}{2}) & \\ \dfrac{\partial v_{\eta,1}}{\partial y}(x,0) = \mu_{\frac{1}{N}}(g_1) & \text{on} \quad \partial\Omega_0 = \{y=0\}. \end{cases}$$

Since $g_1(x,0)$ is periodic on $\{y=0\}$ with period ϵ and $\delta_0 = N\epsilon$, we can apply Lemma 4.2 (i), using the fact that $\delta_0 \geq \epsilon^{1/20}$, to conclude that

$$|w_{\eta,1}(x,y) - (w_{\eta,1}(0,-\frac{\delta_0}{2}) + \mu_{1/N}(g_1)(y+\frac{\delta_0}{2}))| \leq C\delta_0^{1+\beta} \tag{27}$$

on $\{y=-\frac{\delta_0}{2}\}\cap I_1$. Similarly one can define $w_{\eta,k}$ and $v_{\eta,k}$ for $k \in \mathbb{Z}$ to conclude that

$$|w_{\eta,k}(x,y) - (w_{\eta,k}((k-1)\delta_0,-\frac{\delta_0}{2}) + \mu_{1/N}(g_{\tilde{k}})(y+\frac{\delta_0}{2}))| \leq C\delta_0^{1+\beta} \tag{28}$$

on $\{y=-\frac{\delta_0}{2}\}\cap I_k$.

5. We will now construct barriers which bound w_ϵ from above and below, by pasting together the near- boundary and the rest of the region together as follows. First we construct a supersolution of (P_ϵ). Let ρ_ϵ solve the Neumann boundary problem away from the boundary $\{y=0\}$:

$$\begin{cases} F(D^2\rho_\epsilon) = 0 & \text{in} \quad \{-1 \leq y \leq -\delta_0\} \\ \dfrac{\partial \rho_\epsilon}{\partial y} = \Lambda(x) & \text{on} \quad H = \{y=-\delta_0\} \\ \rho_\epsilon = 1 & \text{on} \quad \{y=-1\} \end{cases}$$

Here $\Lambda(x)$ is a hölder continuous function obtained by approximating $\mu_{1/N}(g_k)+2\delta_0^{\alpha_0}$ in each $N\epsilon$-strip, where the constant $0<\alpha_0<1$ will be decided below. Here the hölder continuity of $\Lambda(x)$ is obtained by the fact that g_k and g_j differs from each other by $((k-j)\delta_0)^\beta$ and they are apart by $(k-j)N\epsilon \geq (k-j)\delta_0^{100}$.

Then Theorem 2.4 (b) yields that $\rho_\epsilon \in C^{1,\gamma}$ up to H, where γ depends on β and n. Therefore there exists a constant $0<\alpha_0<1$ such that the following holds: in each $\delta_0^{1-\alpha_0}$-neighborhood of a point $(x_0,-\delta_0)\in H$, we have

$$|\rho_\epsilon(x,-\delta_0) - \rho_\epsilon(x_0,-\delta_0) - \alpha(x_0)(x-x_0)| \leq \delta_0^{1+\alpha_0}, \tag{29}$$

where $\alpha(x_0)$ is the tangential derivative of ρ_ϵ at $(x_0,-\delta_0)$.

6. Next we construct the near-boundary barrier:

$$\begin{cases} F(D^2 f_\epsilon) = 0 & \text{in} \quad \{-\delta_0 \leq y \leq 0\} \\ f_\epsilon = \rho_\epsilon & \text{on} \quad H = \{y=-\delta_0\} \\ \dfrac{\partial f_\epsilon}{\partial y} = g_{\tilde{k}}(\frac{x}{\epsilon}) & \text{on} \quad \{y=0\} \end{cases}$$

Let us now estimate the slope of f_ϵ on H. Let us choose a constant μ_ϵ and the corresponding linear profile ϕ_ϵ such that

$$\begin{cases} F(D^2\phi_\epsilon) = 0 & \text{in} \quad \{-\delta_0 \le y \le 0\} \\ \phi_\epsilon(x, -\delta) = f_\epsilon(0, -\delta_0) & \text{on} \quad H \\ \phi_\epsilon(0, -\frac{\delta}{2}) = f_\epsilon(0, -\frac{\delta_0}{2}) & \\ \dfrac{\partial \phi_\epsilon}{\partial y} = \mu_\epsilon & \text{on} \quad \partial\Omega_0 = \{y = 0\}. \end{cases}$$

(29) and and the comparison principle (Theoren 2.2), as well as the localization argument as in the proof of Lemma 3.1 applied to the rescaled function

$$(\delta_0)^{-1} f_\epsilon(\frac{(x-x_0)}{\delta_0} + x_0, \frac{y}{\delta_0}) - \alpha(x_0)(x - x_0)$$

in the region $\{-1 \le y \le 0\} \cap \{|x| \le \delta_0^{-\alpha_0}\}$ yields that

$$|\phi_\epsilon - f_\epsilon| \le C\delta_0^{1+\alpha_0} \text{ in } \{-\delta_0 \le y \le 0\} \cap \{|x| \le \delta_0^{1-\alpha_0}\} \tag{30}$$

Putting the estimates (28) and (30) together, it follows that for any $(x_0, -\delta_0) \in H$ we have

$$|f_\epsilon(x,y) - (\alpha(x_0)(x-x_0) + \mu_{1/N}(g_k)(y + \tfrac{\delta_0}{2}))| \le \delta_0^{1+\alpha_0}$$
$$\text{on } \{y = -\tfrac{\delta_0}{2}\} \cap \{|x - x_0| \le \delta_0^{1-\alpha_0}\},$$

for appropriate k in each δ-strip. Using (29), (4.2) and the $C^{1,\gamma}$ regularity of f_ϵ up to its Dirichlet boundary, we obtain that

$$\frac{\partial f_\epsilon}{\partial y} \le \Lambda(x),$$

which then makes the following function a supersolution of (P_ϵ):

$$\underline{\rho}_\epsilon := \begin{cases} \rho_\epsilon & \text{in} \quad \{-1 \le y \le -\delta_0\} \\ f_\epsilon & \text{in} \quad \{-\delta_0 \le y \le 0\}. \end{cases}$$

Similarly, one can construct a subsolution $\bar{\rho}_\epsilon$ of (P_ϵ) by replacing $\Lambda(x)$ given in the construction of ρ_ϵ by $\tilde{\Lambda}(x) := \Lambda(x) - 4\delta_0^{\alpha_0}$, such that

$$\bar{\rho}_\epsilon \le w_\epsilon \le \underline{\rho}_\epsilon. \tag{31}$$

7. Parallel arguments as in steps 2 to 4 apply to the other direction ν_2: if we define $\bar{\epsilon}$, M and $\bar{H}$ by

$$|\nu_2 - e_2| = \bar{\epsilon} < \epsilon, \quad M = [\frac{\delta_0}{\bar{\epsilon}}], \text{ and } \bar{H} = \{y = -M\bar{\epsilon}\},$$

then we can construct barriers $\bar{\rho}_{\bar{\epsilon}}$ and $\underline{\rho}_{\bar{\epsilon}}$ such that

$$\bar{\rho}_{\bar{\epsilon}} \leq w_{\bar{\epsilon}}(x) \leq \underline{\rho}_{\bar{\epsilon}} \tag{32}$$

with their corresponding Neumann boundary conditions on H:

$$\frac{\partial}{\partial y}\bar{\rho}_{\bar{\epsilon}}, \quad \frac{\partial}{\partial y}\underline{\rho}_{\bar{\epsilon}} = \mu_{\frac{1}{M}}(g_{\bar{k}}) + O(\delta_0^{\alpha_0}) \text{ on } \bar{H} \cap \bar{I}_k, \tag{33}$$

where their respective derivative is taken as a limit from the region $\{-1 \leq y < -\delta_0\}$.

8. Now we proceed to estimate the averaging behavior of u^ϵ away from the Neumann boundary. By Lemma 4.6,

$$|\mu_{\frac{1}{N}}(g_{\tilde{k}}) - \mu_{\frac{1}{M}}(g_{\tilde{k}})| < m(\frac{1}{N}) + (\frac{1}{M})^\beta, \tag{34}$$

where $m(\frac{1}{N}) = CN^{-\alpha/20}$. Let us denote $\mu_{\frac{1}{N}}(g_{\tilde{k}}) = \mu_{\tilde{k},N}$ and let h and $\bar{h}$ respectively solve

$$\begin{cases} F(D^2 h) = 0 & \text{in} \quad \{-1 \leq y \leq -N\epsilon\} \\ h = 1 & \text{on} \quad \{y = -1\} \\ \dfrac{\partial h}{\partial \nu} = \mu_{\tilde{k},N} & \text{on} \quad H \cap I_k \end{cases}$$

and

$$\begin{cases} F(D^2 \bar{h}) = 0 & \text{in} \quad \{-1 \leq y \leq -M\bar{\epsilon}\} \\ \bar{h} = 1 & \text{on} \quad \{y = -1\} \\ \dfrac{\partial \bar{h}}{\partial \nu} = \mu_{\tilde{k},M} & \text{on} \quad \bar{H} \cap I_k. \end{cases}$$

Let $\mu(h)$ and $\mu(\bar{h})$ be the respective slope of linear approximation for h and $\bar{h}$.

Then it follows from (34) that if $\delta_0 \sim N\epsilon \sim M\bar{\epsilon}$ is sufficiently small,

$$|\mu(h) - \mu(\bar{h})| < C(m(\frac{1}{N}) + (\frac{1}{M})^\beta). \tag{35}$$

Lastly, observe that by (31) and (32), there exists $0 < \gamma < 1$ such that

$$|\mu(w_\epsilon) - \mu(h)| < C\delta^\gamma \text{ and } |\mu(w_{\bar{\epsilon}}) - \mu(\bar{h})| < C\delta^\gamma.$$

The above inequalities and (35) yield

$$|\mu(w_\epsilon) - \mu(w_{\bar{\epsilon}})| < C(\delta^\gamma + m(\frac{1}{N}) + (\frac{1}{M})^\beta).$$

Then we conclude from (26) that

$$|\mu(u_\epsilon) - \mu(u_{\bar\epsilon})| < C(\delta^\gamma + m(\frac{1}{N}) + (\frac{1}{M})^\beta). \quad (36)$$

9. Lastly we estimate the rate of convergence of $\mu(u^\epsilon)$ to $\mu(\nu_1)$ as $\epsilon \to 0$. The claim is that

$$|\mu(\nu_1) - \mu(u_\epsilon)| \leq C(\epsilon_0^\beta + \epsilon_0^{21\alpha/200} + \epsilon_0^{1/20}).$$

We will argue similarly as in the proof of Lemma 4.2 (ii). Let us define v^ϵ, the linear approximation of u^ϵ, as in (P_{μ_j}) of section 4.1, where the reference function u_j is replaced by u^ϵ.

Recall that $\Omega_1 = \{y : -1 \leq y \cdot \nu_1 \leq 0\}$. We define

$$\tilde\Omega_1 := \Omega_1 \cap \{y : y \cdot \nu_1 \leq -N\epsilon\delta^{-1}\nu_1\},$$

and $L := \partial\Omega_1 - N\epsilon\delta^{-1}\nu_1$. Then for any given $x_0 \in L$ and for any $x \in L$, there exists $y \in \mathbb{R}^2$ such that $|x - y| \leq N\epsilon m$, $x_0 - y = 0 \mod \epsilon\mathbb{Z}^2$, and

$$\mathrm{dist}(y, L) \leq \epsilon|\nu_1 - e_2| = \epsilon\epsilon_0.$$

(recall that $m = [\frac{1}{\delta}] + 1$.) Then by arguing as in (15), for $x \in L$,

$$|u_\epsilon(x_0) - u_\epsilon(x)| \leq C\epsilon_0^\beta + C(N\epsilon\delta^{-1})^\alpha(N\epsilon m)^\alpha \leq C(\epsilon_0^\beta + \epsilon^{\alpha/10}).$$

Hence due to the comparison principle (Theorem 2.2) applied to u_ϵ and v_ϵ in the domain $\tilde\Omega_1$, we obtain

$$|u_\epsilon - v_\epsilon| \leq C(\epsilon_0^\beta + \epsilon^{\alpha/10} + N\epsilon\delta^{-1}) = C(\epsilon_0^\beta + \epsilon_0^{21\alpha/200} + \epsilon_0^{1/20}). \quad (37)$$

Following the proof of (21) using (37) instead of (16), to conclude

$$|\mu(u_\epsilon) - \mu(\nu_1)| \leq C(\epsilon_0^\beta + \epsilon_0^{21\alpha/200} + \epsilon_0^{1/20}) \leq \delta.$$

Parallel arguments applies to ν_2. Combing the above inequality with (36),

$$|\mu(\nu_1) - \mu(\nu_2)| \leq C(\delta^\gamma + m(\frac{1}{N}) + (\frac{1}{M})^\beta).$$

Since N and M grow to infinity as ϵ and $\bar\epsilon$ go to zero, the above inequality proves the lemma.

10. For the general dimensions $n > 2$, let us define

$$g_i(x_1, ..., x_{n-1}, x_n) = g_i(x_1, ..., x_{n-1}) = g(x_1, ..., x_{n-1}, \delta(i-1))$$

for $i = 0, 1, ..., m := [\delta^{-1}]$. Let us also define

$$I_{k_1,k_2,...,k_{n-1}} := [(k_1 - 1)N\epsilon, k_1N\epsilon] \times ... \times [(k_{n-1} - 1)N\epsilon, k_{n-1}N\epsilon] \times \mathbb{R}.$$

Then parallel arguments as in steps 1 to 9 would apply to yield the proposition in $\mathbb{R}^n$.

□

Remark 4.9. *The proof breaks down for $F = F(D^2u, \frac{x}{\epsilon})$ since the idea of perturbing the problem by tilting the Neumann boundary and its boundary data, i.e., the approximation of u_η by w_η in step 3., does not apply if the inside operator also depends on $\frac{x}{\epsilon}$.*

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