Homogenization of a Hele-Shaw type problem in periodic and random media

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Abstract

We investigate the homogenization limit of a free boundary problem with space-dependent free boundary velocities. The problem under consideration has a well-known obstacle problem transformation, formally obtained by integrating with respect to the time variable. By making rigorous the link between these two problems, we are able to derive an explicit formula for the homogenized free boundary velocity, and we establish the uniform convergence of the free boundaries.

1 Introduction

This paper is devoted to the homogenization of a Hele-Shaw type problem in periodic and random media.

Let K be a compact subset of $\mathbb{R}^n$ with smooth boundary ∂K and let Ω_0 be a bounded open set with C^2 boundary such that

$$K \subset \Omega_0.$$

We are interested in the asymptotic behavior as $\varepsilon \to 0$ of the solution $v^\varepsilon(x,t)$ of the following free boundary problem:

$$(P_\varepsilon) \qquad \begin{cases} -\Delta v^\varepsilon = 0 & \text{in } \{v^\varepsilon > 0\} \setminus K \\ v^\varepsilon = 1 & \text{on } K \\ v^\varepsilon_t = g(x/\varepsilon)|Dv^\varepsilon|^2 & \text{on } \partial\{v^\varepsilon > 0\} \end{cases}$$

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satisfying the initial condition $v^\varepsilon(x,0) = v_0(x)$, where $v_0(x)$ is a continuous function, harmonic in $\Omega_0 \setminus K$ and satisfying

$$v_0(x) = 1 \quad \text{for } x \in K \quad \text{and} \quad v_0(x) = 0 \quad \text{for } x \notin \Omega_0.$$

Here, v_t denotes the partial derivative with respect to the time variable t, while Dv denotes the spatial derivative with respect to x. Note that if v^ε is smooth up to its free boundary $\partial\{v^\varepsilon > 0\}$, then the velocity of the free boundary is given by $V = \frac{v_t^\varepsilon}{|Dv^\varepsilon|}$ and the last condition in (P_ε) can be rewritten as

$$V = g(x/\varepsilon)|Dv^\varepsilon|.$$

Free boundary problems with velocity law given as in (P_ε) describe various motions in heterogeneous media, including heat transfer and shoreline movements in oceanography (see [P],[R2], [Rou], and [VSKP]).

The function $g : \mathbb{R}^n \to \mathbb{R}$ is a given continuous function satisfying

$$0 < \lambda \leq g(x) \leq \Lambda \tag{1.1}$$

for two positive constants λ and Λ. In order to observe some kind of averaging behavior as ε goes to zero, we need to make further assumptions on g. In this paper, we assume that g is stationary ergodic. More precisely, we consider a probability space $(A, \mathcal{F}, P)$ and we assume that $g(x,\omega)$ is a random variable such that

1. the distribution of the random variable $g(x,\cdot) : A \to \mathbb{R}$ is independent of x (we say that g is stationary). More precisely, we will assume that for every $x \in \mathbb{R}$ there exists a measure preserving transformation $\tau_x : A \to A$ such that:
$$g(x + x', \omega) = g(x, \tau_{x'}\omega) \text{ for all } x' \in \mathbb{R}^n \text{ and } \omega \in A.$$
2. the underlying transformation τ_x is ergodic, that is if $B \subset A$ is such that $\tau_x B = B$ for all $x \in \mathbb{R}^n$, then $P(B) = 0$ or 1.

As a consequence of those hypotheses, we will deduce that there exists a constant, denoted by $\left\langle \frac{1}{g} \right\rangle$, such that

$$\int_{\mathbb{R}^n} \frac{1}{g(x/\varepsilon)} u(x)\, dx \longrightarrow \int_{\mathbb{R}^n} \left\langle \frac{1}{g} \right\rangle u(x)\, dx \qquad \text{a.e. } \omega \in A$$

for any function u in $L^1(\mathbb{R}^n)$.

This properties, which holds for much more general functions $g(x)$, is all that we will need for our result to hold. In particular, in the (simpler) case where $g : \mathbb{R}^n \mapsto \mathbb{R}$ is $\mathbb{Z}^n$-periodic, the quantity $\left\langle \frac{1}{g} \right\rangle$ is the average of $1/g(x)$ over one cell.

Our main result states that the free boundaries $\partial\{v^\varepsilon = 0\}$ locally uniformly converge to $\partial\{v = 0\}$, where v solves of the following homogenized Hele-Shaw equation:

$$(P_0) \qquad \begin{cases} -\Delta v = 0 & \text{in } \{v > 0\} \setminus K \\ v = 1 & \text{on } K \\ v_t = \left\langle \frac{1}{g} \right\rangle^{-1} |Dv|^2 & \text{on } \partial\{v > 0\} \end{cases}$$

with the same initial condition $v(x, 0) = v_0(x)$.

Note that the positive phases of solutions for both problems (P_0) and (P^ε) may go through topological changes such as merging of two fingers. Our result states that the oscillating free boundaries converges uniformly even in the event of such singularities.

It is interesting to note that the homogenized speed is unique and independent of the direction of propagation. The condition (1.1) obviously plays an important role in this result and is crucial in the analysis. In fact, it is known that when the velocity is allowed to vanish, very different asymptotic behaviors are observed. In particular, one of the author proves in [K3] that if the free boundary condition is replaced by

$$u_t^\varepsilon = |Du^\varepsilon|(|Du^\varepsilon| - g(x/\varepsilon))$$

then the homogenized speed depends in a non trivial way of Du, and various phenomena such as pinning and hysteresis take place. Similar phenomena are also derived for the homogenization of the following free boundary problem:

$$\begin{cases} \partial_t u^\varepsilon - \Delta u^\varepsilon = 0 & \text{in } \{u^\varepsilon > 0\} \\ |\nabla u^\varepsilon|^2 = g(x/\varepsilon) & \text{on } \partial\{u^\varepsilon > 0\} \end{cases}$$

(see [CLM1], [CLM2]) and are also strongly related to the fact that the free boundary may stop moving.

In the case of periodic media, [K1] obtains uniform convergence of the free boundary via viscosity solution approach, with a rate of convergence for the free boundaries (see [K2]). But this approach does not yield the explicit form of the free boundary velocity. On the other hand the variational approach, (see section 4.1, and Rodriges [R1] in the case of Stefan problem) yields the explicit formula for the limiting problem but does not yield the uniform convergence of the free boundaries unless the limiting boundaries are smooth.

In this paper we combine these two approaches to obtain our result. The investigation consists of two parts:

In order to obtain the convergence of positive phases almost everywhere as well as to derive the formula for the homogenized free boundary velocity (P_0), we rely on an obstacle problem formulation of the Hele-Shaw problem (P_ε). More precisely, it is well-known, since the work of Elliot-Janovsky [EJ] (also see Gustafsson [G]) that the function

$$u^\varepsilon(x,t) = \int_0^t v^\varepsilon(x,s)\,ds$$

is formally solution to

$$(\tilde{P}_\varepsilon) \qquad \begin{cases} -\Delta u^\varepsilon = -\frac{1}{g(x/\varepsilon)}\chi_{\mathbb{R}^n\setminus\Omega_0} & \text{in } \{u^\varepsilon > 0\} \\ u^\varepsilon = |Du^\varepsilon| = 0 & \text{on } \partial\{u^\varepsilon > 0\} \\ u^\varepsilon = t & \text{on } K \end{cases}$$

which is nothing but the Euler-Lagrange equation for some obstacle problem. We will prove, in section 3, that the "time derivative" of the solution u^ε of $(\tilde{P}_\varepsilon)$ coincides with the viscosity solution v^ε of (P_ε). Variational arguments then allow us to prove the uniform convergence of u^ε to u^0, solution of the obstacle problem corresponding to the Hele-Shaw problem (P_0).

Unfortunately, the convergence of v^ε does not follow from that of u^ε, and in order to prove the convergence of v^ε, solution of (P_ε), to v^0, solution of (P_0), we need to investigate the behavior of the oscillating free boundaries $\partial\{u^\varepsilon > 0\}$. For this, we go back, in the last part of this paper, to the viscosity solutions method and we use maximum principle-type arguments as well as stability properties of viscosity solutions, to prove the uniform convergence of $\partial\{u^\varepsilon > 0\}$ to $\partial\{u^0 > 0\}$. The convergence of v^ε to v^0 follows.

We expect that our method applies to the homogenization of Stefan-type problem (work in progress).

In section 2, we recall the definition of viscosity solutions for the Hele-Shaw problem (P_ε). Section 3 is devoted to making the link between (P_ε)

and $(\tilde{P}_\varepsilon)$ rigorous. The homogenization of the obstacle problem and then of the Hele-Shaw problem is detailed in section 4.

Remark: For initial positive phase Ω_0 with boundary less regular than C^2, all our results, in particular Theorem 2.7 and Theorem 3.1, hold as long as the positive phase immediately expands at $t=0$ with $\partial\Omega_0 = \partial\bar{\Omega}_0$ (the second condition guarantees that $v^\varepsilon(x,t)$ changes continuously at $t=0$.) Our arguments also hold for smooth fixed boundary data $f(x,t)>0$ on K, instead of 1. Since the main focus of this article is on the homogenization, we avoid the most general argument on the initial and fixed boundary data.

Notations: For any nonnegative function $w(x,t): I\!R^n \times [0,\infty) \to I\!R^+$, let us define

$$\Omega(w) = \{w>0\}, \quad \Omega_t(w) = \{x;\, w(x,t)>0\}$$

and

$$\Gamma(w) = \partial\Omega(w), \quad \Gamma_t(w) = \partial\Omega_t(w).$$

We call $\Omega_t(u)$ and $\Gamma_t(u)$ respectively the *positive phase* and the *free boundary* of w.

2 Definition of viscosity solutions

In this section, we recall the definition of viscosity solutions for the Hele-Shaw problems (P_ε) and (P_0) from [K1].

Consider a space-time domain $\Sigma \subset I\!R^n \times [0,\infty)$ with smooth boundary.

For a nonnegative function $w(x,t)$, let us define

$$w_*(x,t) = \liminf_{(y,s)\to(x,t)} w(y,s)$$

and

$$w^*(x,t) = \limsup_{(y,s)\to(x,t)} w(y,s).$$

Definition 2.1. *A nonnegative upper semicontinuous function v defined in Σ is a* viscosity subsolution *of (P_ε) if the followings hold:*

(a) For each $T\in(0,\infty)$, the set $\overline{\Omega(v)}\cap\{t\le T\}\cap\Sigma$ is bounded.

(b) For every $\phi\in C^{2,1}(\Sigma)$ such that $u-\phi$ has a local maximum in $\overline{\Omega(v)}\cap\{t\le t_0\}\cap\Sigma$ at (x_0,t_0), the following holds:

(*i*) *If* $u(x_0,t_0) > 0$ *then* $-\Delta\phi(x_0,t_0) \le 0$.

(*ii*) *If* $(x_0,t_0) \in \Gamma(u), |D\phi|(x_0,t_0) \neq 0$ *and* $-\Delta\varphi(x_0,t_0) > 0$, *then*

$$(\phi_t - g(x_0/\varepsilon)|D\phi|^2)(x_0,t_0) \le 0.$$

Definition 2.2. *A nonnegative lower semicontinuous function v defined in Σ is a* viscosity supersolution *of (P_ε) if for every $\phi \in C^{2,1}(\Sigma)$ such that $v-\phi$ has a local minimum in $\Sigma \cap \{t \le t_0\}$ at (x_0,t_0), the following holds:*

(*i*) *If* $v(x_0,t_0) > 0$ *then* $-\Delta\phi(x_0,t_0) \ge 0$.

(*ii*) *If* $(x_0,t_0) \in \Gamma(v), |D\phi|(x_0,t_0) \neq 0$ *and* $-\Delta\varphi(x_0,t_0) < 0$, *then*

$$(\phi_t - g(x_0/\varepsilon)|D\phi|^2)(x_0,t_0) \ge 0.$$

Let $K, \Omega_0, \Gamma_0, v_0$ be as given in the introduction and let $Q = (I\!R^n - K) \times (0,\infty)$.

Definition 2.3. *v is a viscosity subsolution of (P_ε) in Q with initial data v_0 if*

(a) v is a viscosity subsolution of (P_ε) in Q,

(b) v is upper semicontinuous in $\bar{Q}$, $v = v_0$ at $t = 0$ and $v \le 1$ on ∂K.

(c) $\overline{\Omega(v)} \cap \{t = 0\} = \overline{\Omega(v_0)}$.

Definition 2.4. *v is a viscosity supersolution of (P_ε) in Q with initial data v_0 if*

(a) v is a viscosity supersolution in Q,

(b) v is lower semicontinuous in $\bar{Q}$, $v = v_0$ at $t = 0$ and $v \ge 1$ on ∂K.

Definition 2.5. *v is a viscosity solution of (P_ε) (in Q with boundary data v_0) if v is a viscosity supersolution and v^* is a viscosity subsolution of (P) (in Q with boundary data v_0.)*

Viscosity solutions for (P_0) are defined similarly, replacing $g(\frac{x}{\varepsilon})$ by $< \frac{1}{g} >^{-1}$ in Definitions 2.1 and 2.2.

We conclude this section by stating the standard comparison principle for viscosity solutions:

We say that a pair of functions $u_0, v_0 : \bar{D} \to [0,\infty)$ are *(strictly) separated* (denoted by $u_0 \prec v_0$) in $D \subset I\!R^n$ if

(i) the support of u_0, $\mathrm{supp}(u_0) = \overline{\{u_0 > 0\}}$ restricted in $\bar{D}$ is compact and

(ii)

$$u_0(x) < v_0(x) \text{ in } \mathrm{supp}(u_0) \cap \bar{D}.$$

We then have the following theorem which plays a central role in our analysis.

Theorem 2.6. *(Comparison principle, Theorem 1.7, [K1]) Let h_1, h_2 be respectively viscosity sub- and supersolutions of (P_ε) or (P_0) in Σ. If $h_1 \prec h_2$ on the parabolic boundary of Σ, then $h_1(\cdot, t) \prec h_2(\cdot, t)$ in Σ.*

Theorem 2.7. *(Comparison principle for the limit equation) Suppose u and v are respectively sub- and supersolutions of (P_ε) or (P_0) in $(I\!R^n - K) \times [0, \infty)$. If $\Gamma_0(u)$ or $\Gamma_0(v)$ is C^2 and $u \leq v$ for $t = 0$ and $x \in K$, then $u \leq v^*$ and $u_* \leq v$ in $(I\!R^n - K) \times [0, \infty)$.*

Proof. Suppose $\Gamma_0(v)$ is C^2. Since v_0 is harmonic, by Hopf's principle we obtain $|Dv_0| > 0$ on $\Gamma_0(v)$. Hence $\Gamma(v)$ strictly expands at $t = 0$ and thus $u(x, t) \prec v(x, t + \varepsilon)$ on the parabolic boundary of $\Sigma = (I\!R^n - K) \times [\varepsilon, \infty)$. Thus Theorem 2.6 yields that $u(x, t) \prec v(x, t + \varepsilon)$, and sending $\varepsilon \to 0$ yields the first conclusion. Note that we also have $u(x, t - \varepsilon) \prec v(x, t)$, which gives the second conclusion. $\square$

3 Uniting the notions

The purpose of this section is to make rigorous the connection between the Hele-Shaw problem (P_ε) and the obstacle problem $(\tilde{P}_\varepsilon)$. Throughout this section, we suppose that $g : \Omega \to \mathbb{R}$ is a continuous function satisfying

$$0 < \lambda \leq g(x) \leq \Lambda \quad \text{ for all } x \in \Omega.$$

We denote by $a(\cdot, \cdot)$ the Dirichlet inner product on $H^1(\Omega)$ and by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\Omega)$:

$$a(u, v) = \int_\Omega Du \cdot Dv \, dx$$
$$\langle u, v \rangle = \int_\Omega u \, v \, dx.$$

For all $t > 0$, we denote by $u(\cdot, t)$ the solution of the following variational inequality (obstacle problem):

$$\begin{cases} u \in \mathcal{K}_t \\ a(u, v - u) \geq \left\langle -\frac{1}{g(x)} \chi_{\mathbb{R}^n \setminus \Omega_0}, v - u \right\rangle \quad \text{ for all } v \in \mathcal{K}_t, \end{cases} \tag{3.1}$$

where

$$\mathcal{K}_t = \{v \in H^1_0(\mathbb{R}^n)\ ;\ v(x) \geq 0 \text{ in } \mathbb{R}^n,\ v = t \text{ on } K\}.$$

Then $u(\cdot, t)$ solves (at least formally):

$$\begin{cases} -\Delta u = -\frac{1}{g}\chi_{\mathbb{R}^n \setminus \Omega_0} & \text{in } \Omega_t(u) \\ u = |Du| = 0 & \text{on } \Gamma_t(u) \\ u = t & \text{on } K \end{cases} \tag{3.2}$$

The goal of this section is to establish the following result:

Theorem 3.1. *Let $u(x,t)$ be the unique solution of (3.1), and for all $t > 0$, let $v(\cdot, t)$ the solution of*

$$\begin{cases} \Delta v = 0 & \text{in } \Omega_t(u) \setminus K \\ v = 1 & \text{on } K \\ v = 0 & \text{on } \partial\Gamma_t(u), \end{cases} \tag{3.3}$$

(see the proof for the exact definition of v when $\partial\Gamma_t(u)$ is not smooth). Then $v(x,t)$ is a viscosity solution of the following Hele-Shaw type problem:

$$\begin{cases} \Delta v = 0 & \text{in } \{v > 0\} \setminus K \\ v = 1 & \text{on } K \\ v_t = g(x)|Dv|^2 & \text{on } \partial\{v > 0\} \end{cases} \tag{3.4}$$

with the initial condition $v(x,0) = v_0(x)$ (we recall that v_0 is the continuous function satisfying $v_0(x) = 1$ in K, $v_0(x) = 0$ in $\Omega \setminus \Omega_0$ and $\Delta v_0 = 0$ in $\Omega_0 \setminus K$).

Furthermore, v is equal to the left hand side time derivative of u:

$$v(x,t) = \partial_t^- u(x,t).$$

Before turning to the proof of this theorem, we summarize in the next proposition the main properties of the solution of the obstacle problem (3.1). We refer to Caffarelli [C1],[C2] and Rodrigues [R2] for details on the proof (see also Blank [B]):

Proposition 3.2. *Assume that $K \subset \Omega_0 \subset B_1(0)$ and that ∂K is $\mathcal{C}^{1,1}$. Then the following holds:*

(i) For all $t > 0$, (3.1) has a unique solution $u(\cdot, t)$ in $H_0^1(\mathbb{R}^n)$. For all $T > 0$, there exists $R(T)$ such that

$$\Omega_t(u) \subset B_R(0) \quad \text{for all } t \in (0, T)$$

and there exists a constant C depending only on λ, n and p such that

$$||u(\cdot, t)||_{W^{2,p}(\mathbb{R}^n \setminus K)} \leq Ct \qquad \text{for all } t \geq 0$$

for all $1 < p < \infty$.

The function $x \mapsto u(x, t)$ thus lies in $\mathcal{C}^{1,1-n/p}(\Omega \setminus K)$ for all $p < \infty$. In particular, there exists a constant C (depending only on Ω, K and n) such that

$$|u(x, t) - u(y, t)| \quad \leq \quad Ct\,|x - y| \tag{3.5}$$

for every x and y in $\Omega \setminus K$ and for every $t \geq 0$.

(ii) The function $u(x, t)$ is nonnegative in Ω and satisfies

$$-\Delta u = -\frac{1}{g(x)}(\chi_{\Omega_t(u)} - \chi_{\Omega_0}) \qquad \text{in } \mathbb{R}^n \setminus K$$

for all $t \geq 0$.

(iii) If $0 < t_1 < t_2$, then

$$0 \leq u(x, t_1) \leq u(x, t_2) \leq u(x, t_1) + t_2 - t_1 \qquad \text{for all } x \in \mathbb{R}^n.$$

In particular the function $t \mapsto u(x, t)$ is Lipschitz continuous:

$$|u(x, t_1) - u(x, t_2)| \leq |t_1 - t_2| \tag{3.6}$$

for all $x \in \mathbb{R}^n$.

We also need the following lemmas:

Lemma 3.3.

(i) Let $x_0 \in \overline{\Omega_t(u)}$ and assume that $B_r(x_0) \in \mathbb{R}^n \setminus \Omega_0$ for some $r > 0$. Then there exists $C_1 > 0$ depending only on Λ such that

$$\sup_{B_r(x_0)} u(\cdot, t) \geq C_1 r^2$$

(ii) Let $x_0 \in \Gamma_t(u)$, then there exists $C_2 > 0$ depending only on λ such that

$$\sup_{B_r(x_0)} u(\cdot, t) \le C_2 t\, r.$$

Proof. 1. We first assume that $x_0 \in \Omega_t(u)$. Note that $u(\cdot, t)$ satusfies

$$\Delta u \ge \frac{1}{g(x)} \ge \frac{1}{\Lambda} \quad \text{in } \{u > 0\} \setminus \Omega_0,$$

and so

$$\Delta u \ge \frac{1}{\Lambda} \quad \text{in } \{u > 0\} \cap B_r(x_0).$$

Hence $w(x) := u(x, t) - \frac{1}{2\Lambda}(x - x_0)^2$ is subharmonic in $\{w > 0\} \cap B_r(x_0)$. Since $w(x_0) > 0$, the set $\{w > 0\}$ is not empty and thus the maximum of w in $B_r(x_0)$ is nonnegative and is reached on the boundary $\partial B_r(x_0)$. It follows that

$$\sup_{B_r(x_0)} w = \sup_{\partial B_r(x_0)} u(\cdot, t) - \frac{1}{2\Lambda} r^2 \ge 0,$$

which gives (i) when $x_0 \in \Omega_t(u)$. By density, the result holds for all $x_0 \in \overline{\Omega_t(u)}$.

2. Proposition 3.2 (i) yields that u are uniformly Lipschitz in space with Lipschitz constant $C_2 t$. Since $u(x_0, t_0) = 0$, we deduce (ii). □

We deduce the following result:

Lemma 3.4. *For all $t > 0$, $\Omega_t(u)$ satisfies:*

$$\Omega_t(u) \subset \Omega_0 + B_{Ct^{1/2}}. \tag{3.7}$$

Proof. Let x_0 be a point in $\Omega_t(u)$ at distance δ of Ω_0. Then $B_\delta(x_0) \cap \Omega_0 = \emptyset$ so Lemma 3.3 implies

$$\sup_{B_\delta(x_0)} u(\cdot, t) \ge C\delta^2$$

Since $u(x, t) \le Ct$ for all x, we deduce $\delta \le Ct^{1/2}$ which yields (3.7). □

If we consider the function $w(x, t) = u(x, t) - u(x, s)$ for some $t > s > 0$, we obtain similarly the following result:

Lemma 3.5. *For all $t > s > 0$, $\Omega_t(u)$ satisfies:*

$$\Omega_t(u) \subset \Omega_s(u) + B_{C(t-s)^{1/2}}.$$

Now we are ready to prove our main theorem.

Proof of Theorem 3.1:

1. The function $v(x,t)$ is actually defined as follows: For every t, $v(\cdot,t)$ is the supremum of all lower semicontinuous functions $w(x)$ for which there exists some $s<t$ such that

$$-\Delta w \le 0 \text{ in } \Omega_s(u), \quad v=1 \text{ on } K \text{ and supp } v \subset \Omega_s(u)$$

It is rather easy to check that the function $v(x,t)$ itself is then lower semicontinuous (i.e. $\liminf_{(y,s)\to(x,t)} v(y,s) \ge v(x,t)$), and since the function constant equal to 1 is a supersolution for (3.3), we have $v \le 1$ and $v = v^* = 1$ on K.

Furthermore, $u(\cdot,s)/s$ is a subharmonic in $\Omega_s(u)$ and equal to 1 on K, and so by definition of v, we have

$$v(x,t) \ge u(x,s)/s \quad \text{ for all } s<t.$$

By continuity of u with respect to t, we deduce $v(x,t) \ge u(x,t)/t$ and so $v(x,t) > 0$ for all $x \in \Omega_t(u)$. Since v is lower semi-continuous, we have $v=0$ on $\Gamma_t(u)$, and so

$$\{v(\cdot,t)>0\} = \Omega_t(u), \qquad \partial\{v(\cdot,t)>0\} = \Gamma_t(u). \tag{3.8}$$

Finally, Lemma (3.5) yields $d(\Omega_t(u),\Omega_s(u)) \to 0$ as $s \to t$, and so $v^*(\cdot,t) = 0$ outside of $\bar{\Omega}_t(u)$. We deduce

$$\partial\{v^*(\cdot,t)>0\} = \Gamma_t(u)$$

(Note that v^* may be positive on $\Gamma_t(u)$ at singular points).

2. Since v is the largest subharmonic function, we have

$$-\Delta v(\cdot,t) = 0 \text{ in } \Omega_t(u)$$

Furthermore, $v(\cdot,t)$ can be given as the supremum of a sequence of harmonic functions in a sequence of smooth, smaller domains converging to $\Omega_t(u)$ with zero boundary data. Such a function will be subharmonic in $\mathbb{R}^n$, and so

$$-\Delta v(\cdot,t) \le 0 \text{ in } \mathbb{R}^n \setminus K$$

In particular, we have $-\Delta v^*(\cdot,t) \le 0$ in $\mathbb{R}^n \setminus K$.

We are now going to show that v^* is a viscosity subsolution of (3.4). Similar arguments would prove that v is a viscosity supersolution of (3.4).

3. Suppose that $\phi(x,t)$ is a $C^{2,1}$ function such that $v^* - \phi$ has a local maximum (equal to zero) in at $(x_0,t_0) \in \overline{\Omega(u)}$ in a parabolic neighborhood $\Sigma \cap \{v > 0\}$ of (x_0,t_0). Since we already know that $-\Delta v^* \leq 0$ in $\mathbb{R}^n - K$, it is readily seen that if $(x_0,t_0) \in \Omega_t(u)$, then $-\Delta\phi(x_0,t_0) \leq 0$.

It follows that in order to show that v^* is a viscosity subsolution (see definition 2.1), we only have to check that the conditions are satisfied when $(x_0,t_0) \in \Gamma_t(u)$ and $|D\phi|(x_0,t_0) \neq 0$.

Furthermore, if $v^*(x_0,t_0) > 0$ then $\phi > 0$ in a small neighborhood $\mathcal{N} = B_r^{n+1}(x_0,t_0)$ and thus $v^* - \phi$ has a local maximum in $\mathcal{N}$. Since v^* is subharmonic we easily deduce that $-\Delta\phi(x_0,t_0) \leq 0$.

We may therefore assume that

$$v^*(x_0,t_0) = \phi(x_0,t_0) = 0 \text{ and } v(x,t) \leq \phi(x,t) \text{ in } \Sigma := B_r(x_0) \times [t_0 - \tau, t_0],$$

and supposing that

$$\min\left(-\Delta\phi(x_0,t_0), \left[\phi_t - g(x)|D\phi|^2\right](x_0,t_0)\right) > 0, \tag{3.9}$$

we want to derive a contradiction.

Note that since ϕ is smooth, we can always assume that (3.9) holds in Σ by taking r and τ sufficiently small and thus

$$-\Delta\phi > 0 \quad \text{and} \quad \phi_t - g(x)|D\phi|^2 > 0 \quad \text{in } \Sigma.$$

4. Next, we want to construct a radially symmetric smooth function $\varphi(x,t)$ which is still larger than v^* in $B_r(x_0)$ and satisfies $\varphi(x_0,t_0) = 0$ and

$$-\Delta\varphi \geq 0 \quad \text{and} \quad \varphi_t - g(x)|D\varphi|^2 > 0 \quad \text{in } \Sigma \cap \{\varphi > 0\}. \tag{3.10}$$

To do this first observe that, since ϕ is $C^{2,1}$ and $|D\phi|(x_0,t_0) \neq 0$, the zero set $\Gamma(\phi)$ of ϕ is $C^{2,1}$ near (x_0,t_0). Therefore there exists $y_0 \in \mathbb{R}^n$ and a small radius $r(t)$ such that the ball $B_{r(t)}(y_0)$ is tangent to $\Gamma_t(\phi)$ from outside and

$$r'(t) = -\frac{\phi_t}{|D\phi|}.$$

For small $\varepsilon > 0$ and $\delta > 0$, we define $\varphi(\cdot, t)$ a radially symmetric function satisfying

$$\begin{cases} -\Delta\varphi = -\Delta\phi & \text{in } B_{(1+\varepsilon)r(t)}(y_0) - B_{r(t)}(y_0), \\ \varphi = 0 & \text{on } \partial B_{r(t)}(y_0), \\ \varphi = \max_{\partial B_{(1+\varepsilon)r(t)}(y_0)}(1+\delta)\phi & \text{on } \partial B_{(1+\varepsilon)r(t)}(y_0). \end{cases}$$

It is readily seen that $\varphi > \phi$ in $B_{(1+\varepsilon)r(t)}(y_0) - B_{r(t)}(y_0)$.

Let $z_0 = x_0 + \varepsilon r\eta$ and η is the outward unit normal vector of $B_{r(t)}(y_0)$ at x_0 ($z_0 \in \partial B_{(1+\varepsilon)r(t)}(y_0)$ and $\eta = \frac{D\phi}{|D\phi|}(x_0, t_0)$). Since ϕ is $C^{2,1}$ and $|D\phi|(x_0, t_0) \neq 0$, the level sets of ϕ are $C^{2,1}$ graphs near x_0 with its normal vector continuous in space and time. In particular, we have

$$\phi(z_0, t_0) = |D\phi|(x_0, t_0)|\varepsilon r + \mathcal{O}(\varepsilon^2 r^2)$$

and

$$\begin{aligned} \phi(z, t_0) &= |D\phi|(x_0, t_0)|\eta \cdot (z - z_0) + \mathcal{O}(\varepsilon^2 r^2) \\ &\leq |D\phi|(x_0, t_0)|\varepsilon r + \mathcal{O}(\varepsilon^2 r^2) \end{aligned}$$

for all $z \in \partial B_{(1+\varepsilon)r(t_0)}(y_0)$. We deduce:

$$\varphi(x, t_0) = (1+\delta)|D\phi|(x_0, t_0)|\varepsilon r + \mathcal{O}(\varepsilon^2 r^2) \qquad \text{for all } x \in \partial B_{(1+\varepsilon)r(t_0)}(y_0)$$

and so

$$\nabla\varphi(x, t_0) = (1+\delta)|D\phi|(x_0, t_0)| + \mathcal{O}(\varepsilon r) \qquad \text{for all } x \in \partial B_{r(t_0)}(y_0)$$

Therefore if $r(t), \varepsilon, \delta$ and τ are sufficiently small then $|D\varphi|$ is very close to $|D\phi|(x_0, t_0)$ on $\partial B_{r(t)}(y_0) \times [t_0 - \tau, t_0]$, and so (3.10) holds.

Since v^* is subharmonic, we have $v^* \leq (1+\delta)^{-1}\varphi$ by the maximum principle for harmonic functions.

5. We introduce the function

$$\omega(x, t) = \int_{t_0 - \tau}^{t} \varphi(x, s)ds$$

defined for $(x, t) \in \Sigma$. Since φ is smooth, (3.10) yields by a classical computation (see [G]):

$$\begin{cases} -\Delta\omega(\cdot, t) > -\dfrac{1}{g(x)} & \text{in} \quad (\Omega_t(\omega) - \Omega_{t_0}(\omega)) \cap \Sigma \\ \omega = |D\omega| = 0 & \text{on} \quad \Gamma_t(\omega) \cap \Sigma \\ -\Delta\omega(\cdot, t) = 0 & \text{on} \quad \Omega_{t_0}(\omega) \cap \Sigma. \end{cases}$$

In other words, ω is a supersolution of the obstacle problem starting at $t = t_0 - \tau$ with initial domain $\Omega_{t_0}(\phi)$ and fixed boundary $B_r(x_0)$.

6. In order to conclude, we need the semigroup property of the obstacle problem solution. Let $\tilde{u}$ be the solution of the obstacle problem starting at $t = t_0 - \tau$ with initial domain $\Omega_{t_0-\tau}(u)$, fixed boundary K and $\tilde{u} = (t - t_0 + \tau)$ on K. We claim that $\tilde{u}(x,t) = u(x,t) - u(x, t_0 - \tau)$ and

$$\Omega_t(\tilde{u}) = \Omega_t(u) \text{ for } t > t_0 - \tau.$$

This is because the function $w(x,t) = u(x,t) - u(x, t_0 - \tau)$ satisfies, for all $t \in (t_0 - \tau, t_0)$,

$$\tilde{u} = t - t_0 + \tau \text{ on } K, \quad \Omega_t(\tilde{u}) = \Omega_t(u)$$

and

$$-\Delta \tilde{u} = -\frac{1}{g}(\chi_{\Omega_t(u)} - \chi_{\Omega_{t_0-\tau}(u)}) \text{ in } \mathbb{R}^n \setminus K$$

(the last equality follows from Proposition 3.2 (ii)).

Now we compare $\tilde{u}$ and ω. Note that $\tilde{u}(\cdot, t) \leq (t - t_0 + \tau)v(\cdot, t)$, since $\tilde{u}$ is subharmonic with boundary data $t - t_0 + \tau$ on K and zero on $\Gamma_t(u)$. Therefore in Σ

$$\tilde{u}(x,t) \leq (t - t_0 + \tau)v(x,t) \leq (1 + \delta) \int_{t_0-\tau}^{t} \varphi(x,s)ds + O(\tau^2)$$

Since $\varphi > 0$ on $\partial B_r(x_0)$, by choosing sufficiently small τ we have

$$\tilde{u}(x,t) \leq \int_{t_0-\tau}^{t} \varphi(x,s) \text{ on } \partial B_r(x_0).$$

Hence by comparison principle for obstacle problem, we obtain $\tilde{u}$ stays below ω in Σ. In fact by a perturbation argument we can show that $\tilde{u}$ stays strictly below w which yields the desired contradiction.

We deduce that ϕ must satisfy

$$\min(-\Delta\phi, \phi_t - g(x)|D\phi|^2)(x_0, t_0) \leq 0,$$

which proves that v^* is a subsolution of the Hele-Shaw problem (3.4).

7. We now need to check that v and v^* satisfy the initial condition. First, we have:

Lemma 3.6. *For all $t > 0$, $\Omega_t(u)$ satisfies:*

$$\overline{\Omega_0} \subset \Omega_t(u) \qquad \text{for all } t > 0.$$

Furthermore

$$\overline{\{u > 0\}} \cap \{t = 0\} = \overline{\Omega_0}.$$

Proof. 1. First, it is readily seen that $\Omega_0 \subset \Omega_t(u)$ for all $t > 0$. As a matter of fact, we have $\Delta u(\cdot, t) = 0$ in Ω_0 and $u(\cdot, t) \geq 0$ in Ω_0 for all $t > 0$. If $u(x_0, t) = 0$ for some $x_0 \in \Omega_0$, the strong maximum principle yields $u = 0$ in Ω_0 which contradicts the fact that $u = t$ on ∂K.

We thus have $u(x, t) > 0$ for all $x \in \Omega_0$.

2. Next, we see that if $x_0 \in \partial\Omega_0$ is such that $u(x_0, t_0) = 0$, then Hopf's Lemma implies $|Du(x_0, t_0)| \neq 0$. However, since u is in $\mathcal{C}^{1,\alpha}$ with respect x and has a local minimum at x_0, we have $Du(x_0, t_0) = 0$ hence a contradiction. The first inclusion follows.

3. This implies in particular that

$$\overline{\Omega_0} \subset \overline{\{u > 0\}} \cap \{t = 0\},$$

and the last equality now follows from Lemma 3.4. $\square$

Next, we have:

Corollary 3.7. *The function $v(x, t)$ satisfies the following initial condition:*

$$v(x, 0) = v^*(x, 0) = v_0(x) \qquad \text{for all } x \in \mathbb{R}^n.$$

Proof. By definition of $v(x, 0)$, we have $v(x, 0) = v_0(x)$. So we only have to check that $\lim_{s\to 0^+} v(x, s) = v_0(x)$. Lemma 3.6 yields $v(x, t) \geq v_0(x)$ for all $t > 0$, and so

$$\lim_{s\to 0^+} v(x, s) \geq v_0(x).$$

Next, Equation (3.7) implies that

$$\lim_{s\to 0^+} v(x, s) \leq w_t(x)$$

for all $t > 0$, where w_t is the harmonic function in $\Omega_0 + B_{Ct^{1/2}}$ which vanishes on $\partial(\Omega_0 + B_{Ct^{1/2}})$. Since $\partial\Omega_0$ is smooth, it is readily seen that $w_t \longrightarrow v_0$ as $t \to 0^+$ and so

$$\lim_{s\to 0^+} v(x, s) \leq v_0(x).$$

$\square$

Note that the same proof implies that if $\partial\Omega_t(u)$ is smooth for some t, then v is continuous at t.

8. It now only remains to show the last assertion in Theorem 3.1. For that purpose, we introduce the following function

$$v_h^-(x,t) = \frac{u(x,t) - u(x,t-h)}{h}$$

and we are going to show that $\lim_{h\to 0} v_h^-(x,t) = v(x,t)$ for all t and x.

It is readily seen that $\{v_h^- > 0\} = \Omega(u)$ and so supp $v_h^-(\cdot, s) = \Omega_s(u)$. Furthermore, we have $v_h^- = 1$ on K and $-\Delta v_h^- = -\frac{1}{hg}\chi_{\Omega_t \setminus \Omega_{t-h}} \leq 0$ in $\mathbb{R}^n \setminus K$. It follows from the definition of v that

$$v_h^-(x,s) \leq v(x,t) \quad \text{for all } s < t.$$

By continuity of v_h^- with respect to t, we deduce

$$v_h^-(x,t) \leq v(x,t) \quad \text{for all } x. \tag{3.11}$$

Next, we have the following lemma:

Lemma 3.8. *The function v_h^- is monotone decreasing with respect to h. In particular, there exists a function v^- such that*

$$v_h^-(x,t) \longrightarrow v^-(x,t) \quad \textit{as } h \to 0 \textit{ for all } x \textit{ and } t.$$

Proof. Let $0 < h_1 < h_2$. In $\mathbb{R}^n \setminus \Omega_t(u)$, we have $v_{h_1}^- = v_{h_2}^- = 0$. In $\Omega_t(u) \setminus \Omega_{t-h_1}(u)$, we have $v_{h_1}^- = u/h_1$ and $v_{h_2}^- = u/h_2$ and thus $v_{h_1}^- \geq v_{h_2}^-$. In $\Omega_{t-h_1} \setminus K$, we have $\Delta v_{h_1}^- = 0 \leq \Delta v_{h_2}^-$ and since $v_{h_1}^- = v_{h_2}^- = 1$ on K, we have $v_{h_1}^- \geq v_{h_2}^-$ in Ω_{t-h_1}. We deduce

$$v_{h_1}^- \geq v_{h_2}^- \quad \text{in } \Omega$$

which gives the result. □

Finally, the following lemma completes the proof of Theorem 3.1:

Lemma 3.9.

$$v^-(x,t) = v(x,t) \quad \textit{for all } (x,t).$$

Proof. Equation (3.11) implies $v^- \le v$ so we only have to show the other inequality. We fix $(x_0, t_0) \in \Omega(u)$ and $\varepsilon > 0$. By definition of v, there exists a function $w(x)$ with support in $\Omega_s(u)$ (for some $s < t$), subharmonic in $\Omega_s(u) \setminus K$, equal to 1 in K and such that

$$w(x_0, t_0) \ge v(x_0, t_0) - \varepsilon.$$

For any $h < t-s$, we have $v_h^-(\cdot, t_0)$ positive and harmonic in $\Omega_{t-h}(u) \supset \Omega_s(u)$ and equal to 1 in K. We deduce $v_h^-(\cdot, t_0) \ge w$ and so

$$v_h^-(x_0, t_0) \ge v(x_0, t_0) - \varepsilon.$$

Together with the pointwise convergence of v_h^-, this implies

$$v^-(x_0, t_0) \ge v(x_0, t_0) - \varepsilon.$$

Since this holds for all $\varepsilon > 0$ and for all (x_0, t_0), the lemma follows. $\square$

4 Homogenization

In this section, we assume that $g(y, \omega)$ is a stationary ergodic random variable satisfying

$$0 \le \lambda \le g(y, \omega) \le \Lambda.$$

For any $\varepsilon > 0$ and all $t \in (0, T)$, we define $u^\varepsilon(\cdot, t)$ the solution of

$$\begin{cases} u^\varepsilon \in \mathcal{K}_t \\ \\ a(u^\varepsilon, v - u^\varepsilon) \ge \langle -\frac{1}{g(x/\varepsilon)} \chi_{\Omega \setminus \Omega_0}, v - u^\varepsilon \rangle \quad \text{for all } v \in \mathcal{K}_t \end{cases} \tag{4.1}$$

where

$$\mathcal{K}_t = \{v \in H_0^1(\Omega)\ ,\ v(x) \ge 0 \text{ in } \Omega,\ v = t \text{ on } K\}.$$

Using Theorem 3.1, we then define for each $\varepsilon > 0$ the function $v^\varepsilon(x, t)$, viscosity solution of the following Hele-Shaw problem:

$$\begin{cases} -\Delta v^\varepsilon = 0 & \text{in } \{v^\varepsilon > 0\} \setminus K \\ v^\varepsilon = 1 & \text{on } K \\ v_t^\varepsilon = g(x/\varepsilon, \omega)|Dv^\varepsilon|^2 & \text{on } \partial\{v^\varepsilon > 0\} \cap \Omega \end{cases} \tag{4.2}$$

In this section, we first investigate the asymptotic behavior of u^ε as ε goes to zero, and use that result to derive the equation satisfied by $\lim_{\varepsilon \to 0} v^\varepsilon$.

4.1 Homogenization of the obstacle problem

We begin with the homogenization of the obstacle problem. First we need the following lemma:

Lemma 4.1. *There exists a constant denoted by* $\left\langle \frac{1}{g} \right\rangle$ *such that*

$$\lim_{R\to\infty} \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} \frac{1}{g(y,\omega)}\, dy = \left\langle \frac{1}{g} \right\rangle \qquad a.s.$$

for any x_0*. Furthermore, if* Ω *is any bounded subset of* $\mathbb{R}^n$*, and if* u^ε *is a sequence of functions such that* u^ε *converges to* u *strongly in* $L^2(\Omega)$*, then*

$$\lim_{\varepsilon\to 0} \int_\Omega \frac{1}{g(x/\varepsilon,\omega)} u^\varepsilon(x)\, dx = \int_\Omega \left\langle \frac{1}{g} \right\rangle u(x)\, dx \qquad a.s.$$

Remark 4.2. *In the case where* $g(y)$ *is a* $\mathbb{Z}^n$*-periodic function, we have*

$$\left\langle \frac{1}{g} \right\rangle = \int_{[0,1]^n} \frac{1}{g(y)}\, dy$$

We then deduce the following proposition:

Proposition 4.3. *The sequence* u^ε *converges uniformly with respect to* $x \in \mathbb{R}^n$ *and* $t \in [0,T]$ *to* $u^0(t,x)$ *solution of the following obstacle problem:*

$$\begin{cases} u^0 \in \mathcal{K}_t \\ a(u^0, v-u^0) \geq \langle -\left\langle \frac{1}{g} \right\rangle \chi_{\mathbb{R}^n\setminus\Omega_0}, v-u^0 \rangle \quad \text{for all } v \in \mathcal{K}_t \end{cases} \tag{4.3}$$

Proof of Lemma 4.1: The existence of $\left\langle \frac{1}{g} \right\rangle$ is a direct consequence of the subadditive ergodic theorem. We also deduce

$$\lim_{\varepsilon\to 0} \int_G \frac{1}{g(x/\varepsilon,\omega)}\, dx = \left\langle \frac{1}{g} \right\rangle |G| \qquad \text{a.s.},$$

and the density of piecewise continuous functions in L^2 easily yields

$$\lim_{\varepsilon\to 0} \int_\Omega \frac{1}{g(x/\varepsilon,\omega)} u(x)\, dx = \int_\Omega \left\langle \frac{1}{g} \right\rangle u(x)\, dx \qquad \text{a.s.},$$

for all $u \in L^2(\Omega)$.

Next, we write

$$\int_\Omega \frac{1}{g(x/\varepsilon,\omega)} u^\varepsilon(x)\,dx = \int_\Omega \frac{1}{g(x/\varepsilon,\omega)} u(x)\,dx + \int_\Omega \frac{1}{g(x/\varepsilon,\omega)}(u^\varepsilon - u(x))\,dx.$$

For any $\delta > 0$, there exists ε_0 such that for any $\varepsilon < \varepsilon_0$, we have

$$\left|\int_\Omega \frac{1}{g(x/\varepsilon,\omega)}(u^\varepsilon - u(x))\,dx\right| \leq \Lambda|\Omega|^{1/2}||u^\varepsilon - u||_{L^2(\Omega)} \leq \delta.$$

We deduce that

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{1}{g(x/\varepsilon,\omega)} u^\varepsilon(x)\,dx = \int_\Omega \left\langle \frac{1}{g}\right\rangle u(x)\,dx + \mathcal{O}(\delta)$$

and since this holds for any $\delta > 0$, the Lemma follows. □

Proof of proposition 4.3. Inequalities (3.5) and (3.6) in Proposition 3.2 yield:

$$|u^\varepsilon(x,t) - u^\varepsilon(y,s)| \leq C_T(|x-y| + |t-s|)$$

for all x, y in Ω and t, s in $[0,T]$. Moreover, we have $|u^\varepsilon(x,t)| \leq T$ and supp $u^\varepsilon \subset B_{R(T)}$ for all $t \in [0,T]$. So Ascoli's Theorem yields the uniform convergence, up to a subsequence, of u^ε to some continuous function $u^0(x,t)$.

In particular, for all $t \geq 0$, we have $u^\varepsilon(\cdot,t) \to u^0(\cdot,t)$ uniformly with respect to x. Furthermore, (4.3) yields

$$\begin{aligned}\int |Du^\varepsilon|^2\,dx + \frac{1}{\Lambda}\int \chi_{\Omega\setminus\Omega_0} u^\varepsilon\,dx \quad &\leq \quad \frac{1}{\lambda}\int v\,dx + \int Dv\cdot Du^\varepsilon\,dx \\ &\leq \quad \frac{1}{\lambda}\int v\,dx + \frac{1}{2}\int |Dv|^2\,dx \\ &\qquad +\frac{1}{2}\int |Du^\varepsilon|^2\,dx\end{aligned}$$

for all $v \in \mathcal{K}_t$, and so

$$||u^\varepsilon(\cdot,t)||_{H^1(\mathbb{R}^n)} \leq C(t).$$

It follows that

$$u^\varepsilon(\cdot,t) \longrightarrow u^0(\cdot,t) \qquad H^1(\mathbb{R}^n)\text{-weak and } L^2(\mathbb{R}^n)\text{-strong.}$$

which in turn implies that

$$\liminf_{\varepsilon\to 0} a(u^\varepsilon,u^\varepsilon) \geq a(u^0,u^0)$$

and

$$\lim_{\varepsilon\to 0} a(u^\varepsilon, v) = a(u, v)$$

for all $v \in H^1(\Omega)$.

Taking the limit $\varepsilon \to 0$ in (4.1) (using Lemma 4.1 and the fact that $\operatorname{supp} u^\varepsilon \subset B_{R(T)}$), we deduce that u^0 solves (4.3). Finally the uniqueness of the solution of (4.3) (Proposition 3.2 (i)) implies that the whole sequence u^ε converges to u^0. □

Our next task is to show uniform convergence of the free boundaries $\Gamma(u^\varepsilon)$ to $\Gamma(u^0)$, with respect to Hausdorff distance. The difficulties in showing this are due to the lack of estimates on the infimum of u^ε in Lemma 3.3, which makes it difficult to prevent formation of new zero set in the limit $\varepsilon \to 0$. In the next section we will use maximum-principle type arguments for the time derivatives of u^ε (that is the solution of the Hele-Shaw equation (P_ε), v^ε) to show the convergence of the boundaries.

4.2 Homogenization of the Hele-Shaw problem: uniform convergence of the free boundary

We now denote by v^ε the solution of

$$\begin{cases} \Delta v^\varepsilon = 0 & \text{in } \Omega_t(u^\varepsilon) \setminus K \\ v^\varepsilon = 1 & \text{on } K \\ v^\varepsilon = 0 & \text{on } \partial\Gamma_t(u^\varepsilon). \end{cases} \tag{4.4}$$

defined as in Theorem 3.1 (Theorem 3.1 implies that v^ε is a viscosity solution of (P_ε)). Taking the limit $\varepsilon \to 0$, we can define

$$v^*(x,t) := \limsup_{(y,s),\varepsilon\to(x,t),0} v^\varepsilon(y,s)$$

$$v_*(x,t) := \liminf_{(y,s),\varepsilon\to(x,t),0} v^\varepsilon(y,s).$$

We also introduce $v^0(x,t)$, solution of

$$\begin{cases} \Delta v^0 = 0 & \text{in } \Omega_t(u^0) \setminus K \\ v^0 = 1 & \text{on } K \\ v^0 = 0 & \text{on } \partial\Gamma_t(u^0), \end{cases} \tag{4.5}$$

(Note that v^0 is a viscosity solution of (P_0) thanks to Theorem 3.1), and

$$(v^0)^*(x,t) := \limsup_{(y,s)\to(x,t)} v^0(y,s)$$

$$(v^0)_*(x,t) := \liminf_{(y,s)\to(x,t)} v^0(y,s).$$

We have $(v^0)_* \le v^0 \le (v^0)^*$ and the lower-semicontinuity of v^0 implies $(v^0)_* \ge v^0$. We deduce:

$$(v^0)_* = v^0 \le (v^0)^* \tag{4.6}$$

where v^0 is a supersolution of (P_0) and $(v^0)^*$ is a subsolution.

The goal of this section is to prove the following result:

Theorem 4.4. *$\{\Gamma(v^\varepsilon))\}_\varepsilon$ locally uniformly converges to $\{\Gamma(v^0)\}$ with respect to the Hausdorff distance. Moreover*

$$v_* = v^0 \qquad \textit{and} \qquad v^* = (v^0)^*.$$

This implies in particular that v_* is a supersolution of (P_0) and v^* is a subsolution. In general we do not have $v^0 = (v^0)^*$. However, we have the following result:

Corollary 4.5. *If v^0 is continuous, then $v^0 = (v^0)^*$ and v^ε converges locally uniformly to v_0. If $\Gamma(v^0)$ is continuous in time, then for any $t > 0$ $\{\Gamma_t(v^\varepsilon)\}_\varepsilon$ locally uniformly converges to $\{\Gamma_t(v^0)\}$.*

Remark:
1. We have

$$\Omega(v^\varepsilon) = \Omega(u^\varepsilon), \qquad \Gamma(v^\varepsilon) = \Gamma(u^\varepsilon),$$

and

$$\Omega(v^0) = \Omega(u^0), \quad \Gamma(v^0) = \Gamma(u^0).$$

2. Even with Lipschitz continuity of u^0 in time (Proposition 3.2), one cannot show that $\Gamma(u^0)$ is continuous in time with respect to the Hausdorff distance. The main difficulty is the same as mentioned at the end of section 3.1. Intuitively, what may occur is two fingers of $\Omega(u^0)$ contacts each other at time $t = t_0$ with its contact set $\mathcal{C}$, which is part of $\Gamma_{t_0}(u^0)$, having positive $n-1$ dimensional Hausdorff measure. Then at the next moment $\mathcal{C}$ instantly disappears and we have a discontinuity of $\Gamma(u^0)$ and v^0 at $t = t_0$.

3. If K and Ω_0 are star-shaped with respect to a point in K, then it is

known that v^0 is continuous with respect to x and t, and Corollary 4.5 applies.

The proof of Theorem 4.4 will involve a series of Lemmas. We start with a simple result, which will be useful in the sequel:

Lemma 4.6. *Suppose* $(x_k, t_k) \in \{u^{\varepsilon_k} = 0\}$ *and* $(x_k, t_k, \varepsilon_k) \to (x_0, t_0, 0)$. *Then the following holds:*

(i) $(x_0, t_0) \in \{u^0 = 0\}$.

(ii) If $x_k \in \Gamma_{t_k}(u^{\varepsilon_k})$ *then* $x_0 \in \Gamma_{t_0}(u^0)$.

Proof. (i) The uniform convergence of u^ε to u gives $\lim u^{\varepsilon_k}(x_k, t_k) = u^0(x_0, t_0)$, hence the result.

(ii) Since $x_k \in \Gamma_{t_k}(u^{\varepsilon_k})$, Lemma 3.3 (ii) yields that for all r, there exists $y_k \in B_r(x_k)$ such that

$$u^{\varepsilon_k}(y_k, t_k) > cr^2.$$

Up to a subsequence, we can assume that $y_k \to y_0 \in B_r(x_0)$ and the uniform convergence of u^ε yields

$$u^0(y_0, t_0) > cr^2.$$

So $B_r(x_0) \cap \Omega_{t_0}(u^0) \neq \emptyset$ for all $r > 0$, hence $x_0 \in \Gamma_{t_0}(u^0)$. □

Lemma 4.7.

$$(v^0)^*(\cdot, s) \leq v^0(\cdot, t) \text{ for all } s < t.$$

In particular,

$$((v^0)^*)_* \leq v^0$$

Proof. Corollary 3.7 yields

$$(v^0)^*(x, 0) = v^0(x, 0) = v_0(x) \text{ for all } x,$$

Theorem 2.7 then yields the result. □

The next lemma states that the free boundary may not jump in time:

Lemma 4.8. *For any* $(x_0, t_0) \in \Gamma((v^0)^*)$, *there exists a sequence* $(x_k, t_k) \in \Gamma(v^0)$ *with* $t_k \leq t_0$ *converging to* (x_0, t_0).

Proof. 1. Suppose that the lemma does not hold. Then, there exists $r > 0$ (we can assume $r \leq 1$) such that

$$B_r(x_0) \times [t_0 - r, r_0] \subset \{v^0 = 0\} \quad \text{ or } \quad B_r(x_0) \times [t_0 - r, r_0] \subset \{v^0 > 0\}.$$

The second possibility clearly implies that $(v^0)^* > 0$ in $B_r(x_0) \times (t_0 - r, t_0 + r)$ which is impossible. We thus have $B_r(x_0) \times [t_0 - r, r_0] \subset \{v^0 = 0\}$ and Lemma 4.7 implies

$$(v^0)^*(x, t_0 - \tau) = 0 \quad \text{ for } x \in B_r(x_0) \text{ and } \tau \in (0, r).$$

2. Let $h(x) = h(|x - x_0|)$ be a harmonic function in $B_r(x_0) - B_{r/2}(x_0)$ with boundary data 1 on $\partial B_r(x_0)$ and zero on $\partial B_{r/2}(x_0)$. Let

$$\phi(x, t) = h((1 + M(t - t_0 + \tau))|x - x_0|)$$

where $M = Cr^{-1}$ and $\tau = M^{-1}r = C^{-1}r^2$. If the constant C is chosen large enough, then ϕ is a supersolution of (P_0) for $t \in [t_0 - \tau, t_0]$. We now compare $(v^0)^*$ and h in $B_r(x_0) \times [t_0 - \tau, t_0]$ (we recall that $(v^0)^*$ is a subsolution of (P_0)). We have $(v^0)^*(x, t_0 - \tau) = 0 \leq \phi(x, t_0 - \tau)$ in $B_r(x_0)$, so using Theorem 2.7 and the fact that $v^0 \leq 1$, we deduce:

$$B_{r/4}(x_0) \subset \{(v^0)^*(\cdot, t_0) = 0\},$$

a contradiction. □

We deduce the following result:

Corollary 4.9.

$$\Gamma((v^0)^*) = \Gamma(v^0)$$

Proof. 1. Lemma 4.8 and 4.6 yield

$$\Gamma((v^0)^*) \subset \Gamma(v^0).$$

2. Assume now that $(x_0, t_0) \in \Gamma(v^0)$. Since $(v^0)^* \geq v^0$ it is readily seen that $(x_0, t_0) \in \overline{\{(v^0)^* > 0\}}$. Furthermore, if there exists r such that

$$B_r(x_0) \times (t_0 - r, t_0 + r) \subset \{(v^0)^* > 0\}$$

then Lemma 4.7 implies $v^0 > 0$ in $B_r(x_0) \times (t_0 - r, t_0 + r)$, which contradicts $(x_0, t_0) \in \Gamma(v^0)$. Hence $(x_0, t_0) \in \Gamma((v^0)^*)$. □

Lemma 4.10. *The following inclusion holds:*

$$\{v^0 > 0\} \subset \{v_* > 0\}.$$

In particular

$$v_* \geq v^0.$$

Proof. We recall that $v^\varepsilon(x,t) \geq \frac{1}{t} u^\varepsilon(x,t)$ for all x and $t > 0$. The uniform convergence of u^ε to u^0 thus implies:

$$v_*(x,t) \geq \frac{1}{t} u^0(x,t)$$

and so

$$\{v^0 > 0\} = \{u^0 > 0\} \subset \{v_* > 0\}$$

Since v_* is superharmonic in $\{v_* > 0\}$, we deduce $v_* \geq v^0$. □

Lemma 4.10 also implies that $v^0 \leq v_* \leq v^*$ and so

$$(v^0)^* \leq v^*. \tag{4.7}$$

We now want to show the following proposition:

Proposition 4.11. *v^* is a subsolution of (P_0).*

For that purpose, we first need a couple of technical lemmas:

Lemma 4.12. *Suppose $(x_0, t_0) \in \Omega(v^\varepsilon)$. There exists a constant C_1 independent of ε and r such that $B_r(x_0) \cap \Omega_0 = \emptyset$, we have:*

$$\sup_{B_r(x_0)} v^\varepsilon(\cdot, t_0) \geq \frac{C_1\, r^2}{t_0}.$$

Proof. Since $v^\varepsilon(\cdot, t_0) \geq u^\varepsilon(\cdot, t_0)/t_0$ (u^ε is subharmonic in $\Omega_t(u^\varepsilon)$), the result follows from Lemma 3.3. □

Lemma 4.13.

(i) For any $(x_0, t_0) \in \{v^* = 0\}$, there exists $(x_k, t_k) \in \{v^{\varepsilon_k} = 0\}$ such that $(x_k, t_k, \varepsilon_k) \to (x_0, t_0, 0)$ as $k \to \infty$. In particular $(x_0, t_0) \in \{v^0 = 0\}$.

(ii) For any $(x_0, t_0) \in \Gamma(v^*)$, there exists $(x_k, t_k) \in \Gamma(v^{\varepsilon_k})$ such that $(x_k, t_k, \varepsilon_k) \to (x_0, t_0, 0)$ as $k \to \infty$. In particular $(x_0, t_0) \in \Gamma(v^0)$.

Proof. We only prove (ii), the proof of (i) being slightly simpler.

1. Let $(x_0, t_0) \in \Gamma(v^*)$, and assume that the result does not hold. Then there exists $r > 0$ such that, for all $\varepsilon \le \varepsilon_0$, either

$$B_r(x_0) \times [t_0 - r, t_0 + r] \subset \Omega(v^\varepsilon)$$

or

$$B_r(x_0) \times [t_0 - r, t_0 + r] \subset \{v^\varepsilon = 0\}.$$

In the second case, we would deduce $v^*(x,t) = 0$ for all $x \in B_{r/2}(x_0)$ and $t \in [t_0 - r/2, t_0 + r/2]$, which contradicts $x_0 \in \Gamma_{t_0}(v^*)$. So we can assume that $B_r(x_0) \times [t_0 - r, t_0 + r] \subset \Omega(v^\varepsilon)$.

2. We have $B_r(x_0) \subset \Omega_t(v^\varepsilon)$ and so Lemma 4.12 and Harnack's inequality for harmonic functions, yield that there exists c_1 depending on c_0 and n such that

$$v^\varepsilon(x,t) \ge \frac{c_1 r^2}{t} \text{ for all } x \in B_{r/2}(x_0) \text{ and } t \in [t_0 - r, t_0 + r].$$

This contradicts the fact that $(x_0, t_0) \in \Gamma(v^*)$.

3. Lemma 4.6 and the fact that $\{v^\varepsilon = 0\} = \{u^\varepsilon = 0\}$ implies $(x_0, t_0) \in \{v^0 = 0\}$. □

Lemma 4.14. *If $(x_0, t_0) \in \Gamma(v^*)$, then there exists $(y_k, t_k) \in \Gamma((v^0)^*)$ converging to (x_0, t_0).*

Proof. Lemma 4.13 (ii) implies $\Gamma(v^*) \subset \Gamma(v^0)$ and so $(x_0, t_0) \in \Gamma(v^0)$. Assume now that the result does not hold. Then there exists some small r and τ such that either

$$(v^0)^* = 0 \text{ in } B_r(x_0) \times [t_0 - \tau, t_0 + \tau],$$

or

$$(v^0)^* > 0 \text{ in } B_r(x_0) \times [t_0 - \tau, t_0 + \tau].$$

In the first case, we get a contradiction from the fact that $(v^0)^* \ge v^0$ (by definition of $(v^0)^*$), and so $v^0 = 0$ in $B_r(x_0) \times [t_0 - \tau, t_0 + \tau]$. In the second case, the contradiction is given by Lemma 4.7 which gives

$$0 < (v^0)^*(\cdot, s) \le v^0(\cdot, t) \text{ for all } s < t$$

and so $v^0 > 0$ in $B_r(x_0) \times (t_0 - \tau, t_0 + \tau)$. □

Proof of Proposition 4.11:

1. We recall that $(v^0)^*$ is a subsolution of the limiting problem. Since we already know that v^* is subharmonic in its positive phase, we only need to check for the behavior of v^* on its free boundary $\Gamma(v^*)$. We recall (see step 2 in the proof of Theorem 3.1) that v^ε, and thus v^*, is subharmonic in $\mathbb{R}^n \setminus K$.

Let ϕ be a smooth function in $C^{2,1}$ such that

$$v^* \leq \phi_+ = \max(\phi, 0) \quad \text{ in } \Sigma := B_r(x_0) \times (t_0 - \tau, t_0],$$

$$v^*(x_0, t_0) = \phi(x_0, t_0) \quad \text{ and } \quad (x_0, t_0) \in \Gamma(v^*).$$

If $v^*(x_0, t_0) > 0$, since v^* is subharmonic, we get $-\Delta\phi(x_0, t_0) \leq 0$, so we can assume $v^*(x_0, t_0) = 0$ and $|D\phi|(x_0, t_0) \neq 0$. We need to prove that

$$\min(-\Delta\phi, \phi_t - \langle 1/g \rangle^{-1} |D\phi|^2)(x_0, t_0) \leq 0.$$

2. After adding $\delta(x - x_0)^4$ if necessary, we may assume that $v^* - \phi$ has a strict maximum at (x_0, t_0) in $\overline{\Omega(v^*)} \cap \Sigma$. Using (4.7), we see that $(v^0)^* \leq \phi_+$ in Σ. For $\varepsilon > 0$, we perturb ϕ by

$$\varphi^\varepsilon(x, t) = \phi(x, t) - \varepsilon(t - t_0 + \delta)$$

where $\delta = \min(\tau, r^2/2)$. Then $\phi < \varphi$ on the parabolic boundary of $\partial B_r(x_0) \times [t_0 - \delta, t_0]$ and $\phi(x_0, t_0) = \varphi(x_0, t_0) - \varepsilon\delta$.

Finally, Lemma 4.14 implies that $(v^0)^*$ crosses φ_+ from below at $(x_\varepsilon, t_\varepsilon)$ in $\Sigma \cap \{t \leq t_\varepsilon\}$ for any $\varepsilon > 0$. It follows that

$$\min(-\Delta\varphi^\varepsilon, \varphi^\varepsilon_t - \langle 1/g \rangle^{-1} |D\varphi^\varepsilon|^2)(x_\varepsilon, t_\varepsilon)) \leq 0.$$

3. Since φ^ε converges to ϕ and $v^* - \phi$ has a strict maximum at (x_0, t_0) in $\bar{\Omega}(v^*) \cap \Sigma$, we have $(x_\varepsilon, t_\varepsilon) \to (x_0, t_0)$, and thus we can conclude. $\square$

Proof of Theorem 4.4:

1. Since $\Gamma_0 = \partial\Omega_0$ is smooth, a barrier argument as in the proof of Corollary 3.7 yields:

$$v^*(x, 0) = v_0(x).$$

Since v^* is a subsolution for (P_0) (Proposition 4.11) and v^0 is a supersolution (Theorem 3.1), Theorem 2.7 gives

$$v^* \leq (v^0)^*.$$

Together with Lemma 4.10, it leads to

$$v^0 \le v_* \le v^* \le (v^0)^*.$$

In particular, we have $v_* \le (v^0)^*$ and so $v_* \le ((v^0)^*)_*$. Together with Lemma 4.7 (which gives $((v^0)^*)_* \le v^0$), we deduce $v_* \le v^0$ and so

$$v_* = v^0.$$

Finally, since $v^* \ge v_0$, we have $v^* \ge (v^0)^*$ and so

$$v^* = (v^0)^*.$$

In particular, Lemma 4.7 yields:

$$\Gamma(v^*) = \Gamma(v_*) = \Gamma(v^0) = \Gamma((v^0)^*).$$

2. We now have to prove the uniform convergence of the free boundaries. We recall that the sets $\Gamma(v^0) \cap \{0 \le t \le T\}$ and $\Gamma(v^\varepsilon) \cap \{0 \le t \le T\}$ are compact since the positive phases of v^0 and v^ε are uniformly bounded locally in time.

Let $(x_0, t_0) \in \Gamma(v^0) = \Gamma(v^*)$, for some $t_0 \le T$, and assume that there exists a sequence $\varepsilon_k \to 0$ such that either $v^{\varepsilon_k} > 0$ or $v^{\varepsilon_k} \equiv 0$ in $B_r(x_0) \times (t_0 - r, t_0 + r)$ for some small $r > 0$. In the first case, we get a contradiction by arguing as in Lemma 4.13. In the second case, we deduce $v_* = 0$ in $B_r(x_0) \times (t_0 - r, t_0 + r)$ and so $v^0 \equiv 0$, a contradiction with $(x_0, t_0) \in \Gamma(v^0)$.

This proves that for any $\delta > 0$, there exists ε_0 (depending on (x_0, t_0) such that

$$d((x_0, t_0), \Gamma(v^\varepsilon)) \le \delta \text{ if } \varepsilon < \varepsilon_0.$$

3. Similarly, if we have a sequence $(x_\varepsilon, t_\varepsilon)$ such that $(x_\varepsilon, t_\varepsilon) \in \Gamma(v^\varepsilon) \cap \{t \le T\}$, then Lemma 4.6 (ii) and the fact that the sets $\Gamma(v^0) \cap \{0 \le t \le T\}$ and $\Gamma(v^\varepsilon) \cap \{0 \le t \le T\}$ are compact imply that for any $\delta > 0$ there exists ε_0 such that

$$d(\Gamma(v^0), (x_\varepsilon, t_\varepsilon)) \le \delta \text{ if } \varepsilon < \varepsilon_0.$$

4. Finally, since the sets $\Gamma(v^0) \cap \{0 \le t \le T\}$ and $\Gamma(v^\varepsilon) \cap \{0 \le t \le T\}$ are compact we can prove that the constant ε_0 given in the two inequalities above only depend on δ and T. It follows that $\Gamma(v^\varepsilon)$ converges locally uniformly to $\Gamma(v^0)$ with respect to the Hausdorff distance. $\square$

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