Long time regularity of solutions of the Hele-Shaw problem.

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Abstract

In this paper we study the long-time behavior of solutions of the one phase Hele-Shaw problem without surface tension. We show that after a finite time solutions of the Hele-Shaw problem become Lipschitz continuous with nondegenerate free boundary speed. It then follows that after a finite time the solution and the free boundary become smooth in space and time.

0 Introduction

Let $K = \{x \in I\!R^n : |x| = 1\}$ and suppose that a bounded domain Ω contains K and let $\Omega_0 = \Omega - K$ and $\Gamma_0 = \partial\Omega$ (see figure 1). Note that $\partial\Omega_0 = \Gamma_0 \cup \partial K$. Let u_0 be the harmonic function in Ω_0 with $u_0 = f \equiv 1 > 0$ on K and zero on Γ_0. In addition we suppose u_0 satisfies

$$(I) \qquad |Du_0| > 0 \text{ on } \Gamma_0.$$

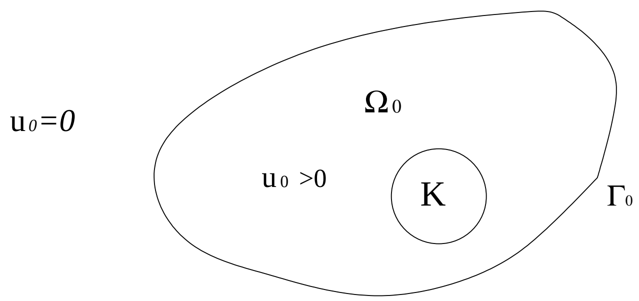

Figure 1.

The classical Hele-Shaw problem models an incompressible viscous fluid which occupies part of the space between two parallel, narrowly placed plates. In this case u_0 denotes the initial pressure of the fluid and f denotes the rate of injection from ∂K into $I\!R^n - K$. As more fluid is injected through a fixed boundary, the region occupied by the fluid will grow as time increases. Assuming no surface tension, then the pressure of the fluid $u(x,t)$ solves the following problem:

$$(HS) \qquad \begin{cases} -\Delta u = 0 & \text{in } \{u>0\}\cap Q, \\ u_t - |Du|^2 = 0 & \text{on } \partial\{u>0\}\cap Q, \\ u(x,0) = u_0(x); \quad u(x,t) = 1 \text{ for } x \in \partial K. \end{cases}$$

where $Q = (I\!R^n - K)\times(0,\infty)$. We refer to $\Gamma_t(u) := \partial\{u(\cdot,t)>0\} - \partial K$ as the *free boundary* of u at time t. Note that if u is smooth up to the free boundary, then the free boundary moves with normal velocity $V = u_t/|Du|$, and hence the second equation in (HS) implies that $V = |Du|$.

The short-time existence of classical solutions when Γ_0 is $C^{2+\alpha}$ was proved by Escher and Simonett [ES]. When $n = 2$, Elliot and Janovsky [EJ] showed the existence and uniqueness of weak solutions formulated by a parabolic variational inequality in $H^1(Q)$. For our investigation we use the notion of viscosity solutions, which have been recently introduced in [K1] (also see section 1.)

In general one cannot expect the free boundary to be smooth, mostly due to the collision between parts of the free boundary. In fact one cannot expect the solution u to be continuous in time (see [K1] for a counterexample). Nevertheless it is observed (see Theorem 2.2) that after $t > T_0$, where T_0 is the time when the positive set of a solution u overflows the smallest ball where its initial support is contained, the free boundary becomes starshaped and thus there would be no more collision of the free boundaries. Based on this observation we will then show that after some time the solution u become Lipschitz continuous and the spatial gradient of u is nondegenerat up to the free boundary. Based on these properties, further regularity analysis in [K2] yields that u is indeed a classical solution and the free boundary is smooth after $t > T_0$.

Similar results are proven in [CVW] for viscosity solutions of the Porous Medium equation

$$(PME) \qquad u_t - \Delta(u^m) = 0; \quad u \geq 0,$$

using barrier arguments and estimates for the derivatives of u. For (HS) our arguments are simpler because our solution is harmonic at each time and therefore the solution is determened solely by the geometry of the free boundary at a fixed time and the fixed boundary data.

In section 1 we recall the properties of viscosity solutions of (HS) and introduce several notations used in the following sections. In section 2 we prove the Lipschitz continuity of the free boundary with bounds for the propagation speed. In particular it follows that u is continuous for $t > T_0$. In section 3 The upper and lower bounds for $|Du|$ are obtained near the free boundary of u. Here we also state the further regularity result obtained in [K2].

Remark.

1. If the fixed boundary data f on ∂K is not constant or if K is not starshaped, then one cannot expect the free boundary to be starshaped even for large times. If $f = f(t) > 0$ is a smooth function of time, then one can use the scaling

$$\bar{u}(x,t) := a(t)u(x, \int_0^t a^2(s)ds), a(t) = 1/f(t),$$

to obtain corresponding results.

2. Throughout the paper condition (I) is assumed to guarantee the lower bound of the propagation speed for the free boundary (see Lemma 2.4). Recently it is proved in [CJK] that such lower bound can be obtained for initially Lipschitz domains with small Lipschitz constant. This result and theorem 2.2 suggests that our result on the large time behavior holds for general initial data without condition (I).

1 Preliminaries

In addition to the terms and conditions introduced in the introduction, we will use the following notations:

- $D_r(x_0) = \{x \in I\!R^n : |x - x_0| < r\}, D_r = D_r(0)$;
- $B_r(x_0,t_0) = \{(x,t) \in I\!R^n \times I\!R : |(x,t) - (x_0,t_0)| < R\}, B_r = B_r(0)$;
- $W(\theta,\nu;x_0) = \{y \in I\!R^n : \cos(y - x_0, \nu) \geq \cos\theta\}, 0 \leq \theta < \pi$.
- Two functions u_0, v_0 are *(strictly) separated* (denoted by $u_0 \prec v_0$) if

(i) the support of u_0, $supp(u_0) = \overline{\{u_0 > 0\}}$, is a compact subset of $I\!R^n$ and

$$\mathrm{supp}(u_0(x)) \subset \mathrm{Int}(\mathrm{supp}(v_0(x))).$$

(ii) inside the support of u_0 the functions are strictly ordered:

$$u_0(x) < v_0(x).$$

• For a nonnegative real valued function $u(x,t)$ defined in a cylindrical domain $D \times (a,b)$,

$$u^*(x,t) = \limsup_{(\xi,s)\in D\times(a,b))\to(x,t)} u(\xi,s);$$

$$\Omega(u) = \{(x,t) : u(x,t) > 0\}, \quad \Omega_t(u) = \{x : u(x,t) > 0\};$$

$$\Gamma(u) = \partial\{(x,t) : u(x,t) = 0\}, \quad \Gamma_t(u) = \partial\{x : u(x,t) = 0\};$$

Next we recall the definition of viscosity solutions of (HS) from [K1]. Let $Q = (I\!R^n - K) \times (0,\infty)$ and let Σ be a cylidrical domain $D \times (a,b)$ in $I\!R^n \times (0,\infty)$.

Definition 1.1 *(1) A nonnegative upper semicontinuous function u defined in in $\bar{\Sigma}$ is a* viscosity subsolution *of (HS) in Σ if*

(a) for each $a < T < b$ the set $\overline{\Omega(u)} \cap \{t \leq T\}$ is bounded;

(b) for every $\phi \in C^{2,1}(\Sigma)$ such that $u - \phi$ has a local maximum in $\overline{\Omega(u)} \cap \{t \leq t_0\} \cap \Sigma$ at (z_0, t_0),

(i) $\quad -\Delta\phi(z_0,t_0) \leq 0 \qquad$ *if* $u(z_0,t_0) > 0$;

(ii) $\quad \min(-\Delta\phi, \phi_t - |D\phi|^2)(z_0,t_0) \leq 0 \qquad$ *if* $(z_0,t_0) \in \Gamma(u),\ u(x_0,t_0) = 0.$

(2) A nonnegative lower semicontinuous function v defined in in $\bar{\Sigma}$ is a viscosity supersolution *of (HS) in Σ if for every $\phi \in C^{2,1}(\Sigma)$ such that $v - \phi$ has a local minimum of in $\overline{\Omega(v)} \cap \{t \leq t_0\} \cap \Sigma$ at (z_0, t_0),*

$$(i) \qquad -\Delta\phi(z_0,t_0) \geq 0 \qquad \text{if } (z_0,t_0) \in \{v>0\},$$

(ii) *If* $(z_0,t_0) \in \partial\{v>0\}, |D\phi|(z_0,t_0) \neq 0$ *and* $\{\phi>0\} \cap \{v>0\} \cap B(z_0,t_0) \neq \emptyset$ *for any ball* $B(z_0,t_0)$,

then

$$\max(-\Delta\phi, \phi_t - |D\phi|^2)(z_0,t_0) \geq 0.$$

(3) u is a viscosity subsolution of (HS) with initial data u_0 and fixed boundary data 1 if

(a) u is a viscosity subsolution in $\bar{Q}$,

(b) $u = u_0$ at $t = 0$; $u \leq 1$ on ∂K;

(c) $\overline{\Omega(u)} \cap \{t=0\} = \overline{\Omega(u_0)}$;

(4) u is a viscosity supersolution of (HS) with initial data u_0 and fixed boundary data 1 if v is a viscosity supersolution in $\bar{Q}$ with $v = v_0$ at $t = 0$ and $v \geq 1$ on ∂K.

(5) u is a viscosity solution of (HS) if u is a viscosity supersolution and u^* is a viscosity subsolution of (HS).

Throughout the paper we keep the conditions on u_0 and notations in the introduction. The following theorem (theorem 2.2 in [K1]) plays an important role in our analysis.

Theorem 1.2 *Let u, v be respectively viscosity sub- and supersolutions of (HS) in Q with initial data $u_0(x), v_0(x)$ and fixed boundary data 1 on K. Moreover assume that $v > 0$ on $\Gamma_0(v)$ for $t > 0$, i.e. the support of v strictly expands at $t = 0$.*

(i) If $u_0 \prec v_0$, then $u(x,t) \prec v(x,t)$ for $t > 0$.

(ii) If $u_0 \leq v_0$, then

$$u(x,t) \leq v(x,t+\epsilon) \text{ for } \epsilon > 0, t \geq 0.$$

(iii) There exists a unique viscosity solution $v(x,t)$ of (HS) in Q with initial data v_0. Moreover the positive set $\Omega_t(v)$ strictly expands in time, i.e.,

$$\Gamma_s(v) \text{ is a compact subset of } \Omega_t(v) \quad \text{for } 0 \leq s < t.$$

We also state the local version of theorem 1.2, whose proof is parallel to theorem 1.2.

Theorem 1.3 (comparison principle) *Let u, v be respectively viscosity sub- and supersolutions in Σ with initial data $u_0 \prec v_0$ in D. If $u \leq v$ on ∂D and $u < v$ on $\partial D \cap \bar{\Omega}(u)$ for $a \leq t < b$, then $u(\cdot, t) \prec v(\cdot, t)$ in D for $t \in [a, b)$.*

For the rest of our paper we denote u as the unique viscosity solution of (HS) with fixed boundary data 1 and the initial data u_0 satisfying (I). Due to theorem 1.2, the following properties holds for u:

Lemma 1.4 *(i) u is continuous at $t = 0$.*
Moreover $u(x, t) = 1$ for $x \in \partial K$ and $u(x, t) = 0$ for $(x, t) \in \Gamma(u)$.

(ii) u is superharmonic in $\Omega(u)$ and u^* is subharmonic in Q.
Moreover $\bar{\Omega}(u) = \bar{\Omega}(u^*)$ and $\Gamma(u) = \Gamma(u^*)$.

(iii) u is harmonic in $\Omega(u)$. Indeed $u(x, t) = h_t(x)$ at each time $t > 0$, where

$$h_t(x) = \inf\{\alpha(x) : -\Delta\alpha \geq 0 \text{ in } \Omega_t(u); \quad \alpha = 1 \text{ on } \partial K; \quad \alpha \geq 0 \text{ in } \Gamma_t(u).\}$$

Proof

1. (i) follows from the definition of u. (ii) follows since due to theorem 1.2 (ii)-(iii)

$$u(x, t) \leq u^*(x, t) \leq u(x, t + \epsilon) \text{ for any } \delta, \epsilon > 0.$$

2. Let $h(x, t) = h_t(x)$ be defined as above for $t \geq 0$. It can be checked that $h(\cdot, t)$ is positive and harmonic in $\Omega_t(u)$ (see Chapter 1.3 of [T] for example.) From definition of h it follows that $h(x, t) \leq u(x, t)$. On the other hand by theorem 1.2 $u^*(x, t - \epsilon) \leq u(x, t)$ for $t > \epsilon$, and thus $u^*(x, t - \epsilon) = 0$ on $\Gamma_t(u)$. Thus again by definition of h we obtain $u^*(x, t - \epsilon) \leq h(x, t)$ for any small $\epsilon > 0$. Now it follows from the lower semicontinuity of u that

$$u(x, t) \leq \lim_{\epsilon \to 0} u^*(x, t - \epsilon) \leq h(x, t) \quad \text{ for } t > 0,$$

which leads to $h = u$ for $t \geq 0$. □

2 Lipschitz continuity of the free boundary

Since Ω_0 is bounded, we may assume that $\Omega_0 \subset D_{R_0}$ for some $R_0 > 0$. In this section we would like to show that the free boundary $\Gamma(u)$ becomes Lipschitz continuous in space and time once it is out of D_{R_0}. First we state the following lemma, which can be proven by a reflection argument and by Theorem 1.2. For detailed proof we refer to [AC], Proposition 2.1 and Lemma 2.2 where a parallel result is proven for u: a solution of (PME).

Lemma 2.1 *For $|x_0|, |x_1| > R_0$ with $(x_1, t) \in \bar{\Omega}(u) \cap Q$ the following holds:*

$$\textit{If } \cos(x_1 - x_0, x_0) \geq R_0/|x_0| \textit{ then } u^*(x_1, t) < u(x_0, t+\delta) \textit{ for } \delta > 0.$$

Let us define

$$T_0 = t(u_0) = \inf\{t > 0 : \Omega_t(u) \supset \bar{D}_{R_0}\} > 0. \qquad (2.0)$$

By comparing u with a radially symmetric solution of (HS) with initial data less than u_0, one can easily show that $T_0 < \infty$.

Theorem 2.2 *For $T_0 < \underline{t} \leq t \leq \bar{t} < \infty$ the free boundary $\Gamma(u)$ is representable in spherical coordinates of the form*

$$r = g(\theta, t)$$

where g is Lipschitz continuous in θ and t. Moreover $g(\cdot, t)$ is uniformly Lipschitz continuous with Lipschitz constant L in θ for $t \geq \underline{t}$ with $L \to 0$ as $t \to \infty$.

Proof

1. First we show that $\Gamma_t(u)$ is starshaped and Lipschitz continuous at each time $t > \underline{t} > T_0$. For $T > T_0$, it follows from Lemma 2.1 that

$$u^*(x, T) < u(\frac{x}{1+\epsilon}, T+\delta) \text{ for } x \in \Gamma_T(u^*).$$

for any $\delta > 0$ and small $0 < \epsilon < \epsilon(T)$. Also

$$u^*(x, T) < 1 = u(x/(1+\epsilon), T+\delta) \text{ on } \{x : |x| = 1+\epsilon\}.$$

Then the maximum principle for harmonic functions yields

$$u^*(x, T) \prec u(x/(1+\epsilon), T+\delta) \quad \text{in } |x| \geq 1+\epsilon.$$

2. Since

$$w(x,t) = u(\frac{x}{1+\epsilon}, \frac{t-T}{1+\epsilon} + T + \delta)$$

is a viscosity supersolution of (HS) in $\{x : |x| \geq 1+\epsilon\} \times [T, \infty)$ with $w = 1 \geq u$ on $\{x : |x| = 1+\epsilon\}$, we have

$$u^*(x,t) \prec w(x,t) \quad \text{for } \{x : |x| \geq 1+\epsilon\} \cap \{t \geq T\}. \tag{2.1}$$

For given $\epsilon > 0$ and $\alpha > 0$, we can choose $\delta = \alpha\epsilon/(1+\epsilon)$ such that $(t-T)/(1+\epsilon) + T + \delta = t$ at $t = T + \alpha$ and thus due to (2.1) it follows

$$u^*(x, T+\alpha) \prec u(x/(1+\epsilon), T+\alpha) \text{ for any } \epsilon, \alpha > 0. \tag{2.2}$$

In particular (2.2) implies that for $t > T_0$ our solution $u(\cdot, t)$ is decreasing in radial directions and thus $\Gamma_t(u)$ is representable by spherical coordinates $r = g(\theta, t) = g(\theta, t) > R_0$.

3. Due to Lemma 2.1 for any direction $p \in I\!R^n$ such that $\cos(p, x) \geq R_0/|x|$ we have

$$u^*(x+\epsilon p, t) < u(x, t+\delta) \text{ for } \epsilon > 0$$

if $|(x+\epsilon p)|, |x| > R_0$ and $(x+\epsilon p, t) \in \bar{\Omega}(u)$.

Hence for $x \in \Gamma_t(u)$,$\mathrm{T}_0 < \underline{t} \leq t$ there exists $\epsilon_0 > 0$ such that for $0 \leq \epsilon < \epsilon_0$ and $\delta > 0$

$$u^*(x+\epsilon p, t-\delta) < u(x,t) \text{ if } (x+\epsilon p, t-\delta) \in \Omega(u). \tag{2.3}$$

This implies that $u(x+\epsilon p, t-\delta) = 0$ for above ϵ, p, δ given in (2.3) and thus $u(x+\epsilon p, t) = 0$ by the lower semicontinuity of u.

In other words at each point $x \in \Gamma_t(u)$ there is $\theta_0 = \theta(|x|) = \cos^{-1}(R_0/|x|)$ such that

$$u(x,t) = 0 \text{ for } y \in W(\theta_0, x/|x|; x), |x-y| < \epsilon_0.$$

On the other hand by similar argument it can be easily seen that

$$u(z,t) > 0 \text{ for } z \in W(\theta_0, -x/|x|; x), |x-z| < \epsilon_0.$$

This leads to the Lipschitz continuity g with respect to θ. Observe that the Lipschitz constant of $g(\cdot, t)$ only depends on θ_0, which can be chosen as

$$\theta_0 = \theta(\underline{t}) = \cos^{-1}[R_0/r(\underline{t})],$$

where $r(\underline{t}) = \inf\{|x| : x \in \Gamma_t(u), t = \underline{t}\} > R_0$. Since $r(t) \to \infty$ as $t \to \infty$, it follows that $\theta_0 \to \pi/2$ as $t\infty$.

4. It remains to show that g is Lipschitz locally in time. We observe that due to (2.1), If $(x_0, t_0) \in \Gamma(u)$, $t_0 > T_0$ then $u((1+\epsilon)x, (1+\epsilon)(t-T_0)+T_0) = 0$ for $\epsilon > 0$. This leads to the local Lipschitz continuity of f in time, i.e.,

$$\frac{|g(\theta, t) - g(\theta, t_0)|}{|t - t_0|} \leq \frac{g(\theta, t_0)}{t_0 - T_0} \text{ if } T_0 < t.$$

□

Due to the Lipschitz continuity of the free boundary, we are now able to show the following properties of u:

Proposition 2.3 *(i) $u(x,t)$ is continuous for $t > T_0$:*

(ii) If $|x_0|, |x_1| > R_0$ and $\cos(x_1 - x_0, x_0) \geq R_0/|x_0|$ then

$$u(x_1, t) \leq u(x_0, t) \text{ for } \delta > 0.$$

(iii) for $T > T_0$, $0 < \epsilon < \epsilon_0(R_0)$ and $|x| \geq R_0$

$$u(x,t) \leq u(x/(1+\epsilon), (t-T)/(1+\epsilon) + T) \quad \text{for } t > T. \tag{2.4}$$

Proof

1. (ii) follows from (i) and Lemma 2.1.
2. By Lemma 1.3 (iii), at each $t > 0$

$$u(x,t) = \inf\{\alpha(x) : -\Delta\alpha \geq 0 \text{ in } \Omega_t(u); \alpha \geq 1 \text{ on } \partial K; \alpha \geq 0 \text{ on } \Gamma_t(u).\}$$

For $t > T_0$, since $\Gamma_t(u)$ is Lipschitz continuous, Perron's method (see Chapter 1.3 of [T] for example) yields that $u(\cdot, t)$ is continuous in $\overline{\Omega_t(u)}$ and harmonic in $\Omega_t(u)$ with $u = 1$ on ∂K and $u = 0$ on $\Gamma_t(u)$.

3. To prove continuity of u in time, let us pick $(x, t_0) \in \Omega(u)$ such that $t_0 > T_0$. due to theorem 1.3 (ii) $u^*(x, t_0 - \delta) \leq u(x, t_0)$ for any $\delta > 0$. On the other hand from (2.2) we have $u^*((1+\epsilon)x, t_0 - \delta) \leq u(x, t_0 - \delta)$ for any $\epsilon > 0$ and $0 < \delta < t_0 - T_0$. Hence due to theorem 1.2 (ii)

$$u^*((1+\epsilon)x, (1+\epsilon)(t - t_0 + \delta) + t_0 - \delta) \leq u(x,t) \text{ for } t \geq t_0 - \delta.$$

Now evaluating above inequality at $t = t_0 - \epsilon/(1+\epsilon)\delta$, we obtain

$$u((1+\epsilon)x, t_0) \leq u(x, t_0 - \frac{\epsilon}{1+\epsilon}\delta.)$$

Since ϵ, δ are arbitrary, we conclude that

$$u(x,t) = \lim_{\epsilon \to 0} u((1+\epsilon)x, t_0) \leq \lim_{\epsilon \to 0} u(x, t_0 - \epsilon).$$

Thus $u(x,t) = \lim_{\epsilon\to 0} u(x, t-\epsilon)$ and from similar argument it follows that

$$u(x,t) = \lim_{\epsilon \to 0} u(x, t+\epsilon).$$

Thus u is continuous in time and now (iii) follows from (2.1).

$\square$

Lemma 2.4 *There exist $A, B, \epsilon_0 > 0$ depending on u_0 such that*

$$u_\epsilon(x,t) = \frac{1+A\epsilon}{(1+\epsilon)^2} u((1+\epsilon)x, (1+A\epsilon)t + B\epsilon) \geq u(x,t) \tag{2.5}$$

for $0 < \epsilon < \epsilon_0$.

Proof.

It is easy to check that u_ϵ is a supersolutions of (HS). Therefore due to theorem 1.2, we only have to show that there is $A, B > 0$ such that

$$\frac{1+A\epsilon}{(1+\epsilon)^2} u((1+\epsilon)x, B\epsilon) \geq u(x,0).$$

For simplicity, let us assume that Ω_0 contains D_4 and let

$$c = \inf\{|Du_0(x)|/4|x| : x \in \Gamma_0\}$$

(Note that $c > 0$ due to condition (I).) Then $w(x,t) = u_0((1+ct)^{-1}x)$ is a viscosity subsolution of (HS) in the domain $\{x : |x| \geq 1 + ct\} \times [0, 1/c]$, since $w(\cdot, t)$ is harmonic in $\Omega_t(w)$ and

$$w_t \leq c|x||Du_0| < (1+ct)^{-2}|Du_0|^2 = |Dw|^2 \text{ on } \Gamma_t(w) \times [0, 1/c].$$

Therefore $w_2(x,t) = c_2 w(x, c_2 t)$, $c_2 > 0$ is also a subsolution of (HS). Now if we choose c_2 such that $c_2 < u_0(x)$ on $|x| = 2$, then $w_2 \leq u$ on $|x| = 2$ for $0 \leq t \leq C = (cc_2)^{-1}$. By Theorem 1.2 then $w_2 \leq u$ in the domain $\{x : |x| \geq 2\} \times (0, C)$. In particular, if $x \in \Omega_0$ and $|x| \geq 2$ then

$$(1 + C^{-1}t)x \in \Omega_t(u) \text{ for } 0 \leq t \leq C.$$

On the other hand if $|x| \leq 2$ then

$$(1+C^{-1}t)|x| \leq 2|x| \in D_4 \subset \Omega_0 \subset \Omega_t(u) \text{ for } 0 \leq t \leq C.$$

In other words if we let $B = C$ then $u(x,0) > 0$ implies that $u((1+\epsilon)x, B\epsilon) > 0$ for small $\epsilon > 0$. Thus if we choose A big enough such that

$$u_0(x) < \frac{1+A\epsilon}{(1+\epsilon)^2} u_0((1+\epsilon)x) \text{ on } |x| = 1 \text{ for small } \epsilon > 0,$$

then we can conclude. (Such A can be chosen, for example, as the maximum of $2|Du_0|$ on $|x| = 1$.)

□

Remark

Heuristically speaking, differentiating (2.4) and (2.5) with respect to ϵ at $\epsilon = 0$ yields that for $t > T_0$

$$\frac{x \cdot \nu(x,t)}{At+B} \leq \frac{u_t}{|Du|} \leq \frac{x \cdot \nu(x,t)}{t - T_0} \text{ on } \Gamma(u), \tag{2.6}$$

where $\nu = -Du/|Du|$. Since formally $V = u_t/|Du| = |Du|$ on $\Gamma(u)$, (2.6) is expected to yield bounds for $|Du|$ on $\Gamma(u)$ for $t > T_0$. In the next section we use barrier arguments based on the geometry (Lipschitz continuity) of Γ_t to obtain the expected bounds on $|Du|$.

3 Gradient Estimates

First we prove the nondegeneracy of $|Du|$ near $\Gamma(u)$. We start with the following lemma.

Lemma 3.1 *Let $(x_0, t_0) \in \Gamma(u)$ and $T_0 < \underline{t} \leq t_0 < \bar{t} < \infty$. Then there exist positive constants h_0, C_1, C_2 depending only on $\underline{t}, \bar{t}$ such that*

$$\sup\{u(x,t) : |x - x_0| < h, 0 \leq t - t_0 \leq C_2 h\} > C_1 h \quad \textit{for } 0 < h < h_0.$$

Proof.

1. Let $x_1 = (1 + \frac{h}{|x_0|})x_0$. By theorem 2.2 for small $h > 0$ there is $L > 1$ depending only on $\underline{t}$ such that $u(\cdot, t_0) = 0$ in $D_{h/L}(x_1)$ (see Figure 2.)

2. Consider $\omega(x)$ such that

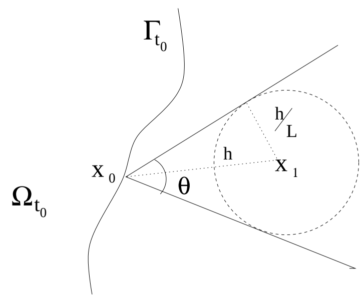

Figure 2.

$$\begin{cases} -\Delta\omega = 0 & \text{in } \Sigma := D_h(x_1) - D_{h/L}(x_1); \\ \omega = 0 & \text{on } \partial D_{h/L}(x_1); \\ \omega = C_1 h & \text{on } \partial D_h(x_1). \end{cases}$$

Notice that $\omega(x) = \omega(r)$, where $r = |x - x_1|$ and $|\omega_r| = C > 0$ on $\partial D_{h/L}(x_1)$ where C is a constant depending only on L and C_1. (In fact $C = k(n)C_1$ for fixed L, where k is a constant depending on the dimension n.) Then $v(x,t) = (1 + Ct)^{-1}\omega((1 + 8C\frac{L}{h}t)r)$ is a supersolution of (HS) in

$$\Sigma_2 = \Sigma \times (0, C_2 h], C_2 = 1/8CL.$$

In fact on the free boundary $(1 + 8C\frac{L}{h}t) = \frac{h}{L}r^{-1}$ and

$$v_t = \frac{8CLr/h}{1+Ct}\omega_r, \quad |Dv| = \frac{1+8CLt/h}{1+Ct}\omega_r.$$

Therefore we have

$$v_t - |Dv|^2 \geq C(8r^3 - (h/L)^3) \geq 0 \text{ if } r \geq \frac{h}{2L},$$

which is true for $0 \leq t \leq \frac{h}{8CL}$.

3. Now if we assume that $u(x,t) \leq C_1 h$ in Σ_2, then $u \leq v$ on the parabolic boundary of Σ_2 and thus $u \leq v$ in Σ_2. Therefore the point $(x_1, t_0 + \frac{h}{8CL})$ belongs to the interior of the set $\{u = 0\}$. But this contradicts Lemma 2.4 if we take C_1 such that

$$\frac{|x_1 - x_0|}{|t_1 - t_0|} = |4C_1 L| \leq \frac{\alpha}{At_0 + B},$$

where A, B is as in Lemma 2.4 and

$$\alpha = \alpha(\bar{t}) = \inf\{|x| : x \in \Gamma_{\underline{t}}\} > R_0.$$

$\square$

Lemma 3.2 *If $T_0 < \underline{t} < t_1 < \bar{t}$ and $g(x_1/|x_1|, t_1) - |x_1| = h$ with $0 < h \leq (|x_1| - R_0)/2$ then we have*

$$u(x_1, t_1) \geq C_3 h,$$

where C_3 only depend on the initial data u_0 and $\underline{t}, \bar{t}$.

Proof.

1. We recall that for $T > T_0$ and for any $\epsilon > 0$

$$w(x,t) = u(x/(1+\epsilon), (t-T)/(1+\epsilon) + T) \geq u(x,t).$$

Let us take $T = T_0 + (t_1 - T_0)/2$. Then

$$\text{if } u(x_1, t_1) \leq ch \text{ then } u((1+\epsilon)x_1, (1+\epsilon)(t_1 - T) + T) \leq ch. \tag{3.1}$$

2. Now assume that $g(x_1/|x_1|, t_1) - |x_1| = M_1 h$, where $M_1 > 0$ is to be decided. Let $\nu = x_1/|x_1|$ and $x_0 = g(\frac{x_1}{|x_1|}, t_1)\nu \in \Gamma_{t_1(u)}$. From Lemma 3.1

$$\sup_{|x-x_0| \leq h, 0 \leq t - t_1 \leq C_1 h} u(x,t) > C_2 h, \tag{3.2}$$

where $C_i = C_i(\underline{t}, \bar{t})$.

We recall that in $\{|x| > R_0\} \times \{t \geq \underline{t}\}$ there is a cone of directions $K(\theta_{\underline{t}}, \nu)$ along which $u(\cdot, t_1)$ is decreasing (see the remark after Proposition 2.3.) Hence there is $M_2 = M_2(\underline{t}) > 0$ such that if we take $M_1 = M_2$ then $D_h(x_0)$ belongs to $W(\theta_{\underline{t}}, \nu; x_1)$. We choose M_1 a bit larger than M_2, that is, we set

$$M_1 = M_2 + \frac{C_1}{\underline{t} - T_0} \sup\{|x| : x \in \Gamma_{\bar{t}}(u)\}$$

so that $D_h(x_0)$ belongs to the cone $W(\theta_{\underline{t}}, \nu; (1+\epsilon)x_1)$ for $0 \leq \epsilon \leq \beta$, where $\beta = \frac{C_1}{t_1 - T_0} h$. (see figure 3.)

3. Due to (3.1) if $u(x_1, t_1) < C_2 h$ then $u((1+\epsilon)x_1, t_1 + \epsilon(t_1 - T)) < C_2 h$ for h small and $0 \leq \epsilon \leq \beta$. Due to the fact that $u(x,t) \leq u(y,t)$ for $x \in W(\theta_{\underline{t}}, \nu; y)$ for $|x|, |y| > R_0$, it follows that $u(x,t) < C_2 h$ in the cylinder

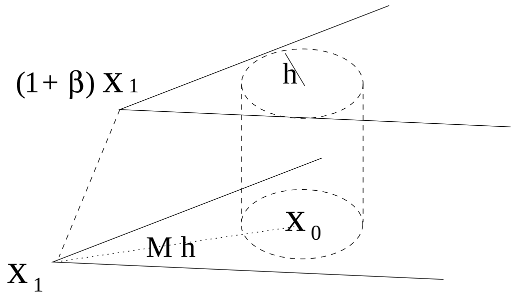

Figure 3.

$D(x_0, h) \times [t_1, t_1 + C_1 h]$. This contradicts (3.2) and thus we can conclude by taking $C_3 = C_2/M_1$.

$\square$

Next we proceed to obtain an upper bound for $|Du|$ near $\Gamma(u)$.

Lemma 3.3 *Let* $g(\frac{x_1}{|x_1|}, t_0) - |x_1| = h$ *with* $0 \leq h \leq |x_1| - R_0/2$ *and* $T_0 < \underline{t} \leq t_0 \leq \bar{t}$. *Then there are constants* L *depending on* $\underline{t}, n$ *and* $M = M(L, \underline{t}, \bar{t}, u_0, n)$ *such that*

$$\inf\{u(x, t_1) : |x - x_1| < h/L\} \leq Mh.$$

Proof 1. Let $x_0 = x_1 + h\nu \in \Gamma_{t_0}(u)$, where $\nu = \frac{x_1}{|x_1|}$. Due to theorem 2.2, there is a constant $L = L(\underline{t}) > 0$ such that $D_{2h/L}(x_1) \in \Omega_{t_0}$. Consider a harmonic function $\omega(x)$ in $D_{2h/L}(x_1) - D_{h/L}(x_1)$ such that

$$\begin{cases} \omega(x) = 0 & \text{on } \partial D_{2h/L}(x_1), \\ \\ \omega(x) = Mh & \text{on } \partial D_{h/L}(x_1), \end{cases}$$

where $M > 0$ is a constant chosen so that $|D\omega| = M_3(1 + M_3)/L$ on $\partial D_{2h/L}(x_1)$, where $M_3 > 0$ is to be decided. Note that $\omega(x) = \omega(r)$, $r = |x - x_1|$ and M only depends on M_3 and n.

Let us extend ω outside $D_{2h/L}(x_1)$ by zero and let

$$v(x, t) = \omega((1 + \frac{M_3}{h}(t - t_0))^{-1}(r - \frac{h}{L}) + \frac{h}{L}).$$

Then at $t = t_0 + h$, $r - \frac{h}{L} = (1 + M_3)\frac{h}{L}$ on $\Gamma_t(v)$ and thus v satisfies

$$\begin{aligned} v_t - |Dv|^2 &= \frac{M_3/h}{(1+\frac{M_3}{h}(t-t_0))^2}(r-\frac{h}{L})|\omega_r| - \frac{|\omega_r|^2}{(1+\frac{M_3}{h}(t-t_0))^2} \\ &\leq c\cdot(\frac{M_3}{h}(r-\frac{h}{L}) - M_3(1+M_3)/L) \leq 0 \quad \text{ on } \Gamma_t(v), t_0 \leq t \leq t_0+h. \end{aligned}$$

Hence v is a supersolution of (HS) in $(I\!R^n - D_{h/L}(x_1)) \times (t_0+t_0+h]$. 2. Now assume that $u > Mh$ on $D_{h/L}(x_1)$. Since u is increasing in time, $v(x,t_0) = \omega(x) \prec u(x,t_0)$ in $I\!R^n - D_{h/L}(x_1)$ and $v = Mh < u$ on $\partial D_{h/L}(x_1) \times [t_0, t_0+h]$. Hence by Theorem 1.2 (i), it follows that $v \prec u$ in $(I\!R^n - D_{h/L}(x_1)) \times [t_0, t_0 + h]$ and thus

$$(3.3) \quad (x_1+(2+M_3)\frac{h}{L}\nu, t_0+h) = (x_0+(2+M_3-L)\frac{h}{L}\nu, t_0+h) \in \bar{\Omega}(v) \subset \Omega(u).$$

3. On the other hand, due to Proposition 2.3 for any $\epsilon > 0$ we have $u((1+\epsilon)x_0, t_0+\epsilon(t_0-T_0)) = 0$. Now this and (3.3) leads to a contradiction if we choose $\epsilon = h/t_0 - T_0$ and $M_3 = L - 2 + L\alpha/|\underline{t} - T_0|$, where

$$\alpha = \alpha(\bar{t}) = \sup\{|x| : x \in \Gamma_{\bar{t}}(u)\}.$$

□

Lemma 3.4 *For $T_0 < \underline{t} \leq t_1 \leq \bar{t}$ there is $h_0 = h(\underline{t}) > 0$ satisfying the following:*

if $g(x_1/|x_1|, t_1) - |x_1| = h$ with $0 < h < h_0$ then we have

$$u(x_1, t_1) \leq M'h,$$

for $M' = M'(\underline{t}, \bar{t}, u_0, n) > 0$.

Proof

1. Take $h_0 = h_0(\underline{t}) > 0$ such that $|x| > R_0 + 3h_0$ on $\Gamma_{\underline{t}}(u)$. Let $\nu = x_1/|x_1|$ and let $x_0 = x_1 + h\nu \in \Gamma_{t_1}(u)$. We recall that, due to proposition 2.3, there is a cone of directions $W(\theta_{\underline{t}}, -\nu; x_1)$ along which $u(\cdot, t_1)$ is increasing in the domain $\{|x| > R_0\} \times (\underline{t}, \infty)$. We choose $L = L(\underline{t}) > 2$ such that $D_{2h/L}(x_2) \in W(\theta_{\underline{t}}, -\nu; x_1)$ where $x_2 = x_1 - h\nu$ (see Figure 4.) By the choice of h_0, $D_{2h/L}(x_2) \subset \{|x| > R_0\}$. Observe that lemma 3.3 holds with this choice of L.

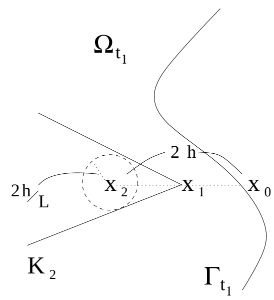

Figure 4.

2. Assume $u(x_1, t_1) > M'h$, where $M' = M(L)$ is given as in Lemma 3.3 with the choice of L in step 1. Then it follows that $u(\cdot, t_1) \geq u(x_1, t_1) > M'h$ in $D_{2h/L}(x_2)$. This contradicts lemma 3.3 since $g(\frac{x_2}{|x_2|}, t_1) - |x_2| = 2h$.

□

Now we are ready to prove the main theorem:

Theorem 3.5 *Let $u_0(x)$ to satisfy (I) and let u be a viscosity solution of (HS) with initial data u_0 and fixed boundary data 1 on K. Then $\Gamma(u)$ is Lipschitz continuous in space and time for $t > T_0$, where T_0 is given in (2.0). Moreover there is a neighborhood $\mathcal{N}$ of $\Gamma(u)$ in each strip of the form $T_0 < \underline{t} \leq t \leq \bar{t}$ and constants $C_1, C_2 > 0$ depending only on $\underline{t}, \bar{t}, u_0$ and n such that*

$$C_1 \leq \frac{u_t}{|Du|}, |Du(x,t)| \leq C_2,$$

if $(x,t) \in \Omega(u) \in \mathcal{N}$.

Proof

We only have to obtain the bounds for $|Du|$. From theorem 2.2 and properties of harmonic functions in a Lipschitz domain (see Lemma 4 of [C1]), for $t \geq \underline{t} > T_0$ there are positive constants c_1, c_2 depending on $\underline{t}$ such that

$$0 < c_1|Du|(x_0 - h\nu, t) \leq u(x_0 - h\nu, t)/h \leq c_2|Du|(x_0 - h\nu, t)$$

where $x_0 \in \Gamma_t(u)$ and $\nu = \frac{x_0}{|x_0|}$. Now lemma 3.3 and 3.4 leads to the conclusion.

□

Based on above theorem the following result is proved in section 6 of [K2]:

Corollary 3.6 *Let u_0, u be given as in theorem 3.5. Then after $t > T$ u is a classical solution. More precisely u and $\Gamma(u) \cap \{t > T\}$ are analytic in space and time and u satisfies the free boundary condition $V = |Du|$ in the classical sense.*

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