THE PINNING EFFECT OF DILUTE DEFECTS

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Abstract. We consider the Bernoulli free boundary problem with "defects", inhomogeneities in the coefficients of compact support. When the defects are small and arrayed periodically there exist plane-like solutions with a range of large-scale slopes slightly different from the background field value. This is known as pinning. By studying the capacity-like pinning effect of a single defect in the Bernoulli free boundary problem, we can compute the asymptotic expansion of the interval of pinned slopes as the defect size goes to zero for lattice aligned normal directions. Our work is motivated by the issue of contact angle hysteresis in capillary contact lines.

Contents

1. Introduction

One of the main issues in the study of capillary droplets on rough surfaces is the phenomenon of contact angle hysteresis [20]. On an idealized smooth surface the balance of surface tension between the liquid, air and the surface yields a constant contact angle between the drop and the surface along the contact line. In this model stationary droplets on on a flat surface will be axisymmetric. However, in practice, even on surfaces which are quite flat at the droplet length scale, there are smaller scale inhomogeneities on the surface. This small scale heterogeneity affects the energy landscape even at the macro scale, the result is a range of metastable large-scale contact angles and resulting non-axisymmetric stationary droplet shapes [5, 8, 11, 15, 20, 24, 25, 37]. Dynamically, a slowly evaporating or spreading drop will get pinned along some directions of propagation, leading to the aforementioned non-round geometries [31]. The interval of pinned slopes has been introduced and studied via a homogenization approach in [6, 22, 31]. However it has only been computed analytically in the setting of laminar media. In this singular case, pinning occurs only in the laminar direction.

Our aim here is thus to establish a more informative description of the pinning interval in a general class of media. We will study the setting of "dilute defects" where small localized defects are arrayed on a periodic lattice in a homogeneous background medium. This setting forms the basis of the most influential model of contact angle hysteresis in the physics literature, originally introduced in the pioneering work of Joanny and de Gennes [30]. Indeed our work can be viewed as a rigorous justification of the Joanny-de Gennes asymptotics.

In this paper we will work in the setting of the Bernoulli free boundary model, which is a partially linearized version of the capillary problem (see Remark 1.10 below):

$$\begin{cases} \Delta u = 0 & \text{in } \ \{u > 0\} \\ |\nabla u| = Q(x) & \text{on } \partial\{u > 0\}. \end{cases} \tag{1.1}$$

Here $u : \mathbb{R}^d \to [0, \infty)$ denotes the height of the liquid, and the free boundary $\partial\{u > 0\}$ corresponds to the liquid-surface contact line. While we will work in all $d \geq 2$, the case $d = 2$ is most important since it is the physical dimension for the capillary problem. The slope condition $|\nabla u| = Q(x)$ at the free boundary corresponds to Young's contact angle condition in capillarity. To model the dilute defect regime we will take

$$Q(x) = Q_\delta(x) = 1 + \sum_{z \in \mathbb{Z}^d} q(\tfrac{x-z}{\delta}),$$

where δ is small. The function $q : \mathbb{R}^d \to (-1, \infty)$ represents a single chemical defect on the surface. We will assume $q \in C_0(\overline{B_1(0)})$.

To state our results, we briefly recall the definition of the pinning interval for (1.1) from [22] (see Section 7.3 for a full discussion): we call $\alpha > 0$ a *pinned slope* at normal direction $e \in S^{d-1}$ if there is a solution v of

$$\begin{cases} \Delta v = 0 & \text{in } \ \{v > 0\}; \\ |\nabla v| = Q_\delta(x) & \text{on } \partial\{v > 0\}; \\ \sup_{x \in \{v>0\}} |v(x) - \alpha(e \cdot x)| < +\infty. \end{cases} \tag{1.2}$$

The pinned slopes at direction e then form a non-empty, closed interval, which we call the *pinning interval*: $[Q^\delta_{\rm rec}(e), Q^\delta_{\rm adv}(e)]$. Here $Q^\delta_{\rm rec}(e)$ is called the *receding slope* and $Q^\delta_{\rm adv}(e)$ is called the *advancing slope*. This terminology arises from the dynamic manifestation of the pinning phenomenon. The slope is smaller than $Q^\delta_{\rm rec}(e)$ where the contact line recedes, and is larger than $Q^\delta_{\rm adv}(e)$ where the contact line advances: see for example [2, 31].

Our aim in this paper is to obtain the asymptotic expansion for the endpoints of this pinning interval as $\delta \to 0$. We will focus on rational directions $e = \xi/|\xi|$ for some $\xi \in \mathbb{Z}^d$ (see Remark 1.9 for a discussion on irrational directions).

Theorem 1.1 (Main result, see Propositions 8.1 and 9.1)**.** *Let $\xi \in \mathbb{Z}^d$ irreducible and $e := \frac{\xi}{|\xi|}$. Then as $\delta \to 0$*

$$Q^{\delta}_{\rm adv}(e;q) = 1 + \gamma_d \delta^{d-1}|\xi|^{-1}k_{\rm adv}(e;q) + o(\delta^{d-1}), \text{ and} \\ Q^{\delta}_{\rm rec}(e;q) = 1 + \gamma_d \delta^{d-1}|\xi|^{-1}k_{\rm rec}(e;q) + o(\delta^{d-1}). \tag{1.3}$$

Here $\gamma_2 = \pi$ and $\gamma_d = \frac{1}{2}d(d-2)|B_1|$ for $d \geq 3$.

Here $k_{\rm adv}$ and $k_{\rm rec}$ are a nonlinear analogue of the electrostatic capacity, measuring the pinning effect of a single defect site via a far-field expansion. See Theorems 1.3, 1.4, and 1.5 for the definitions. The quantity $|\xi|^{-1}$ is the density of $\mathbb{Z}^d$ lattice sites on the hyperplane $\{x \cdot e = 0\}$.

Our analysis will have consequences that are of independent interest, to be discussed below. In particular it leads to the proof that the pinning phenomena is generic (see Theorem 1.7).

1.1. **Statement of main results and ideas.** We are now ready to present the main results obtained in our analysis arriving at (1.3). Let us introduce the single defect problem in $\mathbb{R}^d$:

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \\ |\nabla u| = 1 + q(x) & \text{on } \partial\{u > 0\}. \end{cases} \tag{1.4}$$

The pinning coefficients $k_{\rm adv}$ and $k_{\rm rec}$ will arise from the far field asymptotic expansion of solutions of (1.4). While this is reminiscent of capacities from the study of elliptic PDEs exterior domains, there is a significant difference: since we have a free boundary problem, there is not just one but instead a family of capacitory potentials and capacities associated with a given q.

Our analysis will be focused on *proper solutions* of (1.4):

Definition 1.2. $u \in C(\mathbb{R}^d)$ is *proper* with direction e if the sequence $u_r(x) := r^{-1}u(rx)$ converges to $(x \cdot e)_+$ as $r \to 0$ in the following sense:

$$(u_r(x) - x \cdot e)\mathbf{1}_{\{u_r > 0\}} \to 0 \text{ locally uniformly in } \mathbb{R}^d \text{ as } r \to 0. \tag{1.5}$$

This rather strong definition is introduced so that we can rule out solutions that blow down to singular solutions. We will show that it is enough to consider proper solutions to obtain (1.3). The main concern is the two-plane solution $|x \cdot e|$, a well-known singular profile in the Bernoulli problem [9, 29, 33]. Ruling out this scenario is most challenging in $d = 2$ due to the far-field logarithmic growth of u away from the planar profile. An important step in achieving this is Proposition 5.10, which rules out the occurrence of two-plane blow-downs for advancing solutions in dimension $d = 2$.

In all theorems presented in this section, we will fix q. We begin with $d = 2$.

Theorem 1.3 (Theorem 4.1 and Theorem 5.4)**.** *Let $d = 2$. For any $e \in S^{d-1}$, there exists $-\infty < k_{\rm rec}(e) \leq 0 \leq k_{\rm adv}(e) \leq +\infty$ such that for any proper solution u of (1.4), there is $k \in [k_{\rm rec}, k_{\rm adv}]$ such that*

$$u(x) = x \cdot e + k \log|x| + O(1) \quad \text{as} \quad \{u > 0\} \ni x \to \infty. \tag{1.6}$$

Conversely, for any $k \in [k_{\rm rec}, k_{\rm adv})$ there is a proper solution of (1.4) with (1.6).

It is unclear whether $k_{\rm adv} < +\infty$ in general, with the exception of small $\max q$, where we can show that $k_{\rm adv}(e) \leq C \max q < \infty$. It is also open whether there exist proper solutions with the endpoint capacity $k = k_{\rm adv}$. These challenges, specific to the advancing capacity in $d = 2$, are again tied to the logarithmic tail of solutions.

Next we turn to $d \geq 3$. Here the behavior of proper solutions is a bit different due to the thin tail of the fundamental solution: there is a monotone family of solutions that goes through the defect and varying capacities can be achieved at varying heights.

Theorem 1.4 (Theorem 4.1 and Theorem 6.1)**.** *Let $d \geq 3$. For any $s \in \mathbb{R}$ there exist finite numbers $\kappa_{\rm rec}(s;e) \leq 0 \leq \kappa_{\rm adv}(s;e)$ so that for any proper solution u of (1.4) there is $s \in \mathbb{R}$ and $k \in [\kappa_{\rm rec}(s), \kappa_{\rm adv}(s)]$ such that*

$$u(x) = x \cdot e + s - k|x|^{2-d} + O(|x|^{1-d}) \quad as \quad \{u > 0\} \ni x \to \infty. \tag{1.7}$$

Moreover, for each $s \in \mathbb{R}$ there is a maximal subsolution $u_{\rm rec}(s)$ achieving (1.7) with $k = \kappa_{\rm rec}(s)$ and a minimal supersolution $u_{\rm adv}(s)$ with $k = \kappa_{\rm adv}(s)$.

We refer to Section 6 for further properties of $\kappa_{\rm rec}$ and $\kappa_{\rm adv}$. For example we show that $s \mapsto \kappa_{\rm adv/rec}(s)$ are compactly supported. We also show that $\kappa_{\rm adv/rec}$ are, in a certain sense, continuous with respect to the normal direction $e \in S^{d-1}$.

Now we are ready to characterize the capacities in Theorem 1.1. In dimension $d = 2$ we have already defined $k_{\rm adv/rec}(e)$ in Theorem 1.3. In dimension $d \geq 3$ we define, in terms of $\kappa_{\rm adv/rec}(s)$ from Theorem 1.4,

$$k_{\rm adv}(e) := \max_{s \in \mathbb{R}} \kappa_{\rm adv}(s;e) \quad \text{and} \quad k_{\rm rec}(e) := \min_{s \in \mathbb{R}} \kappa_{\rm rec}(s;e).$$

Theorem 1.5 (Completing the statement of Theorem 1.1)**.** *Let $e \in S^{d-1}$ be a rational direction, and let $k_{\rm adv}(e)$ and $k_{\rm rec}(e)$ be as defined above, depending on $d = 2$ or $d > 2$. Then the expansions (1.3) hold.*

Below we describe the main ideas leading to Theorem 1.5, as well as two consequences of our analysis that are of independent interest.

It has been proved in previous work that $Q^{\delta}_{\rm adv}(e)$ and $Q^{\delta}_{\rm rec}(e)$ are achieved by strong Birkhoff solutions u^{δ} of the periodic problem (1.1), see Section 7.3.

When δ is small, we show that the free boundary of u^{δ} does not deviate much from a fixed plane H with normal e. This means that u^{δ} only feels the effect of the defects $\mathbb{Z}^d \cap H$, and near these defects we expect u^{δ} to look like a solution of the single site problem. Namely we expect, for each $z \in \mathbb{Z}^d \cap H$,

$$\frac{1}{\delta} u^{\delta}(z + \delta x) \approx u_{\rm in}(x) \quad \text{which is a solution of (1.4).} \tag{1.8}$$

We will show (more or less) that $u_{\rm in}$ is proper: then Theorem 1.3 or 1.4 applies to conclude that $u_{\rm in}$ has a far field capacity $k \in [k_{\rm rec}, k_{\rm adv}]$.

To complete the asymptotic expansion of u^{δ} we must describe the behavior of u^{δ} in the outer region away from the holes. We will show that, away from the defects,

$$u^{\delta}(x) \approx (1 + \gamma_d |\xi|^{-1} k \delta^{d-1}) x_d + \delta^{d-1} k \omega(x) \tag{1.9}$$

Here the "corrector" $\omega(x)$ is the solution of an auxiliary cell problem

$$\Delta \omega = 0 \quad \text{in } \{x \cdot e > 0\}, \qquad \partial_{x_d} \omega = 1 - \sum_{z \in \mathbb{Z}^d \cap H} |\xi| \delta_z \quad \text{on } H := \{x \cdot e = 0\}. \tag{1.10}$$

Recall that we have written $e = \xi/|\xi|$ for an irreducible vector $\xi \in \mathbb{Z}^d$, so $|\xi|$ is the area of the fundamental domain of the lattice $\mathbb{Z}^d \cap H$. This means that the average of $\partial_d \omega$ is 0 on H and so the slope of ω at infinity is zero as well. The appearance of fundamental solution type singularities at the lattice sites arises from matching the far-field behavior of the inner solution $u_{\rm in}$.

The above heuristic expansions, in (1.8) and (1.9), are used in two ways to deliver Theorem 1.5. First to construct barriers to obtain a lower bound on the pinning interval (Section 8 And second to study the asymptotics of arbitrary plane-like solutions u^δ, in order to obtain a matching upper bound on the pinning interval (Section 9). As a byproduct we obtain a decomposition of linear and nonlinear part of u^δ as $\delta \to 0$, which appears to be new in the analysis of free-boundary problems:

Theorem 1.6. *(see Section 9.5) For a rational $e \in \mathcal{S}^{d-1}$, suppose $k_{\rm adv}(e) \in (0,\infty)$. If u^δ are strong Birkhoff plane-like solutions with the maximal slope $Q^\delta_{\rm adv}(e)$, then, modulo a period translation of the u^δ, there are $s_\delta \to 0$ so that*

$$\lim_{\delta\to 0} \frac{1}{\delta^{d-1}}[Q^\delta_{\rm adv}(e)\, x\cdot e + s_\delta - u^\delta(x)] = k_{\rm adv}(e)\omega(x) \quad \textit{locally uniformly in} \quad \{x \cdot e > 0\},$$

where ω solves (1.10). A parallel result holds for strong Birkhoff plane-like solutions with the minimal slope with $Q^\delta_{\rm adv}$ and $k_{\rm adv}$ replaced by $Q^\delta_{\rm rec}$ and $k_{\rm rec}$.

Lastly, using (1.3) and our results on single site problem, we can show that the pinning interval is nontrivial for all rational directions, for a generic family of q. In particular it will follows that $k_{\rm adv}$ and $k_{\rm rec}$ are generically non-zero (see Section 10). Here, by the word generic, we mean that it occurs at least on an open set of defect coefficients $q \in C_c(B_1)$.

Theorem 1.7 (Theorem 10.1, Corollary 10.2)**.** *The pinning interval $[Q^\delta_{\rm rec}(e), Q^\delta_{\rm adv}(e)]$ is non-trivial for all rational $e \in \mathcal{S}^{d-1}$ if (a) q from (1.1) is nontrivial and has sign, or if (b) q changes sign with $\int q dx \neq 0$ and with a small Lipschitz constant.*

Remark 1.8. We will reduce the problem to the case $e = e_d$ for the rest of the paper. For the single site problem this normalization can be achieved by rotating q accordingly. Even in Sections 7-9 where we consider the full periodic setting, we reduce to the case $e = e_d$ by rotation, at the cost of considering a rotated periodicity lattice.

Remark 1.9. It is natural to ask what happens for irrational $e \in \mathcal{S}^{d-1}$. In this case there is no longer a $(d-1)$ dimensional periodic structure on the hyperplane $\{x : x \cdot e = 0\}$: instead the hyperplane goes through the periodic structure of the entire $\mathbb{Z}^d$. Due to this averaging, in $d \geq 3$, the extremal solutions will not be able to pass by each lattice site at value of the height s making $k_{\rm adv/rec}(s)$ extremal, and we expect to instead see the coefficients $\int k_{\rm adv}(s)ds$ and $\int k_{\rm rec}(s)ds$ appear in the asymptotic expansion. Also due to the sparsity of lattice sites near the irrational hyperplane we expect the leading order in (1.3) to be order δ^d instead of δ^{d-1}. We expect that the analysis in this case would still build on the single site problem, but would need to also understand the asymptotic distribution of the lattice sites in an $O(\delta)$ neighborhood of an irrational hyperplane.

Remark 1.10. The Bernoulli free boundary problem (1.1) arises from the capillary free boundary problem in the limit as the contact angle approaches 0 or π capillary energy. In that limit u represents a normalized droplet height. For more details see

[10] which also contains the first rigorous application of this limit procedure, there is also some discussion in [23]. In this partially linearized model various computations are simpler, but many of the serious nonlinear issues are still present. We expect parallel results to hold for the capillary free boundary problem, but leave this as an open problem for further study.

1.2. **Comparison with previous literature.** Earlier we discussed the novelty of our results in the context of contact line pinning phenomena. Here we discuss the contribution and relevance of our work in the context of perforated domain homogenization and free boundary regularity theory.

An interesting feature of our work is the new connection it provides the pinning phenomena with cell problems in perforated domain, as well as with the exterior problem for Bernoulli free boundaries. Let us discuss the relevant literature below and our contribution.

Viewing the periodic structure of defects in the context of perforated domains, our analysis establishes, for the first time, a homogenization result when the perforations are also localized in the height variable. In the scenario of capillary drops, the finite height of "perforation" means that the drop is allowed to sit on the defects, a new feature even compared to a few other papers addressing free boundary problems in perforated domain, see [1, 27]. As a result, the single-site problem is driven by a family of pinned solutions, not by a single unique capacitory potential as would be the case in classical perforated domain homogenization problems, for example see [7, 12–14, 28], but this is just a small sample of a huge literature. This scenario generates interesting challenges in contrast to all previous works on perforated domains. In Sections 5 and 6 we describe the full family of capacitory potentials. The content of Section 9 where we analyze the asymptotic expansion of a general plane-like solution is another new aspect caused by the pinning phenomenon.

The study of far field behavior of solutions of the single site problem (1.4) is related to the "exterior problem" for Bernoulli free boundaries. Our usage of hodograph transform in order to study foliating families of solutions indexed via far-field expansions are reminiscent of some works on solutions near singular cones [17–19], see [18] for more discussion of the literature including related literature in the minimal surface theory. There are significant differences with our work. Most works on exterior problems consider variational solutions and have access to monotonicity formulae, whereas we work with viscosity solutions. We also need uniform, quantitative estimates, achieved with explicit barriers, which we carry out in Sections 3 and 4. In addition our two-dimensional results address logarithmic tail behavior, which appear to be new in the study of Bernoulli free boundaries.

Lastly, let us mention that while we are able to describe the pinning interval only in terms of "flat" (or proper) solutions, there could also be singular global solutions of (1.4) that are pinned on the defect. The challenge of understanding two-plane solutions is a serious issue in general for non-minimizing Bernoulli solutions [29, 33]. It also is an issue in the minimal surface capillary model, posing an obstacle for the description of hydrophilic advancing angles as well as hydrophobic receding angles: see e.g. [16].

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2. Setting and preliminary results

In this section we introduce the settings we will work in through the paper, and we will recall several results from the literature which will be useful later.

2.1. **Bernoulli free boundary problems and solution notions.** We will consider continuous viscosity solutions $u \in C(U)$, $u \geq 0$, of the Bernoulli free boundary problem in a domain $U \subset \mathbb{R}^d$ with a continuous and positive coefficient field $Q(x)$

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap U \\ |\nabla u| = Q(x) & \text{on } \partial\{u > 0\} \cap U. \end{cases} \tag{2.1}$$

In many cases we will be able to reduce to studying a homogeneous problem

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap U \\ |\nabla u| = 1 & \text{on } \partial\{u > 0\} \cap U. \end{cases} \tag{2.2}$$

Background on the definition and basic theory of viscosity solutions can be found in Section A. We will also sometimes consider classical solutions of (2.1), u is a classical solution if $u \in C^1(\overline{\{u > 0\}} \cap U) \cap C^2(\{u > 0\} \cap U)$. Of course both viscosity and classical solutions are smooth $\{u > 0\} \cap U$ since they are harmonic in that open set.

2.2. **Sliding comparison.** Note that standard comparison principle in bounded domain does not hold for solutions of (2.2). We will often use the following sliding comparison principle, where we compare a supersolution u with a continuously varying family of regular subsolutions v_t from below. We can also do similar if u is a subsolution and v_t is a continuously varying family of supersolutions. Note that u does not need to be regular, only the sliding family.

Lemma 2.1 (Sliding comparison [9, Theorem 2.2])**.** *Let $u \in C(\bar{U})$ be a viscosity supersolution of (2.1). Let $v_0 \in C(\bar{U})$ satisfy the following:*

(i) $v_t(x) := v_0(x + te_d)$ are classical subsolutions of (2.1) for all $t \in [0, T]$;
(ii) $v_0 \leq u$ in U;
(iii) $v_t \leq u$ on ∂U and $v_t < u$ in $\overline{\{v(x + te_d) > 0\}} \cap \partial U$ for $0 \leq t \leq T$.

Then $v_t \leq u$ in U for $0 \leq t \leq T$.

We will often apply the sliding comparison in the whole space by ensuring the boundary ordering property holds "at ∞", that is on the boundary of all sufficiently large radius balls.

2.3. **Blow-downs of exterior solutions under one-sided flatness.** Much of our analysis is focused on the single-site defect problem (1.4) where u solves (2.2) in the *exterior domain* $U = \mathbb{R}^d \setminus B_1$. The initial stage in the analysis of the asymptotics at ∞ is a blow-down limit.

One cannot expect arbitrary exterior solutions to have a simple blow-down, however we will work in a nicer class. Since we are working with viscosity solutions and sliding comparisons we will typically arrive at a one-sided flatness condition: namely either

$$u(x) \leq (x_d + t)_+ \quad \text{for some} \quad t \in \mathbb{R} \tag{2.3}$$

or

$$u(x) \geq (x_d + t)_+ \quad \text{for some} \quad t \in \mathbb{R}. \tag{2.4}$$

We will show that in these situations the blow-down profile of an exterior solution u is a half-plane solution, modulo some additional hypotheses described below. While the proofs are mostly parallel to the full Bernoulli problem, we need to deal with the presence of the defect in B_1, and we need to rule out the occurrence of two-plane solutions which can appear as blow-downs in the case of (2.4).

Proposition 2.2. *Suppose that u solves (2.2) in $U = \mathbb{R}^d \setminus B_1$, then the following hold:*

(i) If u also satisfies (2.3) and is not identically zero then

$$\lim_{r\to\infty} \frac{1}{r} u(rx) = (x_d)_+ \quad \textit{locally uniformly on } \mathbb{R}^d.$$

(ii) If u also satisfies (2.4) then

$$\lim_{r\to\infty} \frac{1}{r} u(rx) = (x_d)_+ \quad \textit{locally uniformly on } \{x_d \geq 0\}.$$

We provide a proof in Appendix B.1.

2.4. $C^{1,\alpha}$ **regularity for flat solutions.** General viscosity solutions of (2.2) may not be regular. However, sufficiently flat solutions are indeed classical in a slightly smaller domain.

Theorem 2.3 (Caffarelli)**.** *For any $\alpha \in (0,1)$ there is $\eta_0(\alpha, d) > 0$ and $C(d) \geq 1$ so that the following holds. If u is a viscosity solution of (2.2) in B_1 and*

$$(x_d)_+ \leq u(x) \leq (x_d + \eta)_+ \quad \textit{in} \quad B_1 \quad \textit{with} \quad \eta \leq \eta_0$$

then $u \in C^{1,\alpha}(\overline{\{u > 0\}} \cap B_{1/2})$ and

$$\sup_{\overline{\{u>0\}} \cap B_{1/2}} |\nabla u - e_d| \leq C\eta.$$

For exterior solutions which blow down to a half-planar solution we can apply this regularity theory sufficiently large annuli.

Lemma 2.4. *Suppose that u solves (2.2) in $\mathbb{R}^d \setminus B_1$ and blows down to $(x_d)_+$ as in (1.5). Then there is $R_0 > 0$ depending on u so that for any $\alpha \in (0,1)$*

$$u \in C^{1,\alpha}(\overline{\{u > 0\}} \setminus B_{R_0})$$

and for all $r \geq R_0$

$$\sup_{\overline{\{u>0\}} \cap \partial B_r} |\nabla u - e| \leq C \sup_{B_2 \setminus B_{1/2}} |r^{-1} u(rx) - (x \cdot e)_+|.$$

2.5. **Partial hodograph transform.** Now we recall the partial hodograph transform. This transformation was first introduced as a tool to prove higher regularity of free boundaries by [32]. While requiring a C^1, planar-like solution as a starting point, this transformation has served as one of the main tools to obtain higher regularity of the free boundary. For us the transformation puts us in a PDE setting where we can apply classical higher regularity, Harnack, and Kelvin transform techniques to study the exterior asymptotics. It is also convenient for the construction of barriers, especially for $d = 2$ case, where the logarithmic far-field growth of the free boundary makes it challenging to construct barriers in original coordinates.

Let $u \in C^2(\{u > 0\}) \cap C^1(\overline{\{u > 0\}})$ be a classical solution of (2.2) in an open neighborhood U of $x_0 \in \partial\{u > 0\}$. We assume that $\partial_{x_d} u > 0$ in $\overline{\{u > 0\}} \cap U$. We now define the new coordinates $y = (y', y_d)$ by

$$y' := x',\ \ y_d = u(x), \quad \text{and} \quad v(y) := x_d - y_d. \tag{2.5}$$

Under our hypotheses on u the coordinate transform defines a diffeomorphism of $\overline{\{u > 0\}} \cap U$ onto its image, a set $\mathcal{N} \cap \{y_d \geq 0\}$.

We now derive the y-cordinate PDE in the domain $\{y_d \geq 0\} \cap \mathcal{N}$. Observe that

$$\nabla_{y'} y_d = 0 = \nabla_{y'} u + \partial_d u \nabla_{y'} x_d,$$

and so

$$\nabla_{y'} v = \nabla_{y'} x_d = (-\nabla_{x'} u)(\partial_d u)^{-1} \text{ and } \partial_{y_d} v = \partial_{y_d} x_d - 1 = (\partial_d u)^{-1} - 1.$$

We also compute

$$\sqrt{1 + |\nabla_y' v|^2} = |Du|(\partial_d u)^{-1} = |Du|(1 + \partial_{y_d} v) \quad \text{and} \quad (\nabla v)'/(1 + (\nabla v)_d) = -\nabla_{x'} u.$$

Thus, the PDE in the Hodograph coordinates is:

$$\begin{cases} \operatorname{tr}(A(\nabla_y v) D_y^2 v) = 0 & \text{in } \ \{y_d > 0\} \cap \mathcal{N}; \\ \partial_{y_d} v = N(\nabla_y' v) := \sqrt{1 + |\nabla_y' v|^2} - 1 & \text{on } \partial\{y_d > 0\} \cap \mathcal{N}, \end{cases} \tag{2.6}$$

where

$$A(p) = \begin{bmatrix} \mathrm{I}_{d-1} & (1 + p_d)^{-1} p' \\ (1 + p_d)^{-1} (p')^T & \frac{1 + |p'|^2}{(1 + p_d)^2} \end{bmatrix}. \tag{2.7}$$

Remark 2.5. Note that $A(p)$ is elliptic if $|p| < 1$ and satisfies

$$\|A(p) - I\| \leq C|p| \quad \text{for} \quad |p| \leq 1/2 \tag{2.8}$$

and

$$|N(p')| \leq C|p'|^2 \quad \text{for} \quad |p'| \ll 1. \tag{2.9}$$

These are the main properties of A and N that we will use later.

2.6. **Regularity and Harnack for nonlinear oblique boundary value problems.** We use a Harnack inequality and higher regularity estimates for the nonlinear Neumann problem (2.6). In the contexts where we use these results, our solutions v of (2.6) will be at least $C^{1,\alpha}$ with $|\nabla v| \leq \eta \ll 1$. Therefore we can view (2.6) as a linear uniformly elliptic PDE with a linear uniformly oblique boundary condition with measurable coefficients.

Applying Lieberman's Harnack inequality [34, Theorem 3.3] for linear uniformly elliptic PDE with uniformly oblique boundary condition and measurable coefficients, we arrive at the following Harnack inequality for the Hodograph PDE (2.6).

Theorem 2.6 (Corollary of Lieberman's Harnack inequality [34, Theorem 3.3])**.** *For any $\eta < 1$ there is a constant $C(d, \eta) > 1$ such that for any non-negative C^1 solution of (2.6) in $B_1^+ = B_1 \cap \{y_d > 0\}$ called v with $|\nabla v| \leq \eta < 1$,*

$$\sup_{B_{1/2}^+} v \leq C \inf_{B_{1/2}^+} v.$$

We also need elliptic regularity type estimates up to second order with the correct scaling in large balls. The qualitative C^∞ regularity of $C^{1,\alpha}$ solutions of (2.6) was proved in the original paper applying hodograph techniques by Kinderlehrer and Nirenberg [32, p. 386]. Then we can apply Lieberman and Trudinger's [35] a-priori estimates for C^2 solutions of nonlinear uniformly elliptic problems with nonlinear oblique boundary conditions.

Theorem 2.7 (See [35, Theorem 1.1])**.** *If v is a C^2 solution of (2.6) in B_r^+ with $|\nabla v| \leq \eta < 1$ then*

$$r^k |\nabla^k v(0)| \leq C \operatorname*{osc}_{B_r^+} v \quad \textit{for} \quad k = 1, 2.$$

Note that the nonlinear problem (2.6) is invariant under hyperbolic rescaling $v \mapsto rv(\cdot/r)$, which is how we are applying [35, Theorem 1.1].

2.7. **Harnack inequality for the Bernoulli problem.** Sometimes it is convenient to have the Harnack inequality Theorem 2.6 directly available in the original coordinates. By combining Theorem 2.3 with the hodograph transform and Theorem 2.6 one can derive the following Harnack inequality for flat solutions of (2.2).

Corollary 2.8. *Let u solve (2.2) in B_1. Then there is $\eta_0 > 0$ and $C \geq 1$ depending on dimension so that the following holds. If*

$$(x_d)_+ \leq u(x) \leq (x_d + \eta_0)_+ \quad \textit{in} \quad B_1$$

then

$$\sup_{\{u>0\} \cap B_{1/2}} (u(x) - x_d) \leq C \inf_{\{u>0\} \cap B_{1/2}} (u(x) - x_d).$$

Similarly if

$$(x_d - \eta_0)_+ \leq u(x) \leq (x_d)_+ \quad \textit{in} \quad B_1$$

then

$$\sup_{\{u>0\} \cap B_{1/2}} (x_d - u(x)) \leq C \inf_{\{u>0\} \cap B_{1/2}} (x_d - u(x)).$$

3. Close to planar exterior solutions

In this section we analyze the asymptotic expansion of one-sided flat exterior solutions, i.e. of solutions to (2.2) in $U := \mathbb{R}^d \setminus B_1$ with the property $|\nabla u - e_d| \ll 1$ in $\{u > 0\}$.

The following result will be applied in Section 4 to our original solutions in the region that are away from the defects.

Theorem 3.1. *Let u be a classical C^1 solution of (2.2) in $\mathbb{R}^d \setminus B_1$ such that $u(x) - x_d$ is bounded from above or from below in $\overline{\{u > 0\}} \setminus B_1$. If in addition*

$$\sup_{\overline{\{u>0\}} \setminus B_1} |\nabla u - e_d| \leq \eta_1 \quad \textit{for} \quad 0 < \eta_1(d) < 1/2,$$

then the following holds.

(i) *For $d \geq 3$, there is $C(d) \geq 1$ and $s, k \in \mathbb{R}$ so that*

$$\left|u(x) - (x_d + s + k|x|^{2-d})\right| \leq C|x|^{1-d} \operatorname*{osc}_{(B_2 \setminus B_1) \cap \{u>0\}} (u(x) - x_d)$$

and

$$|k| \leq C \operatorname*{osc}_{(B_2 \setminus B_1) \cap \{u>0\}} (u(x) - x_d).$$

(ii) *For $d = 2$, there is $C \geq 1$ and $k \in \mathbb{R}$ so that*

$$|u(x) - (x_d + k \log |x|)| \leq C \max_{(B_2 \setminus B_1) \cap \{u>0\}} |u(x) - x_d|$$

and

$$|k| \leq C \sup_{(B_2 \setminus B_1) \cap \{u>0\}} |u(x) - x_d|.$$

Due to the small gradient condition, we can perform our analysis entirely in the hodograph coordinates (see Section 2.5), which transforms our free boundary problem to a nonlinear elliptic problem in a half-space with a nonlinear Neumann condition.

3.1. **One-sided flat solutions in hodograph variables.**

Now we state a version of Theorem 3.1 in the hodograph variable.

Theorem 3.2. *Let v be a smooth and one-sided bounded solution of (2.6) with $\mathcal{N} = \mathbb{R}^d \setminus B_1$. There is $\eta_0(d) \in (0, 1/2)$ such that if $\sup_{\mathbb{R}^d_+ \setminus B_1} |\nabla v| \leq \eta_0$ then the following holds.*

(1) *For $d \geq 3$, there is $C(d) \geq 1$ and $s, k \in \mathbb{R}$ so that*

$$\left|v(y) - s - k|y|^{2-d}\right| \leq C|y|^{1-d} \operatorname*{osc}_{(B_2 \setminus B_1)^+} v$$

and

$$|s| \leq \fint_{\partial B_1 \cap \mathbb{R}^d_+} v + C \operatorname*{osc}_{(B_2 \setminus B_1)^+} v \quad \text{and} \quad |k| \leq C \operatorname*{osc}_{(B_2 \setminus B_1)^+} v.$$

(2) *For $d = 2$, there is $C \geq 1$ and $k \in \mathbb{R}$ so that*

$$|v(y) - k \log |y|| \leq C \max_{(B_2 \setminus B_1)^+} |v|$$

and

$$|k| \leq C \max_{(B_2 \setminus B_1)^+} |v|.$$

The remainder of Section 3 will be occupied with the proof of Theorem 3.2.

Remark 3.3. Our analysis can be applied to any nonlinear Neumann problem of the type (2.6) with operators $A(p)$ and $N(p')$ satisfying the estimates (2.8) and (2.9).

3.2. **Initial barriers.** First we establish the existence of smooth homogeneous super and subsolution barriers, to be used in this section.

In dimension $d = 2$ it is very convenient to have barriers with the correct logarithmic behavior at highest order. We show the existence of such barriers in the next result.

Lemma 3.4. *Assume that A and N satisfy (2.8) and (2.9) and $d = 2$. Define the barriers*

$$\psi_\pm(x) := \log|x| \pm \log(1 + \log|x|) \pm \frac{x_d}{|x|^2}.$$

There is $\varsigma_0 > 0$ sufficiently small so that if $0 \leq \varsigma \leq \varsigma_0$ then $\varsigma\psi_+$ is a subsolution and $\varsigma\psi_-$ is a supersolution of (2.6) in $\mathbb{R}^2_+ \setminus B_1$.

Note that, although the actual hodograph PDE is not invariant under negation $v \mapsto -v$, the properties (2.8) and (2.9) are. So, for example, we can also conclude under the hypotheses of Lemma 3.4 that $-\varsigma\psi_+$ is a supersolution of (2.6) in $\mathbb{R}^2_+ \setminus B_1$.

Proof. We check the solution properties by direct computation. Note that $\log|x|$ is harmonic and its derivatives $D^k \log|x|$ are homogeneous of order k. Also $\frac{x_d}{|x|^2}$, which is $\partial_d(\log|x|)$, is harmonic, and its kth derivatives are homogeneous of order $k + 1$.

We record the derivative of $\log(1 + \log|x|)$

$$\nabla \log(1 + \log|x|) = \frac{1}{1 + \log|x|} \frac{x}{|x|^2}$$

and

$$D^2 \log(1 + \log|x|) = \frac{1}{|x|^2(1 + \log|x|)}(I - 2\frac{x \otimes x}{|x|^2}) + \frac{1}{|x|^2(1 + \log|x|)^2} \frac{x \otimes x}{|x|^2}.$$

Since the trace of $(I - 2\frac{x \otimes x}{|x|^2})$ vanishes in dimension $d = 2$ we find that

$$\begin{aligned}\operatorname{tr}(A(\varsigma\nabla\psi_\pm)D^2\psi_\pm) &= \operatorname{tr}(D^2\psi_\pm) + \operatorname{tr}((A(\varsigma\nabla\psi_\pm) - I)D^2\psi_\pm) \\ &= \pm\frac{1}{|x|^2(1 + \log|x|)^2} + \operatorname{tr}((A(\varsigma\nabla\psi_\pm) - I)D^2\psi) \\ &= \pm\frac{1}{|x|^2(1 + \log|x|)^2} + O(\varsigma\frac{1}{|x|^3})\end{aligned}$$

For the last equality we used (2.8) and the homogeneous upper bounds, which hold in $\mathbb{R}^d \setminus B_1$, $|\nabla\psi_\pm| \leq C|x|^{-1}$ and $|D^2\psi_\pm| \leq C|x|^{-2}$. Thus, for $|\varsigma| \leq \varsigma_0$ sufficiently small

$$\pm\operatorname{tr}(A(\varsigma\nabla\psi_\pm)D^2\psi_\pm) \geq 0 \ \text{ for } \ |x| \geq 1.$$

Next we check the subsolution properties on $\partial\mathbb{R}^d_+$. Noting the formula

$$\nabla\psi_\pm(x) = \frac{x}{|x|^2}\left(1 + \frac{1}{1 + \log|x|}\right) \pm \frac{1}{|x|^4}\left(e_d|x|^2 - 2x_d x\right) \tag{3.1}$$

and so

$$\partial_{x_d}\psi_\pm(x) = \pm\frac{1}{|x|^2} \ \text{ on } \ x_d = 0.$$

On the other hand, using again the homogeneous upper bound $|\nabla'\psi_\pm| \leq C|x|^{-1}$ in $|x| \geq 1$ and (2.9)

$$N(\varsigma\nabla'\psi_\pm) \leq \varsigma^2\frac{1}{|x|^2}.$$

So, again for $\varsigma > 0$ sufficiently small,

$$\varsigma \partial_{x_d} \psi_+(x) \geq N(\varsigma \nabla' \psi_+) \quad \text{and} \quad \varsigma \partial_{x_d} \psi_-(x) \leq N(\varsigma \nabla' \psi_-).$$

□

The barriers for $d \geq 3$ are given below. We omit the proof, since it is standard (and easier than the case $d = 2$), based on the fact that $A(p)$ has ellipticity ratio close to 1 when $|p|$ is sufficiently small.

Lemma 3.5. *For $d \geq 3$ and any $0 < \delta < d - 1$ there is $c_\delta = c_\delta(d) > 0$ so that, calling $\sigma = \operatorname{sgn}(d - 2 - \delta)$,*

$$\phi_+(y) = \sigma c_\delta |y|^{2-d+\delta} \quad \text{and} \quad \phi_-(y) = -\sigma c_\delta |y + \frac{1}{2} e_d|^{2-d+\delta}$$

are respectively a supersolution and subsolution of (2.6).

3.3. **A growth bound via the homogeneous barriers.** Next we show a barrier argument which, in a certain sense, controls the growth of v on a large annulus $(B_r \setminus B_1)^+$ in terms of its growth on $(B_2 \setminus B_1)^+$.

Lemma 3.6. *Suppose that $d \geq 3$, $r \geq 2$, and v solves (2.6) in $(B_r \setminus B_1)^+$. There is $\eta_0(d) > 0$ sufficiently small so that if $\sup_{\mathbb{R}^d_+ \setminus B_1} |\nabla v| \leq \eta_0(d)$ then*

$$\max_{\partial B_r} v \geq \max_{\partial B_1} v - C(\max_{\partial B_1} v - \max_{\partial B_2} v)_+$$

and

$$\min_{\partial B_r} v \leq \min_{\partial B_1} v + C(\min_{\partial B_2} v - \min_{\partial B_1} v)_+$$

for a universal $C \geq 1$.

Similar estimates are true of harmonic functions in annuli. Note that if v were a harmonic polynomial the conclusions would be trivial, $\max_{\partial B_r} v \geq 0$ and $\min_{\partial B_r} v \leq 0$.

The proof follows a similar idea to [3, Lemma 5.7].

Proof. We will just argue for the lower bound on $\max_{\partial B_r} v$. The upper bound on the minimum is similar. Define

$$m(r) := \max_{\partial B_1} v - \max_{\partial B_r} v.$$

The goal is to show that $m(r) \leq C m(2)_+$.

If $m(r) \leq 0$ we are done. If $m(r) \geq 0$ then maximum principle in $(B_r \setminus B_1) \cap \mathbb{R}^d_+$. implies that

$$v(x) \leq \max_{\partial B_1} v \quad \text{in} \quad (B_r \setminus B_1) \cap \mathbb{R}^d_+.$$

Implying that $m(2) \geq 0$ as well. So we have reduced to the case that $m(r) \geq 0$ and $m(2) \geq 0$.

By Lemma 3.5 the function $\phi(y) = \alpha |y|^{-1/2}$ is a supersolution of (10.2) in $\mathbb{R}^d_+ \setminus B_1$ whenever $0 \leq \alpha \leq c_*(d)$. Define a barrier, with the non-negative (since $m(r) \geq 0$) constant $\alpha = \min\{m(r), c_*\}$,

$$\psi(y) := \max_{\partial B_1^+} v + \alpha(|y|^{-1/2} - 1).$$

By Lemma 3.5 we have that ψ is a supersolution of (10.2) in $\mathbb{R}^d_+ \setminus B_1$ since $0 \leq \alpha \leq c_*$. Also

$$\psi(x) = \max_{\partial B_1^+} v \geq v(x) \quad \text{on } \partial B_1.$$

Note that, since $\alpha \leq m(r)$, on $y \in \partial B_r^+$ we have

$$\psi(y) = \max_{\partial B_1} v + \alpha(r^{-1/2} - 1) \geq \max_{\partial B_1} v - \alpha \geq \max_{\partial B_r} v.$$

Thus $\psi \geq v$ on ∂B_r^+. By comparison principle $\psi \geq v$ on $(B_r \setminus B_1)^+$.

Evaluating ψ on ∂B_2^+ with the fact $\psi \geq v$, we conclude that

$$(1 - 2^{-1/2}) \min\{m(r), c_*\} \leq \max_{\partial B_1} v - v(y) \quad \text{for any } y \in \partial B_2^+$$

and so

$$\min\{m(r), c_*\} \leq (1 - 2^{-1/2})^{-1}(\max_{\partial B_1} v - \max_{\partial B_2} v) \leq (1 - 2^{-1/2})^{-1} c_d \eta_0 \leq \frac{1}{2} c_*$$

as long as we choose η_0 sufficiently small. Since $c_* > \frac{1}{2} c_*$ so we must have $\min\{m(r), c_*\} = m(r)$, and so,

$$m(r) \leq (1 - 2^{-1/2})^{-1} m(2).$$

□

By a very similar argument we can derive a growth bound in dimension $d = 2$.

Lemma 3.7. *Suppose that $d = 2$, $r \geq 2$, and v solves (2.6) in $(B_r \setminus B_1)^+$. There is $\eta_0 > 0$ sufficiently small so that if $\sup_{\mathbb{R}^d_+ \setminus B_1} |\nabla v| \leq \eta_0$ then*

$$\max_{\partial B_r} v \geq \max_{\partial B_1} v - C(1 + \log r)(\max_{\partial B_1} v - \max_{\partial B_2} v)_+$$

and

$$\min_{\partial B_r} v \leq \min_{\partial B_1} v + C(1 + \log r)(\min_{\partial B_2} v - \min_{\partial B_1} v)_+$$

for some $C \geq 1$ universal.

Proof. We will just argue for the lower bound on $\max_{\partial B_r} v$. The upper bound on the minimum is similar. Define

$$m(r) := \max_{\partial B_1} v - \max_{\partial B_r} v.$$

The goal is to show that $m(r) \leq C r^\delta m(2)_+$.

If $m(r) \leq 0$ we are done. If $m(r) \geq 0$ then maximum principle in $(B_r \setminus B_1) \cap \mathbb{R}^d_+$. implies that

$$v(x) \leq \max_{\partial B_1} v \quad \text{in} \quad (B_r \setminus B_1) \cap \mathbb{R}^d_+.$$

Implying that $m(2) \geq 0$ as well. So we have reduced to the case that $m(r) \geq 0$ and $m(2) \geq 0$.

By Lemma 3.5 the function $\phi(y) = -\alpha |y|^\delta$ is a supersolution of (10.2) in $\mathbb{R}^d_+ \setminus B_1$ whenever $0 \leq \alpha \leq c_\delta$. Define a barrier, with the non-negative (since $m(r) \geq 0$) constant $\alpha = \min\{r^{-\delta} m(r), c_\delta\}$,

$$\psi(y) := \max_{\partial B_1^+} v - \alpha(|y|^\delta - 1).$$

By Lemma 3.5 we have that ψ is a supersolution of (10.2) in $\mathbb{R}^d_+ \setminus B_1$ since $0 \leq \alpha \leq c_\delta$. Also

$$\psi(x) = \max_{\partial B_1^+} v \geq v(x) \quad \text{on } \partial B_1.$$

Note that, since $\alpha \leq r^{-\delta} m(r)$, on $y \in \partial B_r^+$ we have

$$\psi(y) = \max_{\partial B_1} v - \alpha(r^\delta - 1) \geq \max_{\partial B_1} v - m(r) \geq \max_{\partial B_r} v.$$

Thus $\psi \geq v$ on ∂B_r^+. By comparison principle $\psi \geq v$ on $(B_r \setminus B_1)^+$.

Evaluating ψ on ∂B_2^+ with the fact $\psi \geq v$, we conclude that

$$(2^\delta - 1) \min\{r^{-\delta} m(r), c_\delta\} \leq \max_{\partial B_1} v - v(y) \quad \text{for any } y \in \partial B_2^+$$

and so

$$\min\{r^{-\delta} m(r), c_\delta\} \leq (2^\delta - 1)^{-1} (\max_{\partial B_1} v - \max_{\partial B_2} v) \leq (2^\delta - 1)^{-1} \eta_0 \leq \frac{1}{2} c_\delta$$

as long as we choose $0 < \eta_0 \leq c c_\delta(d)$ sufficiently small. Since $c_\delta > \frac{1}{2} c_\delta$ so we must have $\min\{r^{-\delta} m(r), c_\delta\} = r^{-\delta} m(r)$, and so,

$$m(r) \leq (2^\delta - 1)^{-1} r^\delta m(2).$$

□

3.4. **Bounds and limit at infinity.** First we consider bounded solutions, and show that bounded solutions have a limit at ∞.

Lemma 3.8. *Suppose that $d \geq 2$ and v solves (2.6) in $\mathbb{R}^d_+ \setminus B_1$, $\sup_{\mathbb{R}^d_+ \setminus B_1} |\nabla v| \leq \frac{1}{2}$, and either $\liminf_{|x| \to \infty} v$ or $\limsup_{|x| \to \infty} v$ is finite. Then $\lim_{|x| \to \infty} v(x)$ exists.*

Note that if v is bounded then the limit hypothesis is satisfied. On the other hand, Lemma 3.8 shows that if either $\liminf_{|x| \to \infty} v > -\infty$ or $\limsup_{|x| \to \infty} v < +\infty$ then v is bounded.

Also note that in the bound $|\nabla v| \leq 1/2$ above the $1/2$ is not critical, it can be replaced with any $0 < c < 1$ for uniform ellipticity of the hodograph PDE.

Proof. Similar to [3, Lemma 5.8]. Let's consider the case $\liminf_{|x| \to \infty} v(x) > -\infty$, the other case is similar.

We may assume that $\liminf_{|x| \to \infty} v(x) = 0$. Let $\varepsilon > 0$. There is $R \geq 1$ sufficiently large so that $v(x) \geq -\varepsilon$ for $|x| \geq R$. Also is a sequence $x_k \to \infty$ with $|x_k| \geq R$ so that $v(x_k) \leq \varepsilon$, and call $r_k = |x_k|$. Since $v + \varepsilon$ is a non-negative solution of (2.6) in $\mathbb{R}^d_+ \setminus B_R$, by Harnack inequality Theorem 2.6,

$$\sup_{\partial B_{r_k} \cap \mathbb{R}^d_+} (v + \varepsilon) \leq C \inf_{\partial B_{r_k} \cap \mathbb{R}^d_+} (v + \varepsilon) \leq 2C\varepsilon$$

or

$$\sup_{\partial B_{r_k} \cap \mathbb{R}^d_+} v \leq (2C + 1)\varepsilon.$$

By maximum principle in each $(B_{r_{k+1}} \setminus B_{r_k}) \cap \mathbb{R}^d_+$ also

$$\sup_{R^d_+ \setminus B_{r_1}} v = \sup_k \sup_{(B_{r_{k+1}} \setminus B_{r_k}) \cap \mathbb{R}^d_+} v \leq (2C + 1)\varepsilon$$

and so

$$\limsup_{|x| \to \infty} v(x) \leq (2C + 1)\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary the limit supremum is 0 agreeing with the limit infimum. □

Next we consider bounded solutions, which have a limit at ∞ due to Lemma 3.8, and make the upper and lower bounds explicit depending on the limit at ∞. Since we will refer to the limit often we define

$$v_\infty := \lim_{|x|\to\infty} v(x). \tag{3.2}$$

Lemma 3.9. *Suppose that $d \geq 2$ and v solves (2.6) in $\mathbb{R}^d_+ \setminus B_1$, $\sup_{\mathbb{R}^d_+ \setminus B_1} |\nabla v| \leq \frac{1}{2}$ and v is bounded, and so v_∞ exists. Then*

$$\min\{\min_{\partial B_1^+} v, v_\infty\} \leq v(x) \leq \max\{\max_{\partial B_1^+} v, v_\infty\} \quad \text{in } \mathbb{R}^d_+ \setminus B_1.$$

In fact the stated bounds on $v(x)$ are exactly the range of v on $\mathbb{R}^d_+ \setminus B_1$.

Proof. Call $M = \max\{\max_{(\partial B_1)^+} v, v_\infty\}$, then the constant function $M + \delta$ is a solution of (2.6) which is above v on ∂B_1 and on all sufficiently large spheres ∂B_R. Comparison principle in $(B_R \setminus B_1)^+$ implies that $v(x) \leq M + \delta$ in $\mathbb{R}^d_+ \setminus B_1$. Since δ is arbitrary $v(x) \leq M$. The lower bound is proved in a symmetric way. □

3.4.1. *One-sided bounded solutions in $d \geq 3$.* Now we consider solutions which are bounded only from one side. First, in dimension $d \geq 3$, we show that such solutions are bounded and control the limit $v_\infty = \lim_{|x|\to\infty} v$ in terms of the values in an annulus $B_2 \setminus B_1$.

Lemma 3.10. *Suppose that $d \geq 3$ and v solves (2.6). There is $\eta_0(d) > 0$ sufficiently small so that if $\sup_{\mathbb{R}^d_+ \setminus B_1} |\nabla v| \leq \eta_0(d)$ and v is bounded either from above or from below, then v is bounded and*

$$\max_{\partial B_1} v - C(\max_{\partial B_1} v - \max_{\partial B_2} v)_+ \leq v_\infty \leq \min_{\partial B_1} v + C(\min_{\partial B_2} v - \min_{\partial B_1} v)_+$$

where $C \geq 1$ is universal.

Proof. First we show that v is bounded. Assume that v is bounded from below, the other case can be argued similarly. Without loss assume that $v \geq 0$. By Harnack inequality, Theorem 2.6, for all $r \geq 2$,

$$\max_{\partial B_r^+} v \leq C \min_{\partial B_r^+} v.$$

By Lemma 3.6

$$\min_{\partial B_r^+} v \leq \min_{\partial B_1} v + C(\min_{\partial B_2} v - \min_{\partial B_1} v)_+,$$

and so $\max_{\partial B_r^+} v$ is bounded independent of r. Thus v is bounded.

Now we show that v bounded implies the quantitative bounds in the statement. Since v is bounded Lemma 3.8 implies that $v_\infty = \lim_{|x|\to\infty} v$ exists and Lemma 3.9 implies that

$$\min\{\min_{\partial B_1^+} v, v_\infty\} \leq v(x) \leq \max\{\max_{\partial B_1^+} v, v_\infty\} \quad \text{in } \mathbb{R}^d_+ \setminus B_1.$$

In the case $v_\infty \in [\min_{\partial B_1^+} v, \max_{\partial B_1^+} v]$ the result is immediate.

Let's consider the case $v_\infty < \min_{(\partial B_1)^+} v$. In this case the claimed upper bound inequality is trivial, so we aim for the lower bound. Note that $v(x) - v_\infty$ is non-negative. Applying Lemma 3.6 to $v(x) - v_\infty$ we find

$$0 = \lim_{r\to\infty} \max_{\partial B_r^+}(v(x) - v_\infty) \geq \max_{\partial B_1}(v(x) - v_\infty) - C(\max_{\partial B_1} v - \max_{\partial B_2} v)_+.$$

Rearranging this gives the desired lower bound on v_∞.

The case $v_\infty > \max_{\partial B_1} v$ is argued symmetrically.

□

Next we show a quantitative convergence rate to the exterior limit. This will follow from a barrier argument using the explicit barriers from Lemma 3.5, we will make use of the qualitative limit information established before to deal with the "boundary at infinity" in the comparison argument.

Lemma 3.11. *Suppose $d \geq 3$. For any $\delta > 0$ there is $\eta_0(\delta, d) > 0$ small so that: if v solves (2.6) is bounded either from below or from above and $\sup_{\mathbb{R}^d_+\setminus B_1} |\nabla v| \leq \eta_0$ then*

$$|y|^{d-2-\delta}|v(y) - v_\infty| \leq C \operatorname*{osc}_{(B_2\setminus B_1)^+} v \quad \text{in} \quad \mathbb{R}^d \setminus B_1.$$

Proof. By Lemma 3.10 and Lemma 3.8 the limit $v_\infty := \lim_{|x|\to\infty} v(x)$ exists. Then Lemma 3.10 implies that

$$\sup_{\partial B_1^+} |v - v_\infty| \leq C \operatorname*{osc}_{(B_2\setminus B_1)^+} v \leq C\eta_0.$$

Let η_0 sufficiently small so that $C\eta_0 \leq c_\delta$ from Lemma 3.5. Then, by Lemma 3.5,

$$\phi(x) := (\sup_{\partial B_1^+} |v - v_\infty|)|y|^{2-d+\delta}$$

is a supersolution of (10.2) in $\mathbb{R}^d_+ \setminus B_1$. Let $\varepsilon > 0$ arbitrary and $R_1 \geq 1$ sufficiently large so that

$$v(y) \leq \varepsilon \quad \text{on} \quad \partial B_R \cap \mathbb{R}^d_+ \quad \text{for all} \quad R \geq R_1.$$

Then by maximum principle in $B_R \setminus B_1$, for every $R \geq R_1$,

$$v(y) \leq \varepsilon + (\sup_{\partial B_1^+} |v|)|y|^{2-d+\delta} \quad \text{in} \quad \mathbb{R}^d_+ \setminus B_1.$$

Since $\varepsilon > 0$ was arbitrary

$$v(y) \leq (\sup_{\partial B_1^+} |v|)|y|^{2-d+\delta} \quad \text{in} \quad \mathbb{R}^d_+ \setminus B_1.$$

The lower bound argument is similar using the subsolution barriers from Lemma 3.5. □

3.4.2. *One-sided bounded solutions in $d = 2$.* The situation in $d = 2$ is slightly different. First we show a maximum principle. In $d = 2$ bounded from below (resp. above) solutions of (2.6) in the exterior of B_1^+ attain their minimum (resp. maximum) value on ∂B_1^+.

Lemma 3.12. *Suppose that $d = 2$, v solves (2.6), and v is bounded from below then*

$$\inf_{\mathbb{R}^2_+\setminus B_1} v(y) = \min_{\partial B_1^+} v,$$

similarly, if v is bounded from above then

$$\sup_{\mathbb{R}^2_+\setminus B_1} v(y) = \max_{\partial B_1^+} v.$$

Remark 3.13. This is where we need the logarithmic barriers from Lemma 3.4. In particular we need a supersolution barrier $\phi(y)$ with $\phi(y) \to +\infty$ as $|y| \to +\infty$. Homogeneous supersolution barriers with a downward pointing singularity only exist with homogeneity $\alpha \leq d-2 = 0$, see [3]. Thus we cannot achieve both the supersolution property and the property $\lim_{|y|\to\infty} \phi(y) = +\infty$ with a non-zero homogeneity as in Lemma 3.5, and we need to work with logarithmic barriers.

Proof. We just consider the bounded from above case, the bounded from below case is similar. By Lemma 3.4 we have the radially symmetric barriers

$$\phi_L(|y|) := L^{-1}(\log|y| - \log(\log L + \log|y|))$$

which are supersolutions of (2.6) in $\mathbb{R}^d_+ \setminus B_1$ as long as $L \geq R_0$. Here $R_0 > e$ is just some universal parameter determined in the proof of Lemma 3.4.

Since v is bounded from above $M := \sup_{\mathbb{R}^d_+\setminus B_1} v < +\infty$. Suppose that

$$\alpha := \min\{M - \max_{\partial B_1^+} v, \tfrac{1}{2}\} > 0,$$

otherwise we are done. Let $r \gg 1$, and define $L(r) := \alpha^{-1}\log r$. Since $\log r \to +\infty$ as $r \to +\infty$, for $r > 1$ sufficiently large $L(r) \geq R_0$ and also

$$L(r) \geq R_0 \quad \text{and also} \quad \frac{\log(\log L(r) + \log r)}{\log r} \leq \frac{\alpha}{2} \leq \tfrac{1}{2}. \tag{3.3}$$

For such large $r \gg 1$ define

$$\psi_r(y) = M - \frac{\alpha}{2} + \phi_{L(r)}(|y|),$$

which is a supersolution of (2.6) in $\mathbb{R}^d_+ \setminus B_1$. On $y \in \partial B_r^+$, using (3.3),

$$\psi_r(y) = M - \frac{\alpha}{2} + \alpha(1 - \frac{\log(\log L(r) + \log r)}{\log r}) \geq M \geq v(y).$$

On $y \in \partial B_1^+$, using $r \gg 1$ from using (3.3) again,

$$\psi_r(y) = M - \frac{\alpha}{2} - L(r)^{-1}\log\log L(r) \geq M - \alpha \geq \max_{\partial B_1} v \geq v(y).$$

So, by comparison, $v(y) \leq \psi_r(y)$ in $(B_r \setminus B_1)^+$. Evaluating at a fixed y, and using that $\phi_L(y) \to 0$ as $L \to \infty$ for fixed y,

$$v(y) \leq \lim_{r\to\infty} \psi_r(y) = M - \frac{\alpha}{2},$$

which contradicts the definition of M. □

Next we use the previous maximum principle with Lemma 3.7 and Harnack inequality to establish quantitative growth bounds on exterior solutions in $d = 2$.

Lemma 3.14. *Suppose that $d = 2$ and v solves (2.6). There is $\eta_0(d) > 0$ sufficiently small so that if $\sup_{\mathbb{R}^d_+\setminus B_1} |\nabla v| \leq \eta_0(d)$ and v is bounded from below, then*

$$0 \leq v(y) - \min_{\partial B_1^+} v \leq C(1 + \log|y|)(\min_{\partial B_2^+} v - \min_{\partial B_1^+} v) \quad \textit{for} \quad |y| \geq 2.$$

Similarly if v is bounded from above then

$$0 \geq v(y) - \max_{\partial B_1^+} v \geq -C(1+\log|y|)(\max_{\partial B_1^+} v - \max_{\partial B_2^+} v) \quad \text{for} \quad |y| \geq 2.$$

Proof. We just do the bounded from below case. The inequality $v(x) \geq \min_{\partial B_1^+} v$ is the content of Lemma 3.12.

Now $w(y) = v(y) - \min_{\partial B_1^+} v$ is a non-negative solution of (2.6). We can apply Lemma 3.7 to find, for all $r \geq 2$,

$$\min_{\partial B_r^+} w \leq C_\delta r^\delta (\min_{\partial B_2^+} v - \min_{\partial B_1^+} v).$$

Since w is non-negative Harnack inequality, Theorem 2.6, implies that, for all $r \geq 2$,

$$\max_{\partial B_r^+} w \leq C \min_{\partial B_r^+} w.$$

Combining with the previous inequality completes the proof. □

3.5. **Quantitative linearization of the hodograph PDE.** Next we combine the growth (Lemma 3.14 in $d=2$) or decay (Lemma 3.11 in $d \geq 3$) estimates with elliptic regularity to establish decay estimates on first and second order derivatives. Then, plugging these derivative estimates into the hodograph PDE, we obtain that v (almost) solves the Laplace equation with Neumann boundary conditions up to an even more quickly decaying error.

Lemma 3.15. *Suppose $d \geq 2$. For any $\delta > 0$ there is $\eta_0(\delta, d) > 0$ small so that: if v solves (2.6) is bounded either from below or from above and $\sup_{\mathbb{R}^d_+ \setminus B_1} |\nabla v| \leq \eta_0$ then for $k = 1, 2$*

$$|y|^{d-2+k-\delta}|\nabla^k v(y)| \leq C(\delta, k) \operatorname*{osc}_{(B_2 \setminus B_1)^+} v \quad \text{in} \quad \mathbb{R}^d \setminus B_2. \tag{3.4}$$

In particular, by applying these derivative estimates in (2.6), it follows that

$$\begin{cases} |\Delta v(y)| \leq C(\delta)(\operatorname*{osc}_{(B_2 \setminus B_1)^+} v)^2 |y|^{1-2d+\delta} & \text{in} \ \mathbb{R}^d_+ \setminus B_2 \\ 0 \leq \partial_d v(y) \leq C(\delta)(\operatorname*{osc}_{(B_2 \setminus B_1)^+} v)^2 |y|^{2-2d+\delta} & \text{on} \ \partial\mathbb{R}^d_+ \setminus B_2. \end{cases}$$

For the purposes of Theorem 3.2 it will suffice to use this result with $\delta = \frac{1}{2}$, but we keep $\delta > 0$ as a parameter here because we believe it clarifies the important growth/decay rates in the present statement.

Proof. First we apply the elliptic regularity estimates of Theorem 2.7 to estimate the derivatives. For $y \in \mathbb{R}^d_+ \setminus B_2$ we have v solving

$$\begin{cases} \operatorname{tr}(A(\nabla_y v) D_y^2 v) = 0 & \text{in} \quad B_{|y|/2}(y)^+ \\ \partial_{y_d} v = \sqrt{1 + |\nabla_y' v|^2} - 1 & \text{on} \ \partial\{y_d > 0\} \cap B_{|y|/2}(y)^+ \end{cases}$$

so Theorem 2.7 yields

$$|y|^k |\nabla^k v(y)| \leq C(\delta, k) \operatorname*{osc}_{B_{|y|/2}(y)^+} v.$$

In $d \geq 3$ Lemma 3.11 gives

$$\operatorname*{osc}_{B_{|y|/2}(y)^+} v \leq C|y|^{2+\delta-d} \operatorname*{osc}_{(B_2 \setminus B_1)^+} v,$$

while in $d = 2$ Lemma 3.14 gives

$$\operatorname*{osc}_{B_{|y|/2}(y)^+} v \leq C(1+\log|y|) \operatorname*{osc}_{(B_2\setminus B_1)^+} v.$$

Plugging these into the right hand side of the elliptic estimate gives the result.

Next we plug in the elliptic estimates into the hodograph PDE (2.6) and put the error terms on the right hand side for estimates on the Laplacian. More specifically

$$\Delta v = \operatorname{tr}((I - A(\nabla v))D^2 v) \ \text{ in } \ \mathbb{R}^d_+ \setminus B_1.$$

By (2.7) and direct estimation

$$|A(p) - \mathrm{I}| \leq C(d)|p| \ \text{ for } \ |p| \leq 1$$

and so

$$|\Delta v| \leq C|\nabla v||D^2 v| \ \text{ in } \ \mathbb{R}^d_+ \setminus B_1.$$

So applying (3.4) with $k = 1$ and $k = 2$ and $\delta/2$ we find

$$|\Delta v| \leq C(\operatorname*{osc}_{(B_2\setminus B_1)^+} v)^2|y|^{1+\delta/2-d}|y|^{\delta/2-d} \leq C(\operatorname*{osc}_{(B_2\setminus B_1)^+} v)^2|y|^{1-2d+\delta} \ \text{ in } \ \mathbb{R}^d_+ \setminus B_2.$$

The argument for the Neumann condition is similar. We use (2.9) and (3.4) with $k = 1$ and $\delta/2$ to find

$$\partial_{y_d} v \leq C(\operatorname*{osc}_{(B_2\setminus B_1)^+} v)^2(|y|^{1+\delta/2-d})^2 = C(\operatorname*{osc}_{(B_2\setminus B_1)^+} v)^2|y|^{2+\delta-2d}.$$

□

3.6. **Precise asymptotics via Kelvin Transform.** We can now establish the asymptotic expansion at ∞ and, in particular, the existence of the capacity. The idea is to perform a Kelvin transform and then use estimate the difference with the harmonic replacement. The proofs model the case of harmonic functions in exterior domains, using the quantitative linearization result Lemma 3.15 to show that the errors are sufficiently small.

Proof of Theorem 3.2. For this proof denote $m := \operatorname{osc}_{(B_2\setminus B_1)^+} v < 1$, we are shortening the notation since this number will appear repeatedly in the estimates below.

(Case $d \geq 3$.) By Lemma 3.8 $s := \lim_{|y|\to\infty} v(y)$ exists, assume without loss that $s = 0$. Take the Kelvin transform of v

$$\tilde{v}(z) = |z|^{2-d} v(\tfrac{z}{|z|^2}) \ \text{ on } \ z \in (B_1^+ \cup B_1') \setminus \{0\}.$$

Let us make a note that, since $v(y) \to 0$ as $|y| \to \infty$, then

$$|z|^{d-2}\tilde{v}(z) \to 0 \ \text{ as } \ z \to 0. \tag{3.5}$$

Which we will use to show that $\tilde{v}$ has a removable singularity at $z = 0$.

We compute

$$\nabla \tilde{v}(z) = (2-d)\frac{z}{|z|^d} v(\tfrac{z}{|z|^2}) + \frac{1}{|z|^d}(I - 2\frac{z\otimes z}{|z|^2})\nabla v(\tfrac{z}{|z|^2}).$$

Evaluating on $z_d = 0$, and using that $e_d \frac{z\otimes z}{|z|^2} = 0$ for such z,

$$\partial_d \tilde{v}(z) = e_d \cdot \nabla \tilde{v}(z) = \frac{1}{|z|^d}\partial_d v(\tfrac{z}{|z|^2}) \ \text{ on } \ z \in B_1' \setminus \{0\}.$$

Combining this with the estimates in Lemma 3.15 we find the following estimate on $z \in B'_{1/2} \setminus \{0\}$

$$|\partial_d \tilde{v}(z)| = |z|^{-d}|\partial_d v(\tfrac{z}{|z|^2})| \leq Cm^2|z|^{-d}|z|^{2d-2-\delta} = Cm^2|z|^{d-2-\delta}. \tag{3.6}$$

Also recall the following formula for the Laplacian under the Kelvin transform

$$\Delta \tilde{v}(z) = |z|^{-d-2}\Delta v(\tfrac{z}{|z|^2}).$$

Combining this formula with Lemma 3.15 we have the following PDE in $B^+_{1/2}$

$$|\Delta \tilde{v}(z)| = |z|^{-d-2}|\Delta v(\tfrac{z}{|z|^2})| \leq Cm^2|z|^{-d-2}||z|^{-2}z|^{1-2d+\delta} = Cm^2|z|^{d-3-\delta}. \tag{3.7}$$

We summarize the two previous estimates, (3.6) and (3.7), in the following PDE

$$\begin{cases} \Delta \tilde{v}(z) = \tilde{f}(z) & \text{in } B^+_{1/2} \\ \partial_d \tilde{v}(z) = \tilde{g}(z) & \text{on } B'_{1/2} \setminus \{0\} \end{cases} \tag{3.8}$$

where

$$|\tilde{f}(z)| \leq Cm^2|z|^{d-3-\delta} \text{ on } B^+_{1/2} \text{ and } |\tilde{g}(z)| \leq Cm^2|z|^{d-2-\delta} \text{ on } B'_{1/2} \setminus \{0\}. \tag{3.9}$$

Now we need to argue that $\tilde{v}$ has a removable singularity at $z = 0$. We subtract off the Neumann kernel of the right hand side of (3.8)

$$\psi(z) = \int_{B^+_{1/2}} \Phi(z-w)\tilde{f}(w)\ dw + \int_{B'_{1/2}} \Phi(z-w)\tilde{g}(w)dS(w)$$

where

$$\Phi(z) = \frac{1}{d(d-2)|B_1|}(|z|^{2-d} + |(z', -z_d)|^{2-d})$$

is the standard Neumann kernel for $\mathbb{R}^d_+$. The bounds (3.9) imply that

$$|\psi(z) - \psi(0)| \leq Cm^2|z| \ \text{ in } \ B^+_{1/2}. \tag{3.10}$$

Next let $\bar{v}$ be the solution of

$$\begin{cases} \Delta \bar{v} = 0 & \text{in } B^+_{1/2} \\ \partial_d \bar{v} = 0 & \text{on } \partial\mathbb{R}^d_+ \cap B_{1/2} \\ \bar{v}(z) = \tilde{v}(z) - \psi(z) & \text{on } (\partial B_{1/2})^+. \end{cases}$$

We claim that

$$\tilde{v}(z) \equiv \bar{v}(z) + \psi(z) \ \text{ in } \ B^+_{1/2}. \tag{3.11}$$

The reasoning is that (with even reflection)

$$w(z) = \tilde{v}(z) - (\bar{v}(z) + \psi(z))$$

is harmonic in $B_{1/2} \setminus \{0\}$, zero on the boundary $\partial B_{1/2}$, and, by (3.5) and (3.10), grows more slowly than $|z|^{2-d}$ at the origin so it is zero.

By even reflection and interior Lipschitz estimates of harmonic functions for $z \in B^+_{1/4}$

$$|\bar{v}(z) - \bar{v}(0)| \leq C|z| \operatorname*{osc}_{(\partial B_{1/2})^+}(\tilde{v} - \psi) \leq Cm|z|$$

using that, by (3.10) and $m \leq 1$, $\operatorname{osc}_{(\partial B_{1/2})^+} \psi \leq Cm^2 \leq Cm$, and that

$$\operatorname*{osc}_{(\partial B_{1/2})^+} \tilde{v} = 2^{2-d} \operatorname*{osc}_{(\partial B_2)^+} v \leq 2^{2-d}m.$$

So now we conclude that

$$|\tilde{v}(z) - \tilde{v}(0)| \leq |\psi(z) - \psi(0)| + |\bar{v}(z) - \bar{v}(0)| \leq Cm|z| \text{ in } B^+_{1/4}.$$

Calling $k := \tilde{v}(0)$ we also find

$$|k| = |\tilde{v}(0)| = |\bar{v}(0) + \psi(0)| = |\fint_{\partial B^+_{1/2}} (\tilde{v} - \psi)(z) dS(z) + \psi(0)| \leq Cm.$$

Undoing the Kelvin transform we arrive at

$$\begin{aligned} |v(y) - k|y|^{2-d}| &= ||y|^{2-d}\tilde{v}(|y|^{-2}y) - \tilde{v}(0)|y|^{2-d}| \\ &= |y|^{2-d}|\tilde{v}(|y|^{-2}y) - \tilde{v}(0)| \\ &\leq |y|^{2-d}Cm|y|^{-1} = Cm|y|^{1-d}. \end{aligned}$$

(Case $d = 2$) In the case $d = 2$ we argue similarly, with inversion, and use a typical complex analysis trick. Again this is following a classical argument for harmonic functions in exterior domains, with an additional error term controlled via Lemma 3.15. We will now use $z = x + iy$ for the complex variable.

Define

$$\tilde{v}(z) := v(\tfrac{1}{z}) \text{ for } z \in B_1 \setminus \{0\}.$$

Define, as before,

$$\psi(z) = \int_{B^+_{1/2}} \Phi(z - w)\tilde{f}(w)\, dw + \int_{B'_{1/2}} \Phi(z - w)\tilde{g}(w) dS(w)$$

where

$$\Phi(z) = \frac{-1}{4\pi}(\log|z| + \log|(z', -z_d)|)$$

and $\bar{v}$ solving

$$\begin{cases} \Delta\bar{v} = 0 & \text{in } B^+_{1/2} \\ \partial_d\bar{v} = 0 & \text{on } \partial\mathbb{R}^d_+ \cap B_{1/2} \\ \bar{v}(z) = \tilde{v}(z) - \psi(z) & \text{on } (\partial B_{1/2})^+. \end{cases}$$

Then call

$$w(z) = \tilde{v}(z) - (\bar{v}(z) + \psi(z)).$$

The even reflection of w, not relabeled, is harmonic in the punctured disk $B_1 \setminus \{0\}$. Thus we can write, for some $k \in \mathbb{R}$,

$$w(z) = k\log 2|z| + \mathrm{Re}(h(z))$$

where h is holomorphic in the punctured disk. Since w is $o(|z|^{-1})$ at the origin and so is $\log|z|$, then so is $|\mathrm{Re}(h(z))|$. Thus 0 is a removable singularity and h is holomorphic in the entire B_1. Since $\mathrm{Re}(h(z)) = w(z) - k\log(2 \cdot \frac{1}{2}) = 0$ on $\partial B_{1/2}$ then $h \equiv 0$. Thus

$$\tilde{v}(z) = \bar{v}(z) + \psi(z) + k\log 2 + k\log|z|$$

and so

$$\tilde{v}(z) = k\log|z| + \bar{v}(z) + \psi(z)$$

and now we conclude since $\bar{v}$ and ψ are continuous at $z = 0$. □

4. General solutions with a single-site defect

In this section we consider solutions of the single-site defect problem in the entire space, namely

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \\ |\nabla u| = 1 + q(x) & \text{on } \partial\{u > 0\}, \end{cases} \tag{4.1}$$

where the defect $q(x) : \mathbb{R}^d \to (-1, \infty)$ is smooth and supported in $\overline{B_1(0)}$. For applications in the rest of the paper, we focus primarily on "proper" solutions.

Our main result in this section regards the far-field asymptotic expansion of proper solutions to (4.1). In later sections we will need to establish that the single-site solutions of interest are indeed proper.

Theorem 4.1. *Let u be a proper solution of (4.1).*

(i) In the case $d = 2$. There is $k(u) \in \mathbb{R}$ so that:

$$\sup_{x \in \overline{\{u>0\}}} |u(x) - (x_d + k \log |x|)| < +\infty$$

and there are universal constants $\sigma_0 \in (0, 1/2]$ and $C \geq 1$ so that

$$\text{if } |q| \leq \sigma_0 \text{ then } \ C \min q \leq k(u) \leq C \max q.$$

Furthermore, for general $q \in C_c(B_1; (-1, \infty))$, there is $C(\min q)$ universal so that $k(u) \geq -C$.

(ii) In the case $d \geq 3$. There are $s(u), k(u) \in \mathbb{R}$ so that:

$$|u(x) - (x_d + s - k|x|^{2-d})| \leq C_0(1 + |x|)^{1-d} \ \text{ for } \ x \in \{u > 0\}$$

and if $k \neq 0$ then

$$|s|, |k| \leq C_0 = C_0(d, \min q, \max q).$$

Furthermore, if $\max |q| \leq \sigma_0(d)$, then $C_0 \leq C(d) \max |q|$.

Note that we lack universal control on the higher order error terms in the asymptotic expansion in $d = 2$, it remains open whether it can be achieved.

We present the proof of this theorem below in Section 4.1. First, since u is proper, at some large (non-quantitative) scale it is sufficiently flat to employ Theorem 3.1. Still, we need quantified information on k, s (and R for $d \geq 3$) for applications for the rest of the paper. This is a significant new challenge, for which we need barriers with the correct asymptotic expansion. Although the barrier constructions are quite concrete, they play an essential role in the theory later.

Let $\sigma \in (-1, +\infty)$. Call $B_1' = \{x' \in \mathbb{R}^{d-1} : |x'| < 1\}$ and $B_1' \times \mathbb{R}$ is the cylinder above B_1' with axis in the e_d direction. The goal is to construct subsolutions, in case $\sigma > 0$, and supersolutions, in case $\sigma < 0$, of the problem

$$\begin{cases} \Delta \phi = 0 & \text{in } \{\phi > 0\} \\ |\nabla \phi| = 1 + \sigma \mathbf{1}_{B_1' \times \mathbb{R}} & \text{on } \partial\{\phi > 0\}, \end{cases} \tag{4.2}$$

with the asymptotic bounds, for some $C_1(\sigma, d) > 0$, in $d \geq 3$

$$0 \leq \operatorname{sgn}(\sigma)(x_d - \phi(x)) \leq C_1(\sigma, d) \min\{1, |x|^{2-d}\} \ \text{ in } \ \{\phi > 0\}. \tag{4.3}$$

and in $d = 2$,

$$\operatorname{sgn}(\sigma)(x_d - \phi(x)) \leq -C_1(\sigma) \max\{1, \log |x|\} \ \text{ in } \ \{\phi > 0\}. \tag{4.4}$$

Note that (4.2) is invariant with respect translations in the x_d variable, this allows the barriers to be used in sliding comparison arguments.

We will construct three different types of barriers in this section. In the nearly linearized regime when $\max q$ and/or $\min q$ are small we can construct barriers via patching a linear function, in B_1, with a slightly tilted fundamental solution outside of B_1. This is a little bit more difficult in dimension $d = 2$ where we were unable to successfully construct the barrier directly in original coordinates, and instead use hodograph coordinates again. In this almost linear regime the sub and supersolution constructions are basically symmetrical. See Section 4.2.1 and Section 4.2.2 for these constructions. In the nonlinear regime, when $\min q$ and/or $\max q$ are large, we have two quite distinct barrier constructions for sub and supersolutions. This asymmetry reflects an important asymmetry between advancing and receding regimes in the truly nonlinear problem. The supersolution constructions works in all $d \geq 2$, but the subsolution construction only works in $d \geq 3$. We are very interested whether analogous subsolutions exist in the nonlinear regime in dimension $d = 2$. See Section 4.2.3 and Section 4.2.4 for the sub and supersolution constructions respectively.

We state the results of the barrier constructions, first in the nearly linear regime and then in the nonlinear regime.

Proposition 4.2. *(i) There is a universal constant $0 < \sigma_0 < 1$ such that, for $|\sigma| \leq \sigma_0$ and $\sigma > 0$ (resp. $\sigma < 0$) there is a smooth subsolution (resp. supersolution) ϕ_σ of (4.2) satisfying (4.3) (or (4.4)) with $C_1(d, \sigma) = C(d)|\sigma|$.*

(ii) In all $d \geq 2$ and for any $-\infty < \sigma < 0$ there is a smooth supersolution of (4.2) ϕ_σ satisfying (4.3) (or (4.4)).

(iii) In $d \geq 3$ and for any $0 < \sigma < \infty$ there is a smooth subsolution of (4.2) ϕ_σ satisfying (4.3).

4.1. **Asymptotics of general solutions.** Before proceeding to the barrier constructions in Section 4.2 we show how to derive Theorem 4.1 using the barriers from Proposition 4.2.

Proof of Theorem 4.1. Note that if $\partial\{u > 0\} \cap \overline{B_1(0)} = \emptyset$ then u globally solves the homogeneous Bernoulli problem (2.2) and is proper and therefore $u(x) = (x_d + s)_+$ for some s. Thus for the remainder of the proof we can assume that

$$\partial\{u > 0\} \cap \overline{B_1(0)} \neq \emptyset. \tag{4.5}$$

Step 1. By hypothesis, for all $\varepsilon > 0$ there is $R(\varepsilon)$ sufficiently large so that for all $r \geq R(\varepsilon)$

$$(x_d - \varepsilon)_+ \leq r^{-1}u(rx) \leq (x_d + \varepsilon)_+ \quad \text{in} \quad B_1(0).$$

Let $\eta_1(d) > 0$ be from Theorem 3.1. Then Lemma 2.4 implies that there is $R_1 > 1$ sufficiently large so that $u \in C^2(\{u > 0\} \setminus B_{R_1}(0))$ and

$$|\nabla u(x) - e_d| \leq \eta_1 \quad \text{in} \quad \{u > 0\} \setminus B_{R_1}(0).$$

Theorem 3.1 in turn implies that, in $d \geq 3$,

$$u(x) = x_d + s - k|x|^{2-d} + E(x) \quad \text{in} \quad \{u > 0\} \setminus B_{R_1} \quad \text{with} \quad |E(x)| \leq C(1 + |x|)^{1-d} \tag{4.6}$$

while in $d = 2$

$$u(x) = x_d + k \log|x| + E(x) \quad \text{in} \quad \{u > 0\} \setminus B_{R_1} \quad \text{with} \quad \sup|E(x)| < +\infty. \tag{4.7}$$

Note that so far s, k, and the error term $E(x)$ depend on R_1, and R_1 depends on the solution u in a non-universal way.

Step 2. Next we use s in a sliding barrier argument to control R_1 quantitatively. We present separate arguments for $d \geq 3$ and $d = 2$.

(Case: $d \geq 3$). Let $\phi = \phi_\sigma$ be the outer regular supersolution barrier from Proposition 4.2 with $\sigma = \min q \in (-1, 0)$. By (4.3) we have

$$(x_d)_+ \leq \phi(x) \leq (x_d + C_1 \min\{1, |x|^{2-d}\})_+.$$

Now we apply a sliding argument with the family $\phi(x+te_d)$. Note that $\phi(x+te_d) \geq (x_d+t)_+ > u(x)$ in $\overline{\{u > 0\}}$ for sufficiently large $t > 0$. Also for every $t > s$, by (4.6) we have $\phi(x+te_d) \geq (x_d+t)_+ > u(x)$ in $\overline{\{u > 0\}}$ for $x \in \mathbb{R}^d \setminus B_{R(t)}$. So by Lemma 2.1 $u(x) \leq \phi(x+te_d)$ for all $t \geq s$ and so we conclude

$$u(x) \leq (x_d + s + C_1 \min\{1, |x + se_d|^{2-d}\})_+. \tag{4.8}$$

We can also bound s, using (4.5). So letting $x^0 \in \partial\{u > 0\} \cap \overline{B_1}$ be a point realizing (4.5), then

$$x^0_d + s + C_1 \min\{1, |x^0 + se_d|^{2-d}\} \geq 0$$

and therefore

$$s \geq -C_1 - 1.$$

Similar arguments using the subsolutions ϕ_σ with $\sigma = \max q > 0$ from Proposition 4.2 yields

$$u(x) \geq (x_d + s - C_2 \min\{1, |x + se_d|^{2-d}\})_+ \ \text{ and } \ s \leq C_2 + 1 \tag{4.9}$$

with a C_2 depending only on d and on $\max q$. The inequalities (4.9) and (4.8) also imply that

$$-C_2 \leq k \leq C_1.$$

Thus, by using the quantitative flatness (4.8) and(4.9) in Theorem 2.3, there is a radius $R_0 = R_0(C_1, C_2, \eta_1) = R_0(d, \min q, \max q)$ so that

$$|\nabla u - e_d| \leq \eta_1 \ \text{ on } \ \{u > 0\} \setminus B_{R_0}(0). \tag{4.10}$$

As before, but now with quantified R_0, Theorem 3.1 implies that, for $x \in \{u > 0\} \setminus B_{R_0}$,

$$\begin{aligned} |x|^{d-1}|u(x) - (x_d + s + k|x|^{2-d})| &\leq C \operatorname*{osc}_{(B_{2R_0} \setminus B_{R_0}) \cap \{u>0\}} (u(x) - x_d) \\ &\leq C(d, \min q, \max q) \end{aligned}$$

where the last inequality again uses (4.10).

(Case: $d = 2$). Here we assume that $\max |q| \leq \sigma_0$ to use the barriers given in Proposition 4.2. Let ϕ_σ be as given in Proposition 4.2 with $\sigma = \min q \in (-1, 0]$ satisfying (4.2) and (4.3). We claim that $k \geq -C_1$. Suppose otherwise, $k < -C_1$. Since u is proper it is one-sided flat, since $k < -C_1$ in (4.6) it must be flat from above, $u(x) \leq (x_d + T)_+$ for some $T \in \mathbb{R}$. We perform a sliding comparison with $\phi(x+te_d)$. For $t > T$ then $\phi(x+te_d) > u(x)$ in $\overline{\{u > 0\}}$. Let $k < k' < -C_1$. For any $t \in \mathbb{R}$ there is R sufficiently large so that, for $x \in \{u > 0\} \setminus B_R$,

$$\phi(x+te_d) \geq (x_d + t - C_1 \max\{1, \log|x + te_d|\})_+ > (x_d + k' \log|x|)_+ > u(x).$$

Then Lemma 2.1 implies $\phi(x+te_d) \geq u(x)$ for all t, yielding a contradiction.

Arguing similarly with the subsolution barriers from Proposition 4.2 shows that $k \leq C_1$.

□

4.2. **Barriers.** In this section we construct the barriers presented in Proposition 4.2. The construction of the barriers are given in the order of increasing difficulty. We begin with the simplest construction, which is for small σ, i.e. small $\|q\|_{L^\infty}$, in $d = 3$.

For small σ, our barrier construction is relatively simple, by patching of inner and outer parts that are $O(\sigma)$-perturbations of the planar profile, based on the Laplace fundamental solution and its derivative. The barrier construction in $d = 2$ is in hodograph coordinates, since we can only construct the two dimensional fundamental solution type barrier in that coordinate system. This approach only works when σ is sufficiently small, below a universal threshold. This is, perhaps, natural since the construction views subsolutions and supersolutions more or less symmetrically.

Both the nonlinearity and the asymmetry between advancing and receding become more severe for large σ. Our perturbations are more nonlinear in this case, and involve planting a sizable sink and source term, respectively, to pull or push the planar profile. Furthermore the construction of subsolution and supersolution barriers are no longer symmetrical and have a slightly different geometry.

4.2.1. *Barriers for small defects: $d \geq 3$.* Let $C > 1$ to be chosen sufficiently large depending on dimension and $|\sigma| \leq \sigma_0$, $\sigma > 0$ in the subsolution case and $\sigma < 0$ in the supersolution case. We define the inner solution

$$\phi_{in}(x) := (1 + \sigma)x_d - C\sigma$$

and the outer solution

$$\phi_{out}(x) := x_d - C\sigma|x|^{2-d} + \sigma\frac{x_d}{|x|^d}.$$

Note that $x_d/|x|^d$ is a constant multiple of $\partial_{x_d}\Phi(x)$, and thus is harmonic away from the origin. Therefore ϕ_{out} is harmonic away from the origin and ϕ_{in}, being linear, is harmonic everywhere. Define the patching

$$\phi(x) := \begin{cases} \phi_{in}(x) & |x| < 1 \\ \phi_{out}(x) & |x| \geq 1. \end{cases}$$

Note that on $|x| = 1$

$$\phi_{out}(x) = x_d - C\sigma + \sigma x_d = \phi_{in}(x).$$

Thus ϕ is continuous across the patch on ∂B_1. The proposed subsolution / supersolution of the Bernoulli problem will be $\phi(x)_+$.

Lemma 4.3. *For $|\sigma| \leq \sigma_0(d)$ the function $\phi(x)_+$ is a subsolution of (4.2), in the case $\sigma > 0$, and is a supersolution of (4.2), in the case $\sigma < 0$.*

Proof. We claim that, in the subsolution case $\sigma > 0$, $\phi(x) = \max\{\phi_{in}(x), \phi_{out}(x)\}$ in a neighborhood of ∂B_1. It suffices to show that

$$x \cdot \nabla\phi_{in}(x) < x \cdot \nabla\phi_{out}(x) \ \text{ on } \ x \in \partial B_1.$$

We will check this by direct computation. First we record

$$\nabla\phi_{out} = e_d + \sigma\left[C(d-2)\frac{x}{|x|^d} + \frac{e_d}{|x|^d} - d\frac{x_d x}{|x|^{d+2}}\right]. \tag{4.11}$$

So on $x \in \partial B_1$

$$x \cdot \nabla \phi_{out}(x) = x_d + \sigma[C(d-2) + (1-d)x_d] \quad \text{and} \quad x \cdot \nabla \phi_{in}(x) = (1+\sigma)x_d.$$

So fixing $C = \frac{d}{d-2} + 1$ then, on $x \in \partial B_1$ using that $1 \geq x_d$ on that set,

$$x \cdot \nabla \phi_{out}(x) > x_d + \sigma[d + (1-d)x_d] \geq x_d + \sigma[dx_d + (1-d)x_d] = (1+\sigma)x_d = x \cdot \nabla \phi_{in}(x).$$

By a symmetrical argument, in the supersolution case $\sigma < 0$, $\phi(x) = \min\{\phi_{in}(x), \phi_{out}(x)\}$ in a neighborhood of ∂B_1.

Next we check the free boundary condition. The free boundary condition inside B_1 is immediate since the solution is linear, so we only need to check for the outer solution. Then, using (4.11) and expanding the quadratic,

$$|\nabla \phi_{out}|^2 = 1 + 2\sigma \left[\frac{1}{|x|^d} + C(d-2)\frac{x_d}{|x|^d} - d\frac{x_d^2}{|x|^{d+2}} \right] + O\left(\frac{\sigma^2}{|x|^{2(d-1)}} \right)$$

Let us use that

$$x_d = C\sigma |x|^{2-d} \frac{1}{1 - \sigma |x|^{-d}} \quad \text{on} \quad \partial\{\phi > 0\} \setminus B_1.$$

So then, for $\sigma \leq 1$ and $|x| \geq 1$,

$$|\nabla \phi_{out}|^2 = 1 + 2\sigma \frac{1}{|x|^d} + O\left(\frac{\sigma^2}{|x|^{2(d-1)}} \right).$$

Note that $2(d-1) > d$ in dimensions $d \geq 3$. Then $|\nabla \phi_{out}|^2 > 1$ for $|x| \geq 1$ as long as we choose $0 < \sigma \leq \sigma_0(d)$ with sufficiently small σ_0 depending on dimension. The supersolution case $-\sigma_0 \leq \sigma < 0$ is symmetrical.

Finally note that

$$(x_d - (C+1)\sigma)_+ \leq \phi(x)_+ \leq (x_d + (C+1)\sigma)_+.$$

□

4.2.2. *Barriers for small defects: $d = 2$.* Here we will utilize the exterior barrier from Lemma 3.4 in hodograph coordinates, to construct barriers. While our construction is similar to the patched barrier in higher dimensions from Section 4.2.1, we face additional technical challenges here since we need to smooth out the solution at the patching in order to invert the hodograph transform. We go around this with standard mollifier and keep the computations to minimum.

Let us first point out the issue in original coordinates via a Lemma, which we will use later for other purposes.

Lemma 4.4. *The function*

$$\phi(x) := (x_d + \sigma \log |x| + s)_+$$

is a supersolution of (2.2) in $\mathbb{R}^d \setminus B_3$ as long as $\sigma s \geq 0$.

Proof. Note that ϕ is harmonic in its positivity set away from 0. The zero set is on

$$0 = \phi(x) = x_d + \sigma \log |x| + s$$

and the slope is

$$|\nabla \phi(x)|^2 = 1 + 2\sigma \frac{x_d}{|x|^2} + \sigma^2 \frac{1}{|x|^2}.$$

Evaluating the slope on the zero set, by plugging in $x_d = -\sigma \log|x|$,

$$|\nabla\phi(x)|^2 = 1 - 2\frac{1}{|x|^2}\left[\sigma^2 \log|x| + \sigma s - 1\right].$$

So, since $\log 3 > 1$ and $\sigma s \geq 0$, we conclude that $|\nabla\phi(x)|^2 < 1$ on $\partial\{\phi > 0\}\setminus B_3$. $\square$

This is actually fine for the purposes of Theorem 4.1 in the case $\sigma < 0$, but, unfortunately, this is the wrong direction in the case $\sigma > 0$. We have not discovered any elementary way to fix this in the original coordinates.

We will work in the same hodograph coordinate system which was introduced in Section 2.5. We will denote y the variable in hodograph coordinates and x the variable in standard coordinates as we did before. Note that our conventions for the hodograph transform does switch the role of sub and supersolutions.

Let $|\sigma| \leq \sigma_0$ a sufficiently small constant to be specified via the computation. We will define an inner solution, to be used inside $B_2(0)^+$, and an outer solution, to be used outside $B_2(0)^+$. In the case $\sigma > 0$ we will construct a supersolution, and in the case $\sigma < 0$ we will construct a subsolution of the hodograph equation

$$\begin{cases} \mathrm{tr}(A(\nabla_y v) D_y^2 v) = 0 & \text{in } \{y_d > 0\}, \\ (1 + \sigma \mathbf{1}_{B_{1/2}})(1 + \partial_{y_d} v) = \sqrt{1 + |\nabla_y' v|^2} & \text{on } \partial\{y_d > 0\}. \end{cases} \tag{4.12}$$

This is analogous to (4.2) on the hodograph side. The choice of putting the defect in $B_{1/2}$ is for convenience, so we can patch on ∂B_1 with a little room. We will need to construct smooth, at least C^1, sub/supersolutions in order to invert the hodograph transform. But we begin with a non-smooth construction.

First define, using $|\sigma| < 1$,

$$\varsigma(\sigma) := -\frac{\sigma}{1+\sigma} \quad \text{so that} \quad (1+\sigma)(1+\varsigma) = 1.$$

Define the inner solution

$$\psi_{in}(y) := \varsigma y_d$$

and the outer solution

$$\psi_{out}(y) := 2\varsigma(\log|y| + \log(1 + \log|y|)) + \varsigma\frac{y_d}{|y|^2}.$$

Then define the patched solution

$$\psi(x) := \begin{cases} \psi_{in}(y) & y \in B_1(0) \\ \psi_{out}(y) & y \notin B_1(0). \end{cases} \tag{4.13}$$

Notice that ψ, thus defined, is continuous on $\mathbb{R}^d$.

As in the previous section, we will need to check interior and boundary sub and supersolution conditions (see Lemma 3.4) as well as the correct gradient discontinuity at the patching on ∂B_1. Finally we will need to do an inverse hodograph transform to get a sub/supersolution in the original coordinates. Implementing this idea is technically tricky, since the hodograph and inverse hodograph transform require C^1 regularity. We achieve this regularity by using a standard mollifier. We do provide full details for this procedure, since this barrier construction plays a key role in our main results in dimension $d = 2$.

Lemma 4.5. *Let $|\sigma| \leq \sigma_0 < 1$, $\varsigma := -\sigma/(1+\sigma)$, and $0 < \varepsilon \leq \frac{1}{2}$, and let $\eta_\varepsilon(x) := \varepsilon^{-d}\eta(\varepsilon x)$, where η is the standard radially symmetric mollifier supported in $B_1(0)$. If $\sigma_0 > 0$ is sufficiently small, then for $\varsigma > 0$ (resp. $\varsigma < 0$) $\psi(x)$ given in (4.13) and $\eta_\varepsilon * \psi$ are subsolutions (resp. supersolutions) of (4.12).*

Then, to conclude the proof of the relevant piece of Proposition 4.2 we can just apply an inverse hodograph transform.

Corollary 4.6. *Let $\tilde{\psi}_\sigma := \eta_{1/2} * \psi_\sigma$ and let ϕ_σ be the inverse hodograph transform of $\tilde{\psi}_\sigma$. Then, up to a hyperbolic rescaling, ϕ_σ is a supersolution (case $\sigma < 0$) or subsolution (case $\sigma > 0$) of (4.2) satisfying the bound (4.4) with $C_1(d,\sigma) = C(d)|\sigma|$.*

Now we turn to the proof.

Proof of Lemma 4.5. We will only present the subsolution proof for $\varsigma > 0$, since the other case can be proved symmetrically.

First we verify the subsolution condition for ψ. Note that ψ_{out} is exactly the barrier in Lemma 3.4, where it is shown that ψ_{out} solves (4.12) in the complement of B_1. The inner function ψ_{in} is linear and thus solves the interior PDE in (4.12). Note that, by definition of ς,

$$(1+\sigma)(1+\partial_{y_d}\psi_{in}(y)) = (1+\sigma)(1+\varsigma) = 1.$$

Thus ψ_{in} solves (4.12). Thus ψ is a subsolution if we can show that

$$\psi(x) = \max\{\psi_{in}(x), \psi_{out}(x)\} \text{ in a neighborhood of } \partial B_2.$$

This follows if we can show that

$$[[y \cdot \nabla\psi]]_{\partial B_1} := y \cdot \nabla\psi_{out}(y) - y \cdot \nabla\psi_{in}(y) > 0 \text{ on } y \in \partial B_1. \tag{4.14}$$

To check this, recalling (3.1),

$$y \cdot \nabla\psi_{out} = \varsigma\,[4 - y_d]\,.$$

Since $2 > y_d$ on $\partial B_1(0)$, we can verify (4.14) by

$$y \cdot \nabla\psi_{out} > \varsigma\,[2y_d - y_d] = \varsigma y_d = y \cdot \nabla\psi_{in} \text{ on } \partial B_1(0).$$

We have shown that ψ is a subsolution of (4.12). Next we consider the subsolution properties of the mollified barriers $\psi_\varepsilon := \eta_\varepsilon * \psi$.

First note that ψ is linear in $B_1(0)$ so $\psi_\varepsilon \equiv \psi$ in $B_{1-\varepsilon}(0)$. Thus it remains to check the properties in $\{|y| \geq 1-\varepsilon\}$.

To check the interior PDE, note that the Hessian of ψ has a singular component due to the jump of normal derivative. Below we will show that the singular component will have the correct sign due to the order in (4.14).

By writing

$$D^2\psi_\varepsilon(y) = \int_{\mathbb{R}^d} D^2\eta_\varepsilon(y-z)\psi(z)\,dz$$

and then integrating by parts twice in $B_1(0)$ and $\mathbb{R}^d \setminus B_1(0)$ we obtain

$$D^2\psi_\varepsilon = H^{\mathrm{reg}} + H^{\mathrm{sing}},$$

where, using that ψ_{in} is linear and has vanishing Hessian,

$$\begin{aligned} H^{\mathrm{reg}}(y) &:= \int \eta_\varepsilon(y-z)(D^2\psi_{in}(z)\mathbf{1}_{B_2}(z) + D^2\psi_{out}(z)\mathbf{1}_{\mathbb{R}^d\setminus B_1}(z))\,dz \\ &= \int_{\mathbb{R}^d\setminus B_1} \eta_\varepsilon(y-z)D^2\psi_{out}(z)\,dz \end{aligned}$$

and

$$H^{\rm sing}(y) := \int_{\partial B_1} \eta_\varepsilon(y-z)[[\nu_z\cdot\nabla\psi]]_{\partial B_1}(\nu_z\otimes\nu_z)dz \ \geq 0, \quad \text{with} \quad \nu_z = \frac{z}{|z|}.$$

where the sign follows from (4.14). Since $A(p)\geq 0$ for all $p_d>-1$, we have

$$\operatorname{tr}(A(D\psi_\varepsilon)D^2\psi_\varepsilon) \geq \operatorname{tr}(A(\nabla\psi_\varepsilon)H^{\rm reg}).$$

So it suffices to just analyze the regular part of the Hessian.

We first decompose, as usual,

$$\operatorname{tr}(A(\nabla\psi_\varepsilon)H^{\rm reg}) = \operatorname{tr}(H^{\rm reg}) + \operatorname{tr}((A(\nabla\psi_\varepsilon)-I)H^{\rm reg}). \tag{4.15}$$

The goal is to control the second term by positivity of the first. Noting that all terms in ψ_{out} are harmonic except for $\log(1+\log|y|)$, we obtain

$$\operatorname{tr}(H^{\rm reg})(y) = 2\varsigma\int_{\mathbb{R}^d\setminus B_1}\eta_\varepsilon(y-z)\frac{1}{|z|^2(\log|z|)^2}\ dz \geq c\varsigma\frac{1}{|y|^2(\log|y|)^2}\rho(y) \tag{4.16}$$

where

$$\rho(y) := \int_{\mathbb{R}^d\setminus B_1}\eta_\varepsilon(y-z)\ dz.$$

Note that

$$|H^{\rm reg}(y)| \leq \rho(y)\sup_{B_\varepsilon(y)}|D^2\psi_{out}(z)| \leq C\varsigma\frac{1}{|y|^2}\rho(y) \ \text{ in } \ |y|\geq 1-\varepsilon\geq\frac{1}{2}.$$

Also in $|y|\geq\frac{1}{2}$,

$$\begin{aligned}|\nabla\psi_\varepsilon(y)| &\leq (1-\rho(y))\sup_{B_1(y)}|\nabla\psi_{in}(z)| + \rho(y)\sup_{B_1(y)}|\nabla\psi_{out}(z)|\\ &\leq C\varsigma(1-\rho(y)) + C\varsigma\frac{1}{|y|}\rho(y) \leq C\varsigma\frac{1}{|y|}.\end{aligned}$$

Combining the previous two upper bounds and using (2.8)

$$|\operatorname{tr}((A(\nabla\psi_\varepsilon)-I)H^{\rm reg})| \leq C\varsigma^2\frac{1}{|y|^3}\rho(y). \tag{4.17}$$

Combining the lower bound (4.16) and the above upper bound (4.17) into (4.15),

$$\operatorname{tr}(A(\nabla\psi_\varepsilon)H^{\rm reg}) \geq c\varsigma\rho(y)\left[\frac{1}{|y|^2(\log|y|)^2} - C\varsigma\frac{1}{|y|^3}\right] > 0$$

as long as ς is sufficiently small, and thus we conclude that ψ_ε is a subsolution of the interior PDE.

Lastly to check the Neumann subsolution condition, let $y\in\partial\mathbb{R}^d_+$ with $|y|\geq 1-\varepsilon$. Note that

$$\int_{\mathbb{R}^d\setminus B_2}\eta_\varepsilon(y-z)\partial_{y_d}\psi_{out}(z)\ dz = \varsigma\int_{\mathbb{R}^d\setminus B_2}\eta_\varepsilon(y-z)\frac{1}{|z|^4}(|z|^2-2z_d^2)\ dz.$$

Here we used symmetry considerations to eliminate the other terms in $\partial_{y_d}\psi_{out}$. Then since $|z|\geq 1$ and $|z_d|\leq\varepsilon\leq\varepsilon|z|$ on the domain of integration,

$$\int_{\mathbb{R}^d\setminus B_1}\eta_\varepsilon(y-z)\frac{1}{|z|^4}(|z|^2-2z_d^2)\ dz \geq \int_{\mathbb{R}^d\setminus B_1}\eta_\varepsilon(y-z)\frac{1}{|z|^2}(1-2\varepsilon^2)\ dz \geq c\frac{1}{|y|^2}\rho(y)$$

since we chose $\varepsilon \le \frac{1}{2} < \frac{1}{\sqrt{2}}$. Therefore

$$\begin{aligned}\partial_{y_d}\psi_\varepsilon &= (1-\rho(y))\varsigma + \varsigma\int_{\mathbb{R}^d\setminus B_1}\eta_\varepsilon(y-z)\frac{1}{|z|^4}(|z|^2-2z_d^2)\ dz\\ &\ge \varsigma[(1-\rho(y)) + c\frac{1}{|y|^2}\rho(y)]\\ &\ge c\varsigma\frac{1}{|y|^2}\ \text{ in }\ \partial\mathbb{R}^d_+\setminus B_{1/2}.\end{aligned}$$

On the other hand it is straightforward to check that

$$|\nabla'\psi_\varepsilon| \le C\varsigma\frac{1}{|y|}\ \text{ and so }\ \sqrt{1+|\nabla'\psi_\varepsilon|^2}-1 \le C\varsigma^2\frac{1}{|y|^2}.$$

Thus for ς sufficiently small we have $1+\partial_{y_d}\psi_\varepsilon \ge \sqrt{1+|\nabla'\psi_\varepsilon|^2}$, and we can now conclude.

□

4.2.3. *Barriers for large defects: subsolutions in $d \ge 3$.* To achieve a subsolution, we will perturb the half-plane solution $(x_d)_+$ by putting a large sink term along a line segment

$$L_R := \{x \in \mathbb{R}^d : |x'| = 0, -R \le x_d \le R\}.$$

Namely we consider, for $d \ge 3$,

$$\phi_R(x) := \left(x_d - \int_{L_R}|x-y|^{2-d}d\mathcal{H}^1(y)\right)_+. \tag{4.18}$$

It is straightforward to check that

$$\phi_R(x) = x_d - 2R|x|^{2-d} + O(|x|^{1-d})\ \text{ as }\ \{\phi_R > 0\} \ni x \to \infty.$$

Lemma 4.7. *For any $\sigma > 0$ and $d \ge 3$, there are R, a, and z depending on σ and d such that $\phi(x) := a^{-1}\phi_R(ax + ze_d)$ is a supersolution of (4.2).*

Note that, although we can define a similar barrier in dimension $d = 2$ it does not seem to satisfy the subsolution condition at the free boundary. The following proof relies on the fact that the free boundary of ϕ_R is contained in $x_d \ge 0$, this cannot be the case any longer in $d = 2$ since for any solution with nontrivial capacity the free boundary will need to grow logarithmically into $x_d < 0$.

Proof. By construction the positive set does not contain L_R, and so ϕ_R is harmonic in its positive set. Now we need to measure $D\phi_R$ on the free boundary $\Gamma := \partial\{\phi_R > 0\}$ to conclude that this is a subsolution. Also note that, since $\phi_R(x) < x_d$, the closure of the positive set of ϕ_R is contained in $\{x_d > 0\}$, in particular $x_d > 0$ on Γ.

Computing the derivative we find the formula

$$\nabla\phi_R(x) = e_d + (d-2)\int_{-R}^{R}\frac{x-y}{|x-y|^d}dy_d. \tag{4.19}$$

We claim that the e_d component of the integral above is positive on Γ. This is clear if $x_d \ge R$, while if $0 < x_d < R$ we use symmetry to cancel parts of the integral, as in Figure 1, and thus we end up with

$$\int_{-R}^{R}\frac{x_d-y_d}{|x-y|^d}dy_d = \int_{-R}^{2x_d-R}\frac{x_d-y_d}{|x-y|^d}dy_d \ge 0.$$

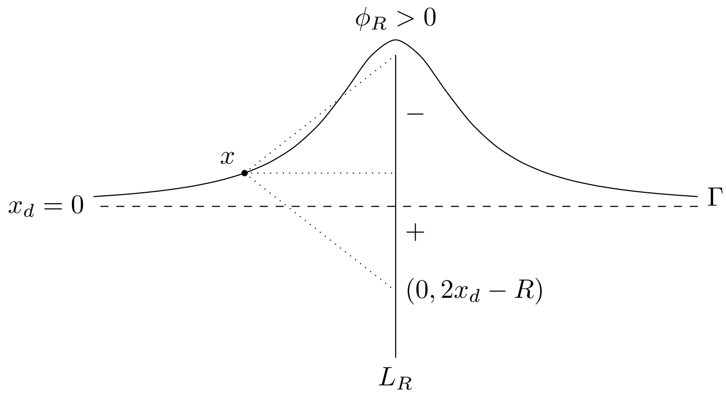

FIGURE 1. Diagram showing the free boundary Γ of the barrier ϕ_R, the line L_R where the sources are placed, and the geometry of the integral cancellation which leads to the subsolution property.

It follows that $|\nabla \phi_R| \geq |\partial_d \phi_R| \geq 1$ on Γ.

We explain the remainder of the proof heuristically, and then proceed with the rigorous details. We will look at the point $z_0 = ze_d \in \Gamma \cap \{|x'| = 0\}$, which is the highest point on the free boundary and where we expect that the slope is most steep. We will use the implicit formula for the location of z to show that $z \to Re_d$ as $R \to \infty$. Using the integral formula for $\nabla \phi_R(z_0)$ this will show that $|\nabla \phi_R(z_0)| \to +\infty$ as $R \to +\infty$. Then we can use continuity of the gradient and hyperbolic rescaling to achieve a barrier with large gradient in a unit neighborhood of the origin.

Now we make this precise. Let $z_0 := ze_d \in \Gamma$. There is a unique such point since the previous arguments show that $z \mapsto \phi_R(ze_d)$ has derivative at least 1 on $R < z < +\infty$ and approaches $-\infty$ as $z \searrow R$. Writing out the equation $\phi_R(z_0) = 0$

$$z = \int_{-R}^{R} (z - y_d)^{2-d} dy_d.$$

By computing the integral in the previous equation we arrive at the implicit formula

$$z = \begin{cases} \log(1 + \frac{2R}{z-R}) & d = 3 \\ (z-R)^{3-d} - (z+R)^{3-d} & d > 3. \end{cases} \tag{4.20}$$

We claim that $z \searrow R$ as $R \to \infty$. From (4.20) it follows that

$$(z - R) \lesssim \begin{cases} Re^{-R} & d = 3 \\ R^{\frac{1}{d}-3} & d > 3. \end{cases}$$

Now note the formula, following from (4.19)

$$\partial_d \phi_R(z_0) = e_d + c(d-2) \int_{-R}^{R} \frac{1}{|z_d - y_d|^{d-1}} \, dy_d. \tag{4.21}$$

Then it follows from (4.21) and $z \searrow R$ that $|\nabla \phi_R(z_0)| \to +\infty$ since $f(a) = |a|^{1-d}$ is not integrable near zero for $d \geq 2$.

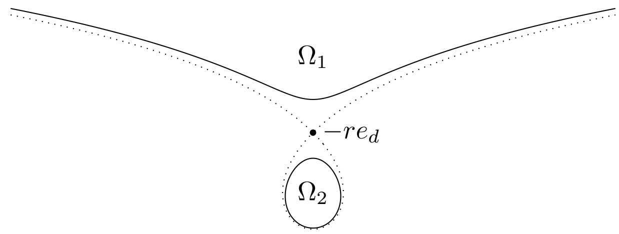

FIGURE 2. Supersolution barrier construction. Zero level set of $\psi_r(x)$ for an $r > \Phi(1)$ plotted with solid line, the level set passing through $-re_d$ is also plotted with a dotted line.

Lastly, by choosing R sufficiently large and performing hyperbolic rescaling by

$$\phi(x) := a^{-1}\phi_R(ax + ze_d)$$

we can generate a subsolution with slope exceeding any particular desired threshold on $\Gamma \cap B_1(0)$.

□

4.2.4. *Barriers for large defects: supersolutions.* To create a supersolution barrier we instead pull the half-plane solution outward via a positive source term in the complement of the positivity set. Unlike the subsolution case before, this construction does work in all dimensions $d \geq 2$. Let $r > 0$ be a free parameter and define

$$\psi_r^0(x) := x_d + \Phi(x + (r+1)e_d)$$

where Φ is the normalized Laplace fundamental solution in dimension d

$$\Phi(x) := \begin{cases} -\log|x| & d = 2 \\ \frac{1}{d-2}\frac{1}{|x|^{d-2}} & d \geq 3. \end{cases} \quad \text{with} \quad \nabla\Phi(x) = -\frac{x}{|x|^d}.$$

Notice that

$$\partial_{x_d}\psi_r^0(-re_d) = 0$$

The heuristic idea is to choose r so that $-re_d$ is (almost) on the free boundary and so the gradient will be very small near $-re_d$, then we can perform a hyperbolic rescaling centered at $-re_d$ to get small gradient in B_1. Note that in that case the zero level set of ψ_r has a saddle-type singularity at $-re_d$ so we want to perturb slightly away from this scenario so that the free boundary is smooth.

We choose $r > \Phi(1)$ so that

$$\psi_r^0(-re_d) = -r + \Phi(1) < 0.$$

For this range of r, along x_d-axis there are three free boundary points $x^i = s_i e_d$, $i = 0, 1, 2$ with

$$s_2 < -(r+1) < s_1 < -r < s_0 < 0.$$

See Figure 2. The positive set $\{\psi_r^0 > 0\}$ then has two connected components, Ω_1 that contains the half space $\{x_d \geq 0\}$ and has x^0 on its boundary, and a bounded component Ω_2 containing x^1 and x^2 on its boundary. Define

$$\psi_r(x) := \psi_r^0(x)\mathbf{1}_{\Omega_1}. \tag{4.22}$$

Lemma 4.8. *For any $\sigma > 0$ and $d \geq 3$, there are $a > 0$ and $r > \Phi(1)$ such that $\psi(x) := a^{-1}\psi_r(ax + s_0 e_d)$ is a supersolution of (4.1).*

Proof. Let $s_0 \in (-r, 0)$ as above and $s_0 e_d$ be the point on the intersection of $\partial\Omega_1 = \partial\{\psi_r > 0\}$ with the x_d-axis. We claim that $s_0 e_d$ is the lowest point on the free boundary, more precisely that

$$x_d \geq s_0 \quad \text{for all} \quad x \in \partial\Omega_1. \tag{4.23}$$

This is clear from Figure 2, and follows from the fact that $\psi_r^0(x', x_d)$ is strictly monotonically decreasing as $|x'|$ increases for fixed x_d.

Now let us show the supersolution property $|D\psi_r| \leq 1$ on $\partial\Omega_1$. Let $x \in \partial\Omega_1$. Denote $M(x) := |x + (r+1)e_d|$ and $A(x) := x_d + r + 1$. Note that (4.23) implies that

$$A = x_d + r + 1 \geq s_0 + r + 1 \geq 1 \quad \text{and also} \quad M = (A^2 + |x'|^2)^{1/2} \geq A \geq 1.$$

Now we can compute the slope

$$\begin{aligned} |D\psi_r(x)|^2 &= \left| e_d - \tfrac{(x+(r+1)e_d)}{|x+(r+1)e_d|^d} \right|^2 \\ &= 1 - 2AM^{-d} + M^{-2(d-1)} \\ &\leq 1 - 2M^{-(d-1)} + M^{-2(d-1)} \\ &= (1 - M^{1-d})^2 \\ &\leq 1. \end{aligned}$$

This verifies the supersolution condition along the free boundary.

Note that as $r \searrow \Phi(1)$ then $|D\psi_r|(s_0 e_d) \to 0$. By continuity of the gradient and hyperbolic rescaling $\psi(x) := a^{-1}\psi_r(ax + s_0 e_d)$ we can ensure that $\sup_{B_1} |\nabla\psi|^2 \leq \sigma$. Note that, following from the definition of ψ_r^0, in $d \geq 3$

$$\psi(x) = x_d + s_0 + \frac{a^{1-d}}{d-2}\frac{1}{|x|^{d-2}} + O(|x|^{1-d}) \quad \text{as} \quad \{\psi > 0\} \ni x \to \infty,$$

and in $d = 2$

$$\psi(x) = x_d - a^{-1}\log|x| + O(1) \quad \text{as} \quad \{\psi > 0\} \ni x \to \infty.$$

□

5. Pinned solutions on a single site in $d = 2$

In this section we will analyze the pinning force of a single-site defect in $d = 2$ via a limit of finite domain approximations and we will prove Theorem 1.3.

More specifically, for R large to be sent to ∞, we consider the pinning problem in B_R

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap B_R(0) \\ |\nabla u| = 1 + q(x) & \text{on } \partial\{u > 0\} \cap B_R(0) \\ u(x) = (x_d + k\log R)_+ & \text{on } \partial B_R(0). \end{cases} \tag{5.1}$$

Intuitively speaking we think of pulling the free boundary through the defect by slowly moving the boundary data via increasing (advancing case) or decreasing (receding case) the parameter k. For a certain range of k there will exist solutions that are pinned on the defect, i.e. their free boundary intersects the support of q, and then past a certain critical value the free boundary jumps and the solution is half-planar and does not see the defect. See Figure 3 for a simulation.

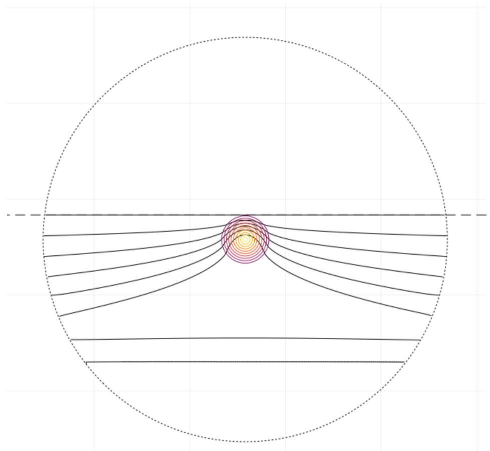

FIGURE 3. Advancing free boundaries pinned on the defect, until the critical value $\kappa^R_{\rm adv}$ where there is a jump discontinuity in the k parameter.

Our plan in this section is to first make a precise definition of this critical pinning force $\kappa^R_{\rm adv}$ and $\kappa^R_{\rm rec}$ in B_R, and then prove a limit theorem as $R \to \infty$ which includes a classification of the $R \to \infty$ limit in terms of the capacities of global proper solutions.

Definition 5.1. Define $u^{k,R}_{\rm adv}$ to be the minimal supersolution of (5.1) above $(x_d - 1)_+$. Define

$$\kappa^R_{\rm adv} := \sup\{k : \ \{u^{k,R}_{\rm adv} = 0\} \cap \overline{B_1(0)} \neq \emptyset\} \geq 0. \tag{5.2}$$

Definition 5.2. Define $u^{k,R}_{\rm rec}$ to be the maximal subsolution of (5.1) below $(x_d+1)_+$. Define

$$\kappa^R_{\rm rec} := \inf\{k : \ \overline{\{u^{k,R}_{\rm rec} > 0\}} \cap \overline{B_1(0)} \neq \emptyset\} \leq 0. \tag{5.3}$$

Definition 5.3. Define

$$k_{\rm adv} := \sup\{k : \exists \text{ a proper solution } u \text{ of (4.1) with capacity } k\}$$

and

$$k_{\rm rec} := \inf\{k : \exists \text{ a proper solution } u \text{ of (4.1) with capacity } k\}.$$

Naturally $k_{\rm adv} \geq 0 \geq k_{\rm rec}$ since half-plane solutions whose free boundary avoids the defect do exist.

Now we are ready to state our main convergence result.

Theorem 5.4. *(i) The limits hold*

$$\lim_{R\to\infty} \kappa^R_{\rm adv} = k_{\rm adv} \quad \textit{and} \quad \lim_{R\to\infty} \kappa^R_{\rm rec} = k_{\rm rec}.$$

(ii) For every $k \in [k_{\rm rec}, k_{\rm adv})$ there exists a proper solution of (1.4) with capacity k.

This result immediately implies Theorem 1.3 and also has some additional, and very useful information, about the behavior of the finite radius pinning problem (5.1) as $R \to \infty$.

Remark 5.5. Note that we have proved in Theorem 4.1 that $k_{\text{rec}} > -\infty$. However, we only know that $k_{\text{adv}} < +\infty$ under the assumption that $\max q$ is sufficiently small. We do not know whether $k_{\text{adv}} = +\infty$ is actually possible, but the limit theorem above holds regardless.

Also notice that we do not show existence of a proper solution of (1.4) with the extremal capacity k_{adv} even when $k_{\text{adv}} < +\infty$. This has to do with the issue of understanding when one-sided flat (from below) solutions are proper. The proof shows that, when $k_{\text{adv}} < +\infty$, there is a solution of (1.4) with a growth lower bound $u(x) \geq (k_{\text{adv}} \log|x| - C)_+$ but it is not clear if this solution is proper.

5.1. **Exterior solutions with a prescribed capacity.** We first prove the following Lemma, which we will use for a sliding barrier argument.

Lemma 5.6. *There are $k_0 > 0$ and $C_0 \geq 1$ universal so that for all $|k| \leq k_0$ there exists ϕ_k with $\phi_k = (x_d)_+$ in B_1 and solving the homogeneous Bernoulli problem (2.2) in $U = \mathbb{R}^2 \setminus B_1$, with the property*

$$|\phi_k(x) - (x_d + k\log|x|)| \leq C_0 \quad \text{for} \quad x \in \{\phi_k > 0\} \cap U. \tag{5.4}$$

Remark 5.7. Note that for $|k_1| \geq k_0$ if we define $R_1(k) := k/k_0$ then

$$\phi_k(x) := R_1\phi_{k_0}(x/R_1)$$

has the expansion

$$\phi_k(x) = x_d + k\log|x| + O(R_1) \quad \text{as} \quad \{\psi > 0\} \ni x \to \infty.$$

Sketch. The solutions can be constructed by solving the approximate problems

$$\begin{cases} \Delta\phi_k^R = 0 & \text{in } \{\phi_k^R > 0\} \cap B_R \setminus B_1 \\ |\nabla\phi_k^R| = 1 & \text{on } \partial\{\phi_k^R > 0\} \cap B_R \setminus B_1 \\ \phi_k^R = (x_d)_+ & \text{on } \partial B_1 \\ \phi_k^R = (x_d + k\log|x|)_+ & \text{on } \partial B_R \end{cases}$$

and sending $R \to \infty$. The barriers from Lemma 3.4 can be used via a comparison argument to guarantee that the limit is nontrivial and has the desired capacity k. The argument is similar to what we do below in the proof of Theorem 5.4. □

5.2. **Bounds on the finite radius pinned solutions.** Preliminary to the proof of Theorem 5.4 we establish several bounds on the solutions and extremal solutions of the finite radius pinning problem (5.1). We will need upper and lower bounds on the capacitory growth. The lower bounds use one-sided flatness and the barriers from Lemma 5.6 along with a sliding argument. The upper bound is much more subtle, and it is especially essential for the advancing case where it is quite challenging to establish that solutions are proper from one-sided flatness from below due to the possibility of two-plane-like solutions.

First we record some (almost) trivial maximum principle upper and lower bounds on $u_{\text{adv}}^{k,R}$ and $u_{\text{rec}}^{k,R}$. At the same time we show, closely related, one sided bounds on proper global solutions of (1.4).

Lemma 5.8 (Maximum principle bounds)**.** *Let $\log R > 1$.*

(a) Assume that $\kappa_{\text{adv}}^R > 0$. For any $0 < k \leq \kappa_{\text{adv}}^R$ there is $t_* \in [-1, 1]$ so that

$$(x_d + t_*)_+ \leq u_{\text{adv}}^{k,R}(x) \leq (x_d + k\log R)_+ \quad \text{in} \quad B_R$$

and $(x_d + t_*)_+$ touches $u_{\text{adv}}^{k,R}$ from below at a point in $\partial\{u_{\text{adv}}^{k,R} > 0\} \cap \overline{B_1}$.

(b) *Assume that* $\kappa^R_{\rm rec} < 0$. *For any* $0 > k \geq \kappa^R_{\rm rec}$ *there is* $t_* \in [-1,1]$ *so that*

$$(x_d + k\log R)_+ \leq u^{k,R}_{\rm rec}(x) \leq (x_d+t_*)_+ \quad in \quad B_R$$

and $(x_d+t_*)_+$ *touches* $u^{k,R}_{\rm rec}$ *from above at a point in* $\partial\{u^{k,R}_{\rm rec} > 0\} \cap \overline{B_1}$.

(c) *If* u *is a proper solution of (1.4) with capacity* $k > 0$ *(resp.* $k < 0$*) there is* $t_* \in [-1,1]$ *so that*

$$(x_d+t_*)_+ \leq u(x) \quad in \quad \mathbb{R}^d, \quad (resp.\ u(x) \leq (x_d+t_*)_+)$$

and $(x_d+t_*)_+$ *touches* u *from below (resp. above) at a point in* $\partial\{u > 0\} \cap \overline{B_1}$. *In particular* $\partial\{u>0\} \cap \overline{B_1} \neq \emptyset$.

Proof. The arguments in advancing and receding cases are almost symmetrical, so we just consider the advancing case. The upper bound of part (a) follows since $(x_d + k\log R)_+$ solves (5.1) and $u^{k,R}_{\rm adv}$ is the Perron's method minimal supersolution of (5.1).

The touching from below in parts (b) and (c) both follow from a very similar sliding comparison. We just present the global case (c). Slide $(x_d+t)_+$ from below until it touches u from below:

$$t_* := \sup\{t : (x_d+t)_+ \leq u(x)\}.$$

Since u is proper and $k > 0$ the above supremum is on a nonempty set, and there is a point $x^* \in \partial\{u>0\}$ with $x^*_d + t_* = 0$. If $x^* \notin \overline{B_1}$ then Lemma B.2 part (ii) then $u(x) \equiv x_d + t_*$ in $x_d + t_* \geq 0$ which contradicts u being proper with capacity $k > 0$. Thus $t_* \in [-1,1]$ and $x^* \in \overline{B_1}$. □

Next we use a sliding argument with the exterior capacitory barriers from Lemma 5.6 to establish that $u^{k,R}_{\rm adv}$ and $u^{k,R}_{\rm rec}$ have the expected capacity behavior from "below", i.e. a lower bound on the growth away from x_d.

Lemma 5.9 (Capacity "lower" bound)**.** *For* $k \geq 0$

$$u^{k,R}_{\rm adv}(x) \geq x_d + k\log|x| - C(k) \quad in \quad \{u^{k,R}_{\rm adv} > 0\}, \tag{5.5}$$

and for $k \leq 0$

$$u^{k,R}_{\rm rec}(x) \leq x_d + k\log|x| + C(k) \quad in \quad \{u^{k,R}_{\rm rec} > 0\}. \tag{5.6}$$

Proof. The case $k=0$ is already implied by Lemma 5.8. We just consider the case $k > 0$ and prove (5.5), the argument to prove (5.6) in the case $k < 0$ is parallel.

Let $k > 0$ and let ϕ_k, $R_0(k)$, and $C_0(k)$ be be from Lemma 5.6.

We perform a sliding comparison between $u^{k,R}_{\rm adv}$ and $\psi_t(x) := \phi_k(x-(1+t)e_d)$ in $B_R(0)$, for $R \geq 2\max\{R_0,k\}$ and for $t \geq 2C_0$, to conclude that

$$\psi_t \leq u^{k,R}_{\rm adv} \ \text{ in } \ B_R(0) \ \text{ for } \ t = 2C_0. \tag{5.7}$$

For all $t > 0$ we have, due to Lemma 5.8,

$$u^{k,R}_{\rm adv}(x) \geq (x_d-1)_+ \geq (x_d-1-t)_+ = \psi_t(x) \ \text{ on } \ B_{R_0}((1+t)e_d).$$

Due to (5.4)

$$\psi_t(x) \leq x_d - (1+t) + k\log|x-(1+t)e_d| + C_0 \ \text{ for } \ x \in \{\psi_t > 0\} \setminus B_{R_0}((1+t)e_d)$$

Evaluating this on $x \in \partial B_R(0) \cap \{\psi_t > 0\}$, for $t \geq 2C_0$, using the concavity of logarithm and $R \geq 2k \geq 0$,

$$\begin{aligned}\psi_t(x) &\leq x_d + k\log(R+1+t) + C_0 - (1+t)\\ &\leq x_d + k\log R + k\frac{1}{R}(1+t) + C_0 - (1+t)\\ &\leq x_d + k\log R + C_0 - \frac{1}{2}(1+t) \leq x_d + k\log R = u^{k,R}_{\mathrm{adv}}(x).\end{aligned}$$

Hence sliding comparison in $\overline{B_R(0)}$ yields (5.7).

Now (5.5) follows by bounds of ψ_{2C_0}. For $|x| \geq 2(1+2C_0)$, (5.4) yields

$$\begin{aligned}u^{k,R}_{\mathrm{adv}}(x) \geq \psi_{2C_0}(x) &\geq x_d + k\log|x - (1+2C_0)e_d| - 1 - 2C_0\\ &\geq x_d + k\log(|x| - (1+2C_0)) - 1 - 2C_0\\ &\geq x_d + k\log|x| - [k\log 2 + 1 + 2C_0]\end{aligned}$$

Lastly recall that for $1 \leq |x| \leq 2(1+2C_0)$

$$u^{k,R}_{\mathrm{adv}}(x) \geq x_d - 1 \geq x_d + k\log|x| - [1 + k\log 2(1+2C_0)].$$

Hence we have shown (5.5). □

Last we show that the finite radius pinned solutions can be extended to global supersolutions at the cost of a slight decrease in the capacitory growth. This is a crucial step which allows us to show that the $u^{k,R}_{\mathrm{adv}}(x)$ behave like a proper solution at all intermediate scales $1 \ll r \ll R$.

Proposition 5.10. *Let $K > 1$. For any $\delta > 0$ there is $R_0(\delta, K) \gg 1$ such that for $R \geq R_0$ and $0 < k \leq \min\{\kappa^R_{\mathrm{adv}}, K\}$ then*

$$w^{k,R}_{\mathrm{adv}}(x) := \begin{cases} u^{k,R}_{\mathrm{adv}}(x) & \text{in } B_R \\ (x_d + (k-\delta)\log|x| + \delta\log R)_+ & \text{in } \mathbb{R}^d \setminus B_R \end{cases} \tag{5.8}$$

is a supersolution of (1.4) in $\mathbb{R}^d$.

Proof. By Lemma 5.8 there is $t_* \in [-1, 1]$ such that

$$x_d + t_* \leq u^{k,R}_{\mathrm{adv}}(x) \leq (x_d + k\log R)_+ \quad \text{in} \quad B_R,$$

and $x_d + t_*$ touches $u^{k,R}_{\mathrm{adv}}$ from below on its free boundary in $\overline{B_1}$.

Let η_0 from Lemma B.3 define $r := \eta_0^{-1}(k\log R + 1)$. By Lemma B.3, applied to $\tilde{u}(x,t) := r^{-1}u(r(x + t_*))$, we find that for any $\delta > 0$ there exists $R_0(\delta, K)$ sufficiently large so that for $R \geq R_0$

$$u^{k,R}_{\mathrm{adv}}(x) \leq (x_d + \tfrac{\eta_0}{8K}\delta r)_+ \quad \text{in} \quad B_r.$$

Then define a barrier

$$\psi^s(x) := x_d + (k-\delta)\log|x| + \delta\log R - s.$$

By Lemma 4.4 $\psi^s(x)_+$ is a supersolution of (2.2) in $\mathbb{R}^2 \setminus B_3$ for $0 \leq s \leq \delta\log R$. Note that

$$\psi^s(x)_+ = (x_d + k\log R - s)_+ \leq u^{k,R}_{\mathrm{adv}}(x) \quad \text{on} \quad \partial B_R.$$

On ∂B_r

$$\psi^s(x) = x_d + (k-\delta)\log r + \delta\log R - s.$$

We compute, letting R_0 larger if needed depending on K so that $\frac{\log r}{\log R} \leq \frac{1}{2}$ and $\eta_0 r \geq 2$,

$$\begin{aligned}(k-\delta)\log r + \delta \log R &\geq (1 - \frac{\log r}{\log R})\delta \log R \\ &\geq \frac{1}{2}\delta \log R \\ &= \frac{1}{2k}\delta(\eta_0 r - 1) \\ &\geq \frac{\eta_0}{4K}\delta r.\end{aligned}$$

So $u^{k,R}_{\text{adv}}(x) \prec \psi^s(x)_+$ on ∂B_r for $0 \leq s < \frac{\eta_0}{4K} r$.

Thus, for $0 < s \leq \frac{\eta_0}{4K} r$,

$$w_s(x) := \begin{cases} u^{k,R}_{\text{adv}}(x) & x \in B_r \\ \min\{\psi_s(x), u^{k,R}_{\text{adv}}(x)\} & x \in B_R \setminus B_r \\ \psi_s(x) & x \in \mathbb{R}^2 \setminus B_R \end{cases}$$

is a supersolution of (1.4) in $\mathbb{R}^2$. By sending $s \to 0$ and uniform stability Lemma A.11 $w_0(x)$ is a supersolution of (1.4) in $\mathbb{R}^2$.

Since $\psi^0_+ \geq u^{k,R}_{\text{adv}}$ on $\partial(B_R \setminus B_r)$, ψ^0_+ is a supersolution of (2.2) in $B_R \setminus B_r$ and $u^{k,R}_{\text{adv}}$ is a minimal supersolution of (2.2) in $B_R \setminus B_r$, we conclude that for $0 \leq s \leq c \log R$,

$$u^{k,R}_{\text{adv}}(x) \leq \psi^0(x)_+ \quad \text{in} \quad B_R \setminus B_r.$$

Thus $w^{k,R}_{\text{adv}}(x) = w_0(x)$ in $\mathbb{R}^2$ and so $w^{k,R}_{\text{adv}}$ is a supersolution of (1.4) in $\mathbb{R}^2$. □

5.3. **Limit of the finite radius pinning thresholds.** Now we have the necessary set-up to return to the proof of Theorem 5.4.

Proof of Theorem 5.4. We will establish the equation for the limit by proving an upper bound for the limit supremum, and a lower bound for the limit infimum. We will focus on the advancing case which has a major additional challenge since it is trickier to establish that global solutions are proper in the flat from below case. The differences in the receding case will be pointed out at the end of the proof.

Step 1. ($\liminf_{R\to\infty} \kappa^R_{\text{adv}} \geq k_{\text{adv}}$) The case $k_{\text{adv}} = 0$ is immediate so we can assume that $k_{\text{adv}} > 0$. It will suffice to show that:

$$\begin{gathered}\text{for any } 0 < k < k_{\text{adv}} \text{ there is } R_1(k) \text{ large so that for } R \geq R_1, \\ \{u^{k,R}_{\text{adv}} = 0\} \cap \overline{B_1} \neq \emptyset \ \text{ and therefore } \ \kappa^R_{\text{adv}} \geq k.\end{gathered} \tag{5.9}$$

Notice that (5.9) implies that $\liminf_{R\to\infty} \kappa^R_{\text{adv}} \geq k$ for all $0 < k < k_{\text{adv}}$ and so $\liminf_{R\to\infty} \kappa^R_{\text{adv}} \geq k_{\text{adv}}$.

Now we proceed to establish the claim (5.9). Let $0 < k < k_{\text{adv}}$. By Definition 5.3 there is $k < k_0 \leq k_{\text{adv}}$, and a proper solution u of (4.1) with capacity k_0. By Lemma 5.8 part (c) we know the free boundary of u must touch the support of q, i.e. $\{u = 0\} \cap \overline{B_1} \neq \emptyset$.

Now we compare u with $u^{k,R}_{\text{adv}}$. Since $u(x) \geq (x_d - 1)_+$, $\psi(x) := \min\{u^{k,R}_{\text{adv}}(x), u(x)\}$ is above $(x_d - 1)_+$. Moreover, since $k < k_0$ we can choose R_1 large such that for $R \geq R_1$

$$u(x) \geq (x_d + k \log R)_+ \quad \text{on } \partial B_R,$$

and thus ψ is a supersolution of (5.1). Since $u^{k,R}_{\mathrm{adv}}(x)$ is the minimal supersolution of (5.1) above $(x_d-1)_+$ so we conclude that $u^{k,R}_{\mathrm{adv}} \leq \psi \leq u$. But this means that

$$\{u^{k,R}_{\mathrm{adv}} = 0\} \cap \overline{B_1} \supset \{u = 0\} \cap \overline{B_1} \neq \emptyset.$$

Thus we proved (5.9) and therefore established $\liminf_{R\to\infty} \kappa^R_{\mathrm{adv}} \geq k_{\mathrm{adv}}$.

Step 2. ($\limsup_{R\to\infty} \kappa^R_{\mathrm{adv}} \leq k_{\mathrm{adv}}$ and existence of solutions with prescribed capacity) Define $\kappa_\infty := \limsup_{R\to\infty} \kappa^R_{\mathrm{adv}}$. It will suffice to show that for every $0 < k < \kappa_\infty$ there is a proper solution of (1.4) with capacity k. We allow the possibility that κ_∞ and/or k_{adv} are equal to $+\infty$.

Let $0 < k < \kappa_\infty$. Let $R_j \to \infty$ along which $\kappa_j := \kappa^{R_j}_{\mathrm{adv}} \to \kappa^\infty$. Up to a further subsequence we can assume that $0 < k < \kappa_j$ for all j. Then define $u_j := u^{k,R_j}_{\mathrm{adv}}$. We will show that, up to taking a further subsequence, the u_j converge to a global proper solution u_∞ with capacity k.

For compactness, note that, since $0 < k < \kappa^{R_j}_{\mathrm{adv}}$ there is at least one point $x_j \in \{u_j = 0\} \cap \overline{B_1}$. So the Lipschitz estimate Lemma A.9 implies that $u_j(x) \leq C|x - x_j|$ in $\mathbb{R}^d$. By uniform Lipschitz continuity, Lemma A.9 again, along a subsequence (not relabeled) the u_j converge locally uniformly in $\mathbb{R}^d$ to some u_∞ and $\{u_\infty = 0\} \cap \overline{B_1} \neq \emptyset$. Also recall that $u_j(x) \geq (x_d - 1)_+$ so $u_\infty(x) \geq (x_d - 1)_+$ and so u_∞ is one-sided flat and nontrivial.

By Lemma 5.9,

$$u_j(x) \geq (x_d + k \log|x| - C)_+ \ \text{ in } \ B_R.$$

Taking the limit of the lower bound

$$u_\infty(x) \geq (x_d + k\log|x| - C)_+ \ \text{ in } \ \mathbb{R}^d. \tag{5.10}$$

Next we aim to establish a similar upper bound, in the advancing case – as we currently consider – we need some kind of upper bound to obtain that u_∞ is proper. But, moreover, we want to show that u_∞ has capacity exactly k so we need a matching upper bound in any case. This is where we make use of Proposition 5.10. Let $\delta := \min\{\frac{1}{2}(\kappa_\infty - k), 1\}$. Then the Proposition 5.10 and the definition of κ_∞ as the limit supremum imply that there is some $R_* > 1$ sufficiently large, depending on $\delta > 0$, k, and on the convergence rate to the limit supremum, so that $k + \delta < \kappa^{R_*}_{\mathrm{adv}}$ and

$$w(x) := \begin{cases} u^{k+\delta,R_*}_{\mathrm{adv}}(x) & \text{in } B_{R_1} \\ (x_d + k\log|x| + \delta \log R_*)_+ & \text{in } \mathbb{R}^d \setminus B_{R_*} \end{cases}$$

is a supersolution of (1.4) in $\mathbb{R}^d$. Then for any j large enough that $R_j > R_*$ we have that w is a supersolution of (5.1) in B_{R_j} and, since u_j is the minimal supersolution of that problem, we conclude that

$$u_j(x) \leq w(x) \ \text{ in } \ B_{R_j} \ \text{ for all } j \text{ large enough that } R_j > R_*.$$

Taking the limit we find

$$u_\infty(x) \leq w(x) \ \text{ in } \ \mathbb{R}^d.$$

Combined with the lower bound (5.10) this implies that u_∞ is proper and has capacity exactly k.

Thus we have proved that $\limsup_{R\to\infty} \kappa^R_{\mathrm{adv}} \leq k_{\mathrm{adv}}$ and also that, for every $k \in (0, k_{\mathrm{adv}})$, there exists a proper solution of (1.4) with capacity k, namely u_∞ constructed above.

□

Remark 5.11. Note that when $k_{\mathrm{adv}} < +\infty$ we could also construct a global solution u_∞ of (1.4) with the same growth lower bound (5.10) exactly at $k = k_{\mathrm{adv}}$, but for the upper bound we do seem to need $k < k_{\mathrm{adv}}$. If u_∞ were proper it would indeed need to have capacity k_{adv}, by the lower bound of the growth and the definition of k_{adv} as the maximal capacity, but we need the upper bound to establish that u_∞ is proper. In the receding case $k < 0$ the upper bound, analogous to (5.10)

$$u_\infty(x) \leq (x_d + k_{\mathrm{rec}} \log|x| + C)_+$$

is sufficient to establish that the solution u_∞ is proper with capacity $k_\infty \leq k_{\mathrm{rec}}$, by the extremal definition of k_{rec} this implies that $k_\infty = k_{\mathrm{rec}}$.

6. Pinned solutions on a single site in $d \geq 3$

In this section we will analyzed the solution pinned on a single site in $d \geq 3$ and prove Theorem 1.4. Namely we will show that there are a pair of laminations of extremal solutions to (4.1). These two families will bound all the proper solutions of (4.1) and will contain all the solutions with extremal values of the capacity.

In the two-dimensional case, since the capacity appears at higher order in the asymptotic expansion, the solution with the extremal pinning force is the last one visible as free boundary is pulled past the defect. The higher dimensional case is a bit different, the oscillation of $u(x) - x_d$ is bounded for all proper solutions and different pinning forces may be viewed at different heights. At a technical level we need to index the pinned solutions by the height s instead of the capacity k. This leads us to use a different approximation procedure to construct the global pinned solutions, based on solving half-space problems indexed by s instead of problems in a large ball B_R as in the two-dimensional case.

Below we use the notion of (weakly) extremal super/subsolutions in the sense of Definition A.4 and Definition A.5.

Theorem 6.1. *Let $d \geq 3$. There exist families of strongly maximal subsolutions $(u_{\mathrm{rec}}(x; s))_{s\in\mathbb{R}}$ and strongly minimal supersolutions $(u_{\mathrm{adv}}(x; s))_{s\in\mathbb{R}}$ of (4.1) such that the following holds:*

(i) Both u_{adv} and u_{rec} are monotone increasing with respect to s and satisfies

$$u_{\mathrm{adv/rec}}(x; s) = x_d + s - \kappa_{\mathrm{adv/rec}}(s)\frac{1}{|x|^{d-2}} + O(\frac{1}{|x|^{d-1}}) \quad \text{as } \{u > 0\} \ni x \to \infty. \tag{6.1}$$

Moreover, u_{rec} is right continuous and $u_{\mathrm{adv}}(\cdot; s)$ is left continuous with respect to s. In particular, both maps have at most countably many jump discontinuities.

(ii) Let u be a proper solution of (4.1) with height s and capacity k. Then

$$u_{\mathrm{adv}}(x; s) \leq u(x) \leq u_{\mathrm{rec}}(x; s) \quad \text{and} \quad \kappa_{\mathrm{rec}}(s) \leq k \leq \kappa_{\mathrm{adv}}(s).$$

If, furthermore, u is minimal then

$$u(x) \leq u_{\mathrm{adv}}(x; s+) \quad \text{and} \quad k \geq \kappa_{\mathrm{adv}}(s+)$$

and if, instead, u is maximal then

$$u_{\mathrm{rec}}(x; s-) \leq u(x) \quad \text{and} \quad k \leq \kappa_{\mathrm{rec}}(s-).$$

Note that by above theorem and Theorem 4.1, it follows that

$$\kappa_{\text{rec}}(s) = \min\{k : u \text{ is a proper solution of } (4.1) \text{ with capacity } k \text{ and height } s\},$$

and $\kappa_{\text{adv}}(s)$ is the maximum of the same set. These quantities are important, as they describe the profile of every plane-like pinning solution, pinned at location s. The following theorem states the behavior of these profiles as the s varies.

Theorem 6.2. *The capacity functions $\kappa_{\text{adv/rec}}(s) : \mathbb{R} \to \mathbb{R}$ satisfy the following properties:*

(i) $\kappa_{\text{adv/rec}}$ are compactly supported.
(ii) κ_{adv} is left continuous and USC, and κ_{rec} is right continuous and LSC.
(iii) $\kappa_{\text{adv/rec}}$ have at most countably many jump discontinuities.

As usual we are just considering the proper solutions at direction e_d, naturally one can define the above laminations and capacities at every direction $e \in S^{d-1}$ by rotating the defect. We discuss the continuity properties of $\kappa_{\text{adv/rec}}(s; q)$ with respect to q, and therefore also with respect to rotations, in Section 6.5.

6.1. **Half-space approximation problem.** In order to construct the laminations of global solutions in Theorem 6.1 we need an approximation procedure by some boundary value problems in subdomains of $\mathbb{R}^d$. It is convenient to use half-spaces of the form $\{x_d < R\}$ due to the good alignment with the expected planar asymptotic $x_d + s$ as $|x'| \to \infty$.

Fix $s \in \mathbb{R}$, let $R > -s$ and consider the half-space problem

$$\begin{cases} \Delta w = 0 & \text{in } \{w > 0\} \cap \{x_d < R\} \\ |\nabla w| = 1 + q(x) & \text{on } \partial\{w > 0\} \cap \{x_d < R\} \\ w = x_d + s & \text{on } \{x_d \geq R\} \end{cases} \tag{6.2}$$

with the additional boundedness requirement

$$\sup_{\{w>0\}} |w(x) - (x_d + s)| < +\infty. \tag{6.3}$$

Note that the condition $w = x_d + s$ in $\{x_d \geq R\}$ is essentially a Dirichlet condition on $\{x_d = R\}$, but we just find it convenient to extend the definition of w to all of $\mathbb{R}^d$ in a natural way.

We begin by pointing out a simple a-priori maximum principle bound. For any w solving (6.2) and (6.3)

$$(x_d + s \wedge (-1))_+ \leq w(x) \leq (x_d + s \vee 1)_+. \tag{6.4}$$

This can be easily proved by sliding comparison, Lemma 2.1, similar to the proof of Lemma 5.8.

In this section we show that solutions w of (6.2)-6.3 limit to $(x_d+s)_+$ as $|x'| \to \infty$ in a quantitative way which is uniform in R. This is preparation for taking the limit $R \to \infty$ in the next section.

The proof of this quantitative asymptotics will be split into two lemmas. First, in Lemma 6.3, we prove the qualitative limit as $|x'| \to \infty$ via a compactness argument. Then, in Lemma 6.4 we can quantify the limit using barriers and using the qualitative limiting behavior for the boundary ordering at tangential ∞.

Lemma 6.3. *If w solves (6.2)- (6.3), then*

$$\lim_{r \to \infty} \sup_{\{w>0\} \setminus B_r(0)} |w(x) - (x_d + s)| = 0.$$

Note that this qualitative result works in all dimensions $d \geq 2$.

Proof. Take $s = 0$, the general $s \in \mathbb{R}$ case is the same. We will show that

$$\liminf_{\{w>0\}\ni x\to\infty}(w(x) - x_d) = 0, \quad \limsup_{\{w>0\}\ni x\to\infty}(w(x) - x_d) = 0.$$

Note that both limits are finite since $w(x) - x_d$ is bounded. The two arguments are similar, so we consider the limit infimum case. Suppose that

$$\liminf_{\{w>0\}\ni|x|\to\infty}(w(x) - x_d) = -\varepsilon < 0.$$

Then there is a sequence $x_k \in \{w > 0\} \to \infty$ so that $w(x_k) - (x_k)_d \to -\varepsilon$. Let x_k' be the projection of x_k onto $x \cdot e_d = 0$. Taking a further subsequence we can assume that $(x_k)_d \to a \in [\varepsilon, R]$, since $w \geq 0$ and so

$$a := \lim_{k\to\infty}(x_k)_d = \lim_{k\to\infty}(w(x_k) + \varepsilon) \geq \varepsilon.$$

Define

$$w_k(x) := w(x_k' + x).$$

Since the w_k are uniformly Lipschitz, up to a subsequence, we can assume that the w_k converge locally uniformly to w_∞ which is also Lipschitz continuous. By uniform stability of viscosity solutions, w_∞ is a viscosity solution of the finite strip problem without a defect

$$\begin{cases} \Delta w_\infty = 0 & \text{in } \{w > 0\} \cap \{x_d < R\} \\ |\nabla w_\infty| = 1 & \text{on } \partial\{w > 0\} \cap \{x_d < R\} \\ w_\infty(x) = x_d & \text{on } \{x_d \geq R\}. \end{cases} \tag{6.5}$$

Furthermore

$$w_\infty(0, a) - a + \varepsilon = 0 \ \text{ and } \ w_\infty(x) - x_d + \varepsilon \geq 0 \ \text{ in } \ \{w_\infty > 0\}.$$

This is already a contradiction of strong maximum principle if $(0, a) \in \{w > 0\}$, and if $x_d - \varepsilon$ touches w_∞ from below at $(0, a) \in \partial\{w > 0\}$ then the strong maximum principle Lemma B.2 also leads to a contradiction.

□

Next we show a stronger, quantitative estimate for Lemma 6.3.

Lemma 6.4. *Let $d \geq 3$. There exists $C(\min q, \max q) \geq 1$ such that, if w solves (6.2) and (6.3) and $R + s > 0$ then:*

$$|w(x) - (x_d + s)| \leq C(1 + |x + se_d|)^{2-d} \ \text{ in } \ \{w > 0\}.$$

Proof. We will use the sub and supersolution barriers constructed in Proposition 4.2. Let us just argue for the lower bound, since the upper bound is similar.

By Proposition 4.2 there exists ϕ a subsolution of (4.2) with $\sigma = \max q$ and satisfying

$$\phi(x) \geq x_d - C(d, \max q)(1 + |x|)^{2-d} \ \text{ in } \ \{\phi > 0\}.$$

By sliding comparison Lemma 2.1 $\phi(x + te_d) \leq w(x)$ for $t \leq s$. Therefore

$$w(x) \geq \phi(x + se_d) \geq (x_d + s) - C(1 + |x + se_d|)^{2-d}.$$

□

6.2. **Construction of the lamination the whole space.** Now we consider the minimal supersolutions $u^R_{\rm adv}(x;s)$ and maximal subsolutions $u^R_{\rm rec}(x;s)$ of (6.2). Existence of these extremal solutions follows from Perron's method, see Section A.4. More precisely

$$u^R_{\rm adv}(x;s) := \inf_{w\in\mathcal{S}^R_{\rm super}(s)} w(x) \ \text{ and } \ u^R_{\rm rec}(x;s) := \sup_{w\in\mathcal{S}^R_{\rm sub}(s)} w(x)$$

where

$$\mathcal{S}^R_{\rm super/sub}(s) := \{w\in C(\mathbb{R}^d) : w \text{ satisfies (6.3) and is a super/subsolution of (6.2)}\}.$$

The plan is to take the limit $R\to\infty$ for each s to construct a monotone family of minimal supersolutions on the whole space with height indexed by s.

Lemma 6.5. *For fixed x, R the maps $s\mapsto u^R_{\rm rec}(x;s)$ and $s\mapsto u^R_{\rm adv}(x;s)$ are monotone increasing.*

Proof. Let $-R\le s<s'$ then $u^R_{\rm adv}(x;s')\wedge u^R_{\rm adv}(x;s)$ is a supersolution of (6.2) and so $u^R_{\rm adv}(x;s')\wedge u^R_{\rm adv}(x;s)\ge u^R_{\rm adv}(x;s)$. Similar arguments show the monotonicity of $s\mapsto u^R_{\rm rec}(x;s)$. □

We will first fix s and send $R\to\infty$.

Lemma 6.6. *Let $s\in\mathbb{R}$ and $R_k\to\infty$, there is a subsequence R_{k_j} so that $u^{R_{k_j}}_{\rm adv}(\cdot;s)$ locally uniformly converges to a minimal supersolution $w(\cdot;s)$ of (4.1) in $\mathbb{R}^d$ which is proper and has height s.*

Proof. By (6.4) and Lemma A.9, $u^R_{\rm adv}(\cdot;s)$ are uniformly bounded and uniformly Lipschitz continuous. Thus the subsequential convergence follows. Note that $w(x;s) = x_d+s+o(1)$ in $\{w>0\}$ as $x\to\infty$ from Lemma 6.4. Moreover (6.4)

$$(x_d+s\wedge(-1))_+\le w(x;s)\le (x_d+s\vee 1)_+.$$

Hence by Lemma A.11 and Lemma A.12, we conclude.

□

Now we define

$$u_{\rm adv}(x;s) := \lim_{s'\nearrow s} w(x;s'). \tag{6.6}$$

The limit exists due to monotonicity of v in s, the convergence is indeed uniform in $\mathbb{R}^n$ due to Lemma 6.4.

Similarly we can define $z:=\lim_{R\to\infty} u^R_{\rm rec}(x;s)$ and

$$u_{\rm rec}(x;s) := \lim_{s'\searrow s} z(x;s'). \tag{6.7}$$

6.3. **Properties of extremal solutions.** Now we are ready to Theorem 6.1.

Proof. **Part (i).** We claim that $u_{\rm adv}$ is a strongly minimal supersolution for each s. If not, by Definition A.6, there is a global supersolution $w\le u_{\rm adv}$ with $w=u_{\rm adv}$ outside of some ball B and $w<u_{\rm adv}$ somewhere.

By (6.6), there is $s'<s$ sufficiently close so that $u_{\rm adv}(x_0;s')>w(x_0)$ for some x_0. On the other hand, $u_{\rm adv}(x;s') = x_d+s'+o(1) < u_{\rm adv}(x;s) = x_d+s+o(1)$ if $|x|\ge M$ for a sufficiently large $M>|x_0|$. Since $v_R(\cdot,s')$ converges to $v(\cdot,s)$ locally uniformly, we conclude that $v_R(x,s')<u_{\rm adv}(x,s')=w$ on $|x|=M$ for a sufficiently

large R and yet $v_R(x_0, s') > w(x_0)$. This contradicts the minimality of v_R and thus we can conclude. The left/right continuity is a direct consequence of (6.6)-(6.7).

Part (ii). Let u be a proper solution of (4.1) with height s and capacity k. First we claim that

$$u_{\text{adv}}(x; s') \leq u(x) \quad \text{if} \quad s' < s.$$

Given $s' < s$ let r sufficiently large so that

$$u(x) > u_{\text{adv}}(x; s') \quad \text{on} \quad \overline{\{u_{\text{adv}}(\cdot; s) > 0\}} \setminus B_r.$$

Then minimality of $u_{\text{adv}}(x; s')$ implies that $u_{\text{adv}}(x; s') \leq u(x)$ in B_r.

Since $u_{\text{adv}}(x; \cdot)$ is left continuous in s we can conclude that

$$u_{\text{adv}}(x; s) \leq u(x) \quad \text{on} \quad \mathbb{R}^n$$

but this implies the ordering of the capacities since

$$\begin{aligned}\kappa_{\text{adv}}(s) &= \lim_{x_d \to +\infty} |x_d|^{d-2}[(x_d + s) - u_{\text{adv}}(x_d e_d; s)] \\ &\geq \lim_{x_d \to \infty} |x_d|^{d-2}[(x_d + s) - u(x_d e_d)] = k.\end{aligned}$$

If, additionally, $u(x)$ is weakly minimal, then arguing as above yields $u(x) \leq u_{\text{adv}}(x; s')$ for any $s' > s$, and thus $u(x) \leq u_{\text{adv}}(x; s+)$. Note that, due to uniform convergence estimates from Lemma 6.4 and (6.4), $u_{\text{adv}}(x; s+)$ is itself a proper solution of (4.1) with capacity $\kappa_{\text{adv}}(s+)$. Thus the same arguments as above show that

$$k \leq \kappa_{\text{adv}}(s+).$$

The arguments are similar for the bounds with respect to u_{rec}.

□

6.4. **Properties of the capacities.** Here we prove Theorem 6.2.

Proof of Theorem 6.2. Part (i) follows immediately from the universal bound on $|s| \leq C_0(d, \max |q|)$ for any proper solution with $k \neq 0$ which was proved already in in Theorem 4.1.

Parts (ii) and (iii). By Theorem 4.1

$$\kappa_{\text{adv/rec}}(s) = \lim_{x_d \to \infty} |x_d|^{n-2}[(x_d + s) - u_{\text{adv/rec}}(x_d e_d; s)]$$

and the limit is uniform over $s \in \mathbb{R}$. Therefore $\kappa_{\text{adv/rec}}(s)$ share the right/left continuity properties of $u_{\text{adv/rec}}(x; s)$ respectively. In particular $\kappa_{\text{adv/rec}}(s)$ can only have a jump discontinuity at a jump discontinuity of $u_{\text{adv/rec}}(\cdot; s)$.

Finally, we mention that the semi-continuity properties of k_{adv} and k_{rec} follows from those of u_{adv} and u_{rec}: let us present the argument for u_{rec}. Note that $u_{\text{rec}}(\cdot; s-) := \lim_{s' \nearrow s} u_{\text{rec}}(\cdot; s)$ limit exists and solves (4.1) with height s. Let k be the capacity of $u_{\text{rec}}(\cdot; s-)$ and, by the same arguments as in the previous paragraph,

$$k = \lim_{s' \nearrow s} \kappa_{\text{rec}}(s).$$

So, since $u_{\text{rec}}(\cdot; s-)$ is another proper solution of (4.1) with height s, Theorem 6.2 implies that $k \leq \kappa_{\text{rec}}(s)$ and we conclude that κ_{rec} is USC.

□

6.5. **Continuity with respect to q.** We finish this section with the continuity of the capacities with respect to perturbation of q. We do not technically need this anywhere in the present paper, but it is an important property nonetheless.

Let us denote $\kappa_{\mathrm{adv/rec}}(\cdot,q)$ as the capacity functions $k_{\mathrm{adv/rec}}$ with given defect function q, and define

$$k_{\mathrm{rec}}(q) := \max_{s\in\mathbb{R}} \kappa_{\mathrm{rec}}(s;q), \quad k_{\mathrm{adv}}(q) := \min_{s\in\mathbb{R}} \kappa_{\mathrm{adv}}(s;q).$$

Proposition 6.7 (Continuity)**.** *If $q_n \in C_c(B_1)$ and $q_n \to q$ uniformly then*

$$\lim_{n\to\infty} \kappa_{\mathrm{adv}}(s;q_n) \subset [\kappa_{\mathrm{adv}}(s-;q), \kappa_{\mathrm{adv}}(s+;q)]$$

and

$$\lim_{n\to\infty} k_{\mathrm{adv}}(q_n) = k_{\mathrm{adv}}(q).$$

The same statements hold for κ_{rec} and k_{rec}.

Proof. Let $u^n_{\mathrm{adv}}(x;s)$ be the monotone family of minimal supersolutions guaranteed by Theorem 6.1 for the defect function q_n, and $u_{\mathrm{adv}}(x;s)$ the corresponding family for q. By Lemma A.9 the functions $u^n_{\mathrm{adv}}(\cdot,s)$ are uniformly Lipschitz continuous in s and n. Fix $s \in \mathbb{R}$ and choose a subsequence so that $u^n_{\mathrm{adv}}(\cdot;s)$ and $\kappa^n_{\mathrm{adv}}(s)$ converge uniformly to $u^\infty(\cdot;s)$ and $\kappa^\infty(s)$.

By Lemma A.11, $u^\infty(\cdot;s)$ solves (4.1) and is (weakly) minimal. By Theorem 4.1 the convergence rates in the asymptotic expansion at ∞ are uniform in n, and so we can interchange limits and find

$$u^\infty(x;s) = x_d + s - \kappa^\infty(s)|x|^{2-d} + O(|x|^{1-d}) \ \text{ as } \ \{u>0\} \ni x \to \infty.$$

Since u_∞ is minimal, Theorem 6.1 part (ii) implies that

$$u_{\mathrm{adv}}(x;s-) \le u_\infty(x;s) \le u_{\mathrm{adv}}(x;s+)$$

and

$$\kappa_{\mathrm{adv}}(s+) \le \kappa^\infty(s) \le \kappa_{\mathrm{adv}}(s-).$$

In particular $(u^\infty(\cdot;s),\kappa^\infty(s)) = (u_{\mathrm{adv}}(\cdot;s),\kappa_{\mathrm{adv}}(s))$ at every continuity point of the latter. Now Theorem 6.2 yields that

$$k_{\mathrm{adv}}(q) = \kappa_{\mathrm{adv}}(s_*) = \lim_{s'\nearrow s_*} \kappa_{\mathrm{adv}}(s') = \lim_{s'\nearrow s_*} \kappa^\infty(s',)$$

where the first and last equality is due to the left continuity of k_{adv} and the fact that $\kappa_{\mathrm{adv}}(s') = \kappa^\infty(s')$ except at countably many values. □

7. Pinned slopes in a periodic medium

In this section we outline the proof of Theorem 1.1, and provide necessary ingredients to prepare for Section 8 and Section 9, where the proof will be carried out.

7.1. **Outline of the proof of Theorem 1.1.** The proof of Theorem 1.1 is divided into two parts, the lower bound away from 1 and the upper bound away from 1, respectively: the lower bounds

$$\liminf_{\delta\to 0} \frac{Q^\delta_{\mathrm{adv}} - 1}{\delta^{d-1}} \ge |\xi|^{-1} k_{\mathrm{adv}}, \quad \liminf_{\delta\to 0} \frac{1 - Q^\delta_{\mathrm{rec}}}{\delta^{d-1}} \ge -|\xi|^{-1} k_{\mathrm{rec}} \tag{7.1}$$

and the upper bounds

$$\limsup_{\delta\to 0}\frac{Q^\delta_{\mathrm{adv}}-1}{\delta^{d-1}}\leq|\xi|^{-1}k_{\mathrm{adv}},\quad \limsup_{\delta\to 0}\frac{1-Q^\delta_{\mathrm{rec}}}{\delta^{d-1}}\leq-|\xi|^{-1}k_{\mathrm{rec}}. \tag{7.2}$$

Both of these bounds rely on a certain cell problem (7.7), to be introduced in Section 7.4.

In Section 8, we establish (7.1). For this it suffices to construct a plane-like supersolution with slope α. Such a barrier will be constructed in Lemma 8.3, by patching together an outer solution, based on ω from (7.7), with an inner solution near the defect based on one of the single site profiles constructed in Section 6.

In Section 9 we proceed to (7.2): Here we show that the barrier constructed above is essentially optimal. This is naturally a more difficult direction, and will be the content of Section 9.2 - 9.5. Due to the characterization of the pinning interval endpoints in Theorem 7.3 from [22], there is a plane-like solution u_δ of (7.5) achieving the maximal macroscopic slope Q_{adv} and satisfying the strong Birkhoff property (see Definition 7.2 below). We will show that u_δ matches the profile of our barrier away from the defects, which amounts to Theorem 1.5. An important ingredient in achieving this is the asymptotic decomposition of u_δ via the semipermeable membrane cell problem (7.9), based on refined compactness properties of u_δ using its periodicity and near-defect behavior understood from earlier sections. The uniqueness / rigidity properties of (7.9) then allow us to compute the asymptotic upper bound.

7.2. **Rotation to $e = e_d$ case.** As usual in this paper, we will normalize the problem to reduce to the direction $e = e_d$, paying attention to the fact that the periodicity lattice is now rotated as well. We use the fact that $e = \xi/|\xi|$ is rational and $\xi \in \mathbb{Z}^d$ is irreducible to explain the structure of the rotated lattice, especially the intersection of the lattice with the plane $\partial\mathbb{R}^d_+$.

For $v \in \mathbb{R}^d$, denote $v^\perp$ to be the linear subspace orthogonal to v. For $\xi \in \mathbb{Z}^d$, irreducible, the lattice $\mathbb{Z}^d \cap \xi^\perp$ is $(d-1)$-dimensional. The area of the unit period cell of this lattice is $|\xi|$. We will now perform a rotation O sending e to e_d. This also rotates the underlying periodicity lattice to $O\mathbb{Z}^d$ We define

$$\mathcal{Z} := O(\mathbb{Z}^d \cap \xi^\perp) \subset \{x_d = 0\} = \partial\mathbb{R}^d_+.$$

Note that $\mathcal{Z}$ is a $(d-1)$-dimensional integer lattice. Take a basis of successive minima for $\mathcal{Z}$, i.e.

$$\zeta_j \in \operatorname{argmin}\{|\zeta| : \ \zeta \in \mathcal{Z} \setminus \operatorname{span}(\zeta_1, \dots, \zeta_{j-1})\}.$$

Let $\square_{\mathcal{Z}} \subset \{x_d = 0\}$ be the fundamental domain / parallelopiped of the lattice $\mathcal{Z}$ centered at $0 \in \mathcal{Z}$ corresponding to the basis of successive minima. The volume of the fundamental domain is (independent of the choice of basis actually)

$$|\square_{\mathcal{Z}}| = |\xi|. \tag{7.3}$$

We also denote

$$⧈^{\mathcal{Z}} := \square_{\mathcal{Z}} \times [-1,1] \ \text{ and } \ ⧈^{\mathcal{Z}}_r := r⧈^{\mathcal{Z}}. \tag{7.4}$$

These fattened cells are convenient since the lattice translations (almost) disjointly cover the channel region $\{|x_d| \leq 1\}$. Finally we note that there is some $\rho_0(\xi) > 0$ so that $B_{\rho_0}(0) \subset ⧈_{\mathcal{Z}}$. In $d = 2$ the lattice $\mathcal{Z}$ is generated by $\xi^\perp = (\xi_2, -\xi_1)$ so we can take $\rho_0 = |\xi|/2 \geq 1/2$ in that case.

7.3. **Pinning interval in periodic homogenization.** In this section we collect some known results on pinning interval and plane-like solutions for the Bernoulli free boundary problem in periodic media:

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \\ |\nabla u| = Q(x) & \text{on } \partial\{u > 0\}, \end{cases} \tag{7.5}$$

where $Q(x) > 0$ is $\mathbb{Z}^d$-periodic and continuous.

Definition 7.1. Given $e \in S^{d-1}$ we define the *maximal pinned slope*

$$Q^*(e) := \sup\{\alpha : \exists \text{ a supersolution } u \in C(\mathbb{R}^d; \mathbb{R}_+) \text{ of (7.5) with } u(x) \geq \alpha(e \cdot x)_+\}$$

and the *minimal pinned slope*

$$Q_*(e) := \inf\{\alpha : \exists \text{ a nontrivial subsolution } u \in C(\mathbb{R}^d; \mathbb{R}_+) \text{ of (7.5) with } u(x) \leq \alpha(e \cdot x)_+\}.$$

The *pinning interval* is the interval

$$I(e) := [Q_*(e), Q^*(e)].$$

Note that $Q_*(e; Q)$, $Q^*(e; Q)$, and $I(e; Q)$ depend implicitly on the periodic field $Q(x)$, but we usually hide the dependence.

It is straightforward to check that

$$0 < \min Q \leq Q_*(e) \leq Q^*(e) \leq \max Q < +\infty.$$

Next we define the Birkhoff property which is an analogue of periodicity in this context.

Definition 7.2. u satisfies the *Birkhoff property* in direction e if

$$u(x + k) \geq u(x) \ \text{ for all } \ k \in \mathbb{Z}^d \text{ s.t. } \ k \cdot e > 0.$$

u satisfies the *strong Birkhoff property* if the same holds for all $k \in \mathbb{Z}^d$ with $k \cdot e \geq 0$.

In particular any u with the strong Birkhoff property are periodic with respect to tangential lattice translations (if there are any such).

We quote a result now summarizing several of the results in [22] on the pinning interval and the existence of corresponding plane-like solutions of (7.5).

Theorem 7.3 ([22]). *The following hold:*

- *(a) (Energy minimizing slope) For all $e \in S^{d-1}$ we have $\langle Q^2 \rangle^{1/2} \in I(e)$.*
- *(b) (Existence of plane-like solutions in the pinning interval) For any $\alpha \in I(e)$ there exists a solution u of (7.5) satisfying the strong Birkhoff property in direction e, and satisfying the flatness condition*

$$\alpha(x \cdot e - C)_+ \leq u(x) \leq \alpha(x \cdot e + C)_+ \tag{7.6}$$

and $C \geq 0$ depends only on d, $\min Q$ and $\max Q$. Such solutions are called plane-like solutions*. Furthermore if $\alpha = Q^*(e)$ then u can be taken to be a minimal supersolution, if $\alpha = Q_*(e)$ then u can be taken to be a maximal subsolution, and if $\alpha = \langle Q^2 \rangle^{1/2}$ then u can be taken to be an energy minimizer.*

7.4. **A cell problem of semi-permeable boundaries.** In this section we describe a certain cell problem, related to the homogenization theory of perforated domains, for example see [36] and many others. The solutions of this problem describe the asymptotic behavior of (1.1) between the defects. The main issue we want to clarify is the relation between integral solutions and pointwise solutions. The former characterization is preferable for computing the asymptotic slope, while the latter is easier to relate to the viscosity solutions of (1.1).

Recall that we will work in a normalized setting, with the inward normal vector $e = e_d$ and a $(d-1)$-dimensional integer lattice $\mathcal{Z} \subset \partial\mathbb{R}^d_+$. Consider the semipermeable boundary cell problem

$$\begin{cases} \Delta\omega = 0 & \text{in } \mathbb{R}^d_+, \\ \partial_{x_d}\omega = 1 - \sum_{z\in\mathcal{Z}} c_*\delta_z & \text{on } \partial\mathbb{R}^d_+, \end{cases} \tag{7.7}$$

with

$$\langle \partial_{x_d}\omega(\cdot, x_d)\rangle' = 0 \text{ for all } x_d > 0. \tag{7.8}$$

Here δ_z denotes the Dirac mass at z. For now this PDE boundary value problem should be interpreted in the weak divergence form sense. Note that $c_* := |\square_{\mathcal{Z}}|$ is a unique constant for which (7.7)-(7.8) are solvable: since $\langle \partial_{x_d}\omega(\cdot, x_d)\rangle'$ is independent of x_d by Lemma C.1, formally the existence of a solution to (7.7)-(7.8) requires that

$$0 = \langle 1 - \sum_{z\in\mathcal{Z}} c_*\delta_z\rangle' = 1 - \frac{1}{|\square_{\mathcal{Z}}|}c_* \quad \text{i.e.} \quad c_* = |\square_{\mathcal{Z}}| = |\xi|.$$

The formal argument can be made rigorous for variational weak solutions of (7.7) using a standard test function choice. Note that ω is also just a modified Green's function for a periodic problem for the half-Laplacian.

The pointwise formulation of (7.7) and (7.8) is

$$\begin{cases} \Delta\omega = 0 & \text{in } \mathbb{R}^d_+, \\ \partial_{x_d}\omega = 1 & \text{on } \partial\mathbb{R}^d_+ \setminus \mathcal{Z} \\ \lim_{x\to z} \Phi(x-z)^{-1}\omega(x) = \frac{1}{\gamma_d}c_* & \text{for } z \in \mathcal{Z}, \\ \langle \partial_{x_d}\omega(\cdot, x_d)\rangle' = 0 & \text{for } x_d > 0. \end{cases} \tag{7.9}$$

where Φ denotes the, un-normalized, upward pointing fundamental solution

$$\Phi(x) := \begin{cases} -\log|x| & d = 2 \\ |x|^{2-d} & d \geq 3 \end{cases} \tag{7.10}$$

and the normalizing constant γ_d is

$$\gamma_2 = \pi \text{ and } \gamma_d = \tfrac{1}{2}(d-2)|\partial B_1| \text{ in } d \geq 3. \tag{7.11}$$

The following lemma explains the relation between distributional and pointwise notions and has some useful information about removing singularities.

Lemma 7.4. *Suppose that $w \in C(\overline{B_1^+} \setminus \{0\})$ solves, for some $A \leq B$,*

$$\begin{cases} \Delta w = 0 & \textit{in } \ B_1^+ \\ \partial_{x_d} w = 0 & \textit{on } \ (\partial\mathbb{R}^d_+ \cap B_1) \setminus \{0\} \\ \limsup_{x\to 0} w(x)/\Phi(x) \leq B \\ \liminf_{x\to 0} w(x)/\Phi(x) \geq A. \end{cases}$$

Then, calling h to be the harmonic function in B_1 which is equal to w on ∂B_1 (extended to the lower half sphere by even reflection),

$$w(x) = c(\Phi(x) - \Phi(e_d)) + h(x) \quad \text{for some } A \leq c \leq B. \tag{7.12}$$

Furthermore w solves, in the standard distributional weak sense,

$$\begin{cases} \Delta w = 0 & \text{in } \ B_1^+ \\ \partial_{x_d} w = -\gamma_d c \delta_0 & \text{on } \ \partial \mathbb{R}^d_+ \cap B_1. \end{cases}$$

Proof. We give a quick sketch of this classical result. By even reflection we can suppose that w is harmonic in $B_1 \setminus \{0\}$ with the same bounds. Then there is $A \leq k \leq B$ such that $\lim_{x \to 0} w(x)/\Phi(x) = k$. This can be proved via a Kelvin transform argument similar to the proof of Theorem 3.2. Then let $h(x) := w(x) - k(\Phi(x) - \Phi(e_d))$. Then $h(x) = w(x)$ on ∂B_1. Note that $\lim_{|x| \to 0} h(x)/\Phi(x) = 0$, thus standard removable singularity arguments show that h is harmonic in B_1. Then we conclude by using the distributional derivatives of Φ, noting that $\Delta \Phi = -2\gamma_d \delta_0$ and the factor of $\frac{1}{2}$ arises from integrating over an upper half sphere instead of the full sphere.

□

One useful consequence of the previous Lemma is the local decomposition / extension for solutions of (7.9) and (7.8):

$$\omega(x) = \tfrac{1}{\gamma_d} c_* \Phi(x) + x_d + h(x) \ \text{ in } \ \square^{\mathcal{Z}}_{3/2} \cap \mathbb{R}^d_+ \tag{7.13}$$

with h a bounded harmonic function in $\square^{\mathcal{Z}}_{3/2}$ which is even symmetric with respect to $x_d \mapsto -x_d$. Since the right hand side naturally extends to the lower half cube $\square^{\mathcal{Z}} \cap \mathbb{R}^d_-$ we can also view that as an extension of the left hand side.

Theorem 7.5. *There exists a $\mathcal{Z}$-periodic solution ω of (7.9) if and only if $c_* = |\square_{\mathcal{Z}}|$. Furthermore any two solutions of (7.9) which are bounded on $\{x_d \geq 1\}$ differ by a constant.*

Proof. We omit the proof of existence: the existence of modified Green's functions for elliptic problems on compact manifolds is classical, see for example [4, Section 2.3]. Alternatively we direct to the constructions in Theorem 1.6 and Corollary 10.2, which provide an existence proof.

If ω solves (7.9) then Lemma 7.4 implies that ω is a variational weak solution of (7.7)-(7.8) and then Lemma C.1 implies that $c_* = |\square_{\mathcal{Z}}| = |\xi|$.

Finally we discuss uniqueness, suppose that ω_0 and ω_1 both solve (7.9) and (7.8). Let $h(y) := \omega_1(y) - \omega_0(y)$. Since h is harmonic in $\mathbb{R}^d_+$ with zero Neumann data on $\partial \mathbb{R}^d_+ \setminus \mathcal{Z}$ we can extend h by even reflection to be harmonic on $\mathbb{R}^d \setminus \mathcal{Z}$.

Due to the asymptotic condition at the holes

$$\lim_{y \to z} \frac{h(y)}{\Phi(y - z)} = 0 \ \text{ for each } \ z \in \mathcal{Z},$$

a standard removable singularity result for harmonic functions implies that h is an entire harmonic function. Since h is $\mathcal{Z}$-periodic and bounded on $|x_d| \geq 1$, it is also bounded on $|x_d| \leq 1$ by maximum principle, and so by Liouville it is constant. □

8. Asymptotic lower bound

In this section we create periodic sub and supersolution barriers for the full problem (1.1), by taking a solutions of the single-site problem and patching them with a solution of (7.7) As a corollary we will obtain the asymptotic lower bounds:

Proposition 8.1. *Let $\xi \in \mathbb{Z}^d$ irreducible and $e = \xi/|\xi|$ then*

$$\lim_{\delta\to 0} \frac{Q^\delta_{\rm adv}(e) - 1}{\delta^{d-1}} \geq k_{\rm adv}(e)|\xi|^{-1}\delta^{d-1} \quad \textit{and} \quad \lim_{\delta\to 0} \frac{1 - Q^\delta_{\rm rec}(e)}{\delta^{d-1}} \geq -k_{\rm rec}(e)|\xi|^{-1}\delta^{d-1}.$$

To construct our patched barrier, first we discuss the *outer solution*, namely the part of the barrier away from the defects, based on (7.7). This will be based on a translation of the profile

$$\tilde{w}_{out}(x) := (1+\sigma\alpha)x_d + \gamma_d \delta^{d-1}|\xi|^{-1}k[x_d - \omega(x)]. \tag{8.1}$$

Here $\sigma \in \{-1, +1\}$, $\alpha > 0$, and $k \in \mathbb{R}$ are parameters to be adjusted, and ω is the unique $\mathcal{Z}$-periodic solution of (7.9) normalized by $\min_{\mathbb{R}^d_+} \omega = 0$, see Theorem 7.5.

Lemma 8.2. *Let $\tilde{w}_{out}$ be given by (8.1) with $0 < \alpha \leq 1/2$, and suppose that for some $r \in [\delta, 1/2]$ and for a sufficiently large $C_0 = C_0(d, \mathcal{Z})$ we have*

$$C_0 k^2 (\delta/r)^{2(d-1)} r^{d-2}\Phi(r) \leq \alpha. \tag{8.2}$$

(i) If $\sigma = -1$ then $\tilde{w}_{out}$ is a supersolution of (2.2) in $\mathbb{R}^d \setminus \cup_{z\in\mathcal{Z}} B_r(z)$, i.e.,

$$|\nabla \tilde{w}_{out}(x)|^2 \leq 1 \quad \textit{on} \quad \partial\{\tilde{w}_{out} > 0\} \setminus \cup_{z\in\mathcal{Z}} B_r(z).$$

(ii) If $\sigma = 1$ then $\tilde{w}_{out}$ is a subsolution of (2.2) in $\mathbb{R}^d \setminus \cup_{z\in\mathcal{Z}} B_r(z)$, i.e.,

$$|\nabla \tilde{w}_{out}(x)|^2 \geq 1 \quad \textit{on} \quad \partial\{\tilde{w}_{out} > 0\} \setminus \cup_{z\in\mathcal{Z}} B_r(z).$$

Proof. By periodicity it suffices to show the result in $\square^{\mathcal{Z}}$. First we estimate the location of the free boundary. Denoting $\beta := (1+\sigma\alpha)$, For $x \in \partial\{\tilde{w}_{out} > 0\} \cap \square^{\mathcal{Z}}$ we have

$$0 = \tilde{w}_{out}(x) = \beta x_d + \delta^{d-1}|\xi|^{-1}k[x_d - \omega(x)].$$

(7.9) and Lemma 7.4 yields that

$$0 = \beta x_d + \delta^{d-1}|\xi|^{-1}k[-\gamma_d^{-1}|\xi|\Phi(x) + O(1)].$$

Since we are in the unit cell we can bound the $O(1)$ term by $|x|^{2-d}$ up to a constant multiple depending on the unit cell diameter. Rearranging the previous displayed equation, and using that $|\beta| \geq 1/2$, then gives

$$|x_d| \leq C\delta^{d-1}k\Phi(x) \ \text{ for } \ x_d \in \partial\{\tilde{w}_{out} > 0\} \cap (\square^{\mathcal{Z}} \setminus B_r(0)). \tag{8.3}$$

On the same set we can use the expansion from (7.13) to write

$$\nabla \tilde{w}_{out}(x) = \beta e_d - \delta^{d-1}k\left[\nabla\Phi \cdot x_d + \nabla h(x)\right] \ \text{ in } \ \square^{\mathcal{Z}}.$$

Here h is a harmonic function in $\square^{\mathcal{Z}}_{3/2}$ which is even with respect to x_d and so

$$|\nabla h(x) \cdot e_d| \leq C|x_d| \ \text{ in } \ \square^{\mathcal{Z}}. \tag{8.4}$$

Now we compute the gradient along $\partial\{\tilde{w}_{out} > 0\}$ via

$$|\nabla \tilde{w}_{out}(x)|^2 = \beta^2 - 2\beta\delta^{d-1}k\left[\nabla\Phi(x) + \nabla h(x)\right] \cdot e_d + O(k^2\delta^{2(d-1)}r^{2(1-d)}). \tag{8.5}$$

Using (8.4) we can bound

$$|\left[\nabla\Phi(x) + \nabla h(x)\right] \cdot e_d| \leq C|x_d|(\frac{1}{|x|^d} + 1) \leq C\frac{|x_d|}{|x|^d} \ \text{ in } \ \square^{\mathcal{Z}}.$$

Due to (8.3), on $x \in \partial\{\tilde{w}_{out} > 0\} \cap \square^{\mathcal{Z}} \setminus B_r(0)$,

$$\left|2\beta\delta^{d-1}k\left[\nabla\Phi(x) + \nabla h(x)\right]\cdot e_d\right| \leq Ck^2\delta^{2(d-1)}r^{-d}\Phi(r).$$

This is of the same order as the quadratic term in (8.5), except in $d = 2$ where it is larger by a factor of $\log r$.

So, in the case $\sigma = -1$ and thus $\beta = (1-\alpha)$, using that $\beta^2 \leq \beta$ for $\alpha \leq 1/2$,

$$|\nabla\tilde{w}_{out}(x)|^2 \leq 1-\alpha+Ck^2\delta^{2(d-1)}r^{-d}\Phi(r) \leq 1 \ \text{ for } \ x \in \partial\{\tilde{w}_{out} > 0\}\cap(\square^{\mathcal{Z}} \setminus B_r(0))$$

where the right-most inequality is from (8.2). By $\mathcal{Z}$-periodicity we derive the supersolution property on the entire $\mathbb{R}^d \setminus \cup_{z\in\mathcal{Z}}B_r(z)$. The argument in the case $\sigma = +1$ is even easier since the quadratic error term is positive and $(1+\alpha)^2 \geq 1+2\alpha$ for all $\alpha > 0$.

□

Next we show that any proper solution of (1.4) in $\mathbb{R}^d \setminus B_1$ with capacity k can be patched together with an appropriate outer solution to create a global periodic sub- or supersolution with the desired asymptotic slope.

Lemma 8.3. *Suppose that w_{in} is a solution of (1.4) in $\mathbb{R}^d$, for example as constructed in Sections 5 or 6, satisfying the expansion, in $d \geq 3$,*

$$w_{in}(x) = x_d + s - k|x|^{2-d} + O(|x|^{1-d}) \quad \textit{as} \quad \overline{\{w_{in} > 0\}} \ni x \to \infty,$$

or in $d = 2$

$$w_{in}(x) = x_d + k\log|x| + O(1) \quad \textit{as} \quad \overline{\{w_{in} > 0\}} \ni x \to \infty.$$

Then for any $\varepsilon > 0$ and $\delta > 0$ sufficiently small depending on ε there is a supersolution (resp. subsolution) $w_{per}(x)$ of (1.1) which is $\mathcal{Z}$-periodic with asymptotic slope at least $1+\gamma_d|\xi|^{-1}(k-\varepsilon)\delta^{d-1}$ (resp. at most $1+\gamma_d|\xi|^{-1}(k+\varepsilon)\delta^{d-1}$).

Proof. We will only present the construction of the supersolution w_{per}, the subsolution construction is parallel.

Let $0 < \varepsilon \leq 1$. For the outer solution, we use a translated version of $\tilde{w}_{out}$ from (8.1) with $\sigma = -1$:

$$\begin{aligned} w_{out}(x) :=& (1-\alpha)(x_d + \delta(s+s_\Delta)) \\ &+ \gamma_d|\xi|^{-1}\delta^{d-1}(k-\varepsilon)[x_d + \delta(s+s_\Delta) - \omega(x + \delta(s+s_\Delta)e_d)]. \end{aligned} \tag{8.6}$$

Here the constants $0 < \alpha < 1$, $s_\Delta \in \mathbb{R}$, and also s (only free in $d = 2$), are to be chosen later in the proof.

Next for $\Lambda \geq 2$, another constant to be chosen later, define

$$w_{per}(x) := \begin{cases} \delta w_{in}(\delta^{-1}(x-z)) & |x-z| \leq \frac{1}{\Lambda}r \text{ for some } z \in \mathcal{Z}, \\ \min\{\delta w_{in}(\delta^{-1}(x-z)), w_{out}(x)\} & \frac{1}{\Lambda}r \leq |x-z| \leq \Lambda r \text{ for some } z \in \mathcal{Z}, \\ w_{out}(x) & \operatorname{dist}(x,\mathcal{Z}) \geq \Lambda r. \end{cases}$$

Here $r = r(\Lambda)$ is chosen sufficiently small so that $\delta \ll r/\Lambda$ and $\Lambda r \ll 1$ so that $B_{\Lambda r} \subset \square^{\mathcal{Z}}_{1/2}$. We now choose

$$\alpha := C\Lambda^{2(d-1)}(\delta/r)^{2(d-1)}r^{d-2}\Phi(r)$$

to enable Lemma 8.2 in the region $\operatorname{dist}(x,\mathcal{Z}) \geq \Lambda^{-1}r$.

To show w_{per} is a supersolution, the goal is to obtain

$$w_{out}(x) < \delta w_{in}(\delta^{-1}x) \quad \text{on} \quad x \in \partial B_{\Lambda r}(0), \tag{8.7}$$

and

$$w_{out}(x) > \delta w_{in}(\delta^{-1}x) \quad \text{on} \quad x \in \partial B_{\frac{1}{\Lambda}r}(0). \tag{8.8}$$

(8.7)-(8.8) yields that w_{per} is a supersolution in $\square_{\mathcal{Z}} \times \mathbb{R}$, and we can conclude using the $\mathcal{Z}$-periodicity of w_{out}.

For $d = 2$, for patching purposes, we compare the expansions of inner and outer solutions on $|x| = \rho$:

$$\delta w_{in}(\delta^{-1}x) = x_d + k\delta \log \frac{\rho}{\delta} + O(\delta)$$

and

$$w_{out}(x) = (1-\alpha)\left[x_d + \delta(s + s_\Delta)\right] + \delta(k-\varepsilon)[\log \rho + O(1)].$$

So, choosing $s := -k\delta \log \frac{1}{\delta}$ for the second equation,

$$\begin{aligned} w_{out}(x) - \delta w_{in}(\delta^{-1}x) &= \delta\left[-\varepsilon \log \rho + s_\Delta + s + k\delta \log \frac{1}{\delta}\right] + O(\alpha\rho + \delta) \\ &= \delta\left[-\varepsilon \log \rho + s_\Delta\right] + O(\alpha\rho + \delta) \\ &=: A(\rho) + E(\rho). \end{aligned}$$

So if we can choose parameters so that

$$A(\Lambda r) < 0,\ A(r/\Lambda) > 0, \quad \text{and} \quad |E(\rho)| < |A(\rho)| \quad \text{for} \quad \rho \in \{r/\Lambda, \Lambda r\}, \tag{8.9}$$

then we will achieve (8.7) and (8.8).

Let us now choose

$$s_\Delta := \varepsilon \log r \quad \text{and} \quad r := c\Lambda^{-1}. \tag{8.10}$$

With these choices

$$A(\rho) = \delta\varepsilon \log \frac{r}{\rho},\ A(r/\Lambda) = \delta\varepsilon \log \Lambda > 0 \quad \text{and} \quad A(\Lambda r) = -\delta\varepsilon \log \Lambda < 0.$$

Moreover, $\alpha = C\Lambda^4\delta^2 \log \frac{1}{\delta}$ and

$$|E(\rho)| \le C(\alpha\rho + \delta) \le C_1(\Lambda^3\delta^2 + \delta).$$

So for any $\varepsilon > 0$ we can choose Λ sufficiently large so that $\varepsilon \log \Lambda \ge 2C_1$ and then for δ sufficiently small $|E(\rho)| \le |A(\rho)|$ for $\rho \in \{R/\Lambda, \Lambda r\}$.

Lastly, recall that from (7.13) the asymptotic slope of this supersolution will be

$$\partial_{x_d} w_{per}(x) \to 1 + (\gamma_d|\xi|^{-1}(k-\varepsilon) - \alpha)\delta^{d-1} \quad \text{as} \quad x_d \to +\infty, \tag{8.11}$$

and since $\alpha \to 0$ as $\delta \to 0$, by its choice in (8.10), we can choose δ smaller if necessary so that the asymptotic slope is at least $1 + \gamma_d|\xi|^{-1}(k - 2\varepsilon)\delta^{d-1}$.

For $d \ge 3$, we take $\Lambda = 2$, and as before compare the expansions on $|x| = \rho$:

$$\delta w_{in}(\delta^{-1}x) = x_d + \delta s + k\delta^{d-1}\rho^{2-d} + O(\delta^d\rho^{1-d})$$

and

$$w_{out}(x) = (1-\alpha)\left[x_d + \delta(s + s_\Delta)\right] + \delta^{d-1}(k-\varepsilon)[-\rho^{2-d} + O(1 + \delta\rho^{1-d})].$$

So, still evaluating on $|x| = \rho$,

$$\begin{aligned} w_{out}(x) - \delta w_{in}(\delta^{-1}x) &= \delta\left[\varepsilon(\delta/\rho)^{d-2} + s_\Delta\right] + O(\alpha\rho + \delta^{d-1} + \delta^d\rho^{1-d}) \\ &=: A(\rho) + E(\rho). \end{aligned}$$

As in the $d = 2$ case it will suffice to show

$$A(2r) < 0,\ A(r/2) > 0, \quad \text{and} \quad |E(\rho)| < |A(\rho)| \quad \text{for} \quad \rho \in \{r/2, 2r\}, \tag{8.12}$$

then we will achieve (8.7) and (8.8).

Let us now choose $s_\Delta := -\varepsilon(\delta/r)^{d-2}$ so that

$$A(\rho) = \delta^{d-1}\varepsilon\left[\tfrac{1}{\rho^{d-2}} - \tfrac{1}{r^{d-2}}\right]$$

and

$$A(r/2) = (2^{d-2}-1)\varepsilon\delta^{d-1}r^{2-d} > 0 \quad \text{and} \quad A(2r) = -(1-2^{2-d})\varepsilon\delta^{d-1}r^{2-d} < 0.$$

Indeed we have

$$\min\{|A(r/2)|, |A(r)|\} \geq c_1\varepsilon\delta^{d-1}r^{2-d}$$

and the error term bound

$$\begin{aligned} |E| &\leq C(\delta^{d-1} + \alpha r + \delta^d r^{1-d}) \\ &\leq C(\delta^{d-1} + (\delta/r)^{2(d-1)}r + \delta^d r^{1-d}) \\ &= C_2\delta^{d-1}(1 + \delta^{d-1}r^{3-2d} + \delta r^{1-d}). \end{aligned} \tag{8.13}$$

Now choose r sufficiently small so that $c_1\varepsilon r^{2-d} \geq 2C_2$, and then choose δ sufficiently small so that $C_2(\delta^{d-1}r^{3-2d} + \delta r^{1-d}) \leq C_2$. We then achieve (8.12) to conclude. □

Proof of Proposition 8.1. By Theorem 5.4 (in $d = 2$) or Theorem 6.1 (in $d \geq 3$), for any $\varepsilon > 0$ there is a proper solution w_{in} of (1.4) with capacity $k \geq k_{\rm adv}(e) - \varepsilon$. Lemma 8.3 in turn delivers, for all sufficiently small $\delta > 0$, a $\mathcal{Z}$-periodic supersolution $w_{per}(x)$ of (1.1) with asymptotic slope at least $1 + \gamma_d|\xi|^{-1}(k-\varepsilon)\delta^{d-1}$. Then, by the definition of $Q^\delta_{\rm adv}(e)$ in Definition 7.1, we have

$$Q^\delta_{\rm adv}(e) \geq 1 + \gamma_d|\xi|^{-1}(k_{\rm adv}(e) - 2\varepsilon)\delta^{d-1}.$$

Rearranging, taking first the limit $\delta \to 0$, and then $\varepsilon \to 0$ gives the desired bound for $Q^\delta_{\rm adv}(e)$. The bound of $Q^\delta_{\rm rec}(e)$ is proved symmetrically with the subsolution counterpart of ω_{per}. □

9. Asymptotic upper bound

Now we complete the proof of Theorem 1.1 by showing the upper bound part (7.2) of the asymptotic expansion.

Proposition 9.1. *Let $\xi \in \mathbb{Z}^d$ irreducible and $e = \xi/|\xi|$ then*

$$\limsup_{\delta\to 0} \frac{Q^\delta_{\rm adv}(e) - 1}{\delta^{d-1}} \leq |\xi|^{-1}k_{\rm adv}(e) \quad \text{and} \quad \liminf_{\delta\to 0} \frac{1 - Q^\delta_{\rm rec}(e)}{\delta^{d-1}} \leq -|\xi|^{-1}k_{\rm rec}(e).$$

Along the route to prove this Proposition we will also prove Theorem 1.6.

9.1. **Set-up and normalization.** Throughout this section u_δ will denote a strong Birkhoff plane-like solution of (1.1) with asymptotic slope $\alpha = \alpha_\delta = 1 + \mu_\delta \in [Q^\delta_{\rm rec}, Q^\delta_{\rm adv}]$. The existence of such solutions is guaranteed by Theorem 7.3 part (b) with $Q(x) = Q_\delta(x) = 1 + \sum_{z\in\mathcal{Z}} q(\frac{x-z}{\delta})$. In particular, due to (7.6), there a constant $C(d,q)$ independent of δ and ξ such that

$$(1+\mu_\delta)(x_d - C)_+ \leq u_\delta(x) \leq (1+\mu_\delta)(x_d + C)_+. \tag{9.1}$$

We will denote

$$\Omega_\delta := \{u_\delta > 0\}.$$

Remark 9.2. For most of this section we do *not* need to assume that $1+\mu_\delta$ is one of the extremal slopes, however we will assume that $\mu_\delta \neq 0$. As we will see, this is sufficient to prove Proposition 9.1 and Theorem 1.6.

Define

$$s_\delta^+ := \inf\{s : u_\delta(x) \leq (1+\mu_\delta)(x_d+s)_+\} \text{ and}$$
$$s_\delta^- := \sup\{s : (1+\mu_\delta)(x_d+s)_+ \leq u_\delta(x)\}. \tag{9.2}$$

Since u_δ are plane-like in the sense of (7.6), $s_\delta^\pm$ are well-defined finite values, bounded independent of δ. One can also check that $(1+\mu_\delta)(x_d+s_\delta^\pm)_+$ must touch u_δ from above/below at the free boundary $\partial\Omega_\delta$: by maximum principle and periodicity it remains to rule out infimum occurring only at $x_d=\infty$, which would contradict the definition of $s_\delta^\pm$ by Harnack inequality.

Lemma 9.3. *Let u_δ, μ_δ and $s_\delta^\pm$ as given above. If $\mu_\delta > 0$ (resp. $\mu_\delta < 0$) then, up to a lattice translation,*

$$-\delta \leq s_\delta^- \text{ (resp. } s_\delta^+ \leq \delta) \text{ and } \partial\Omega_\delta \cap \overline{B_\delta(0)} \neq \emptyset. \tag{9.3}$$

Proof. We will prove for $\mu_\delta > 0$: the other is symmetric. Since $\mu_\delta > 0$ we have $(x_d+s_\delta^-)_+ \leq (1+\mu_\delta)(x_d+s_\delta^-)_+$, so $(x_d+s_\delta^-)_+$ touches u_δ from below at the same free boundary point where $(1+\mu_\delta)(x_d+s_\delta^-)_+$ touches from below. By a lattice translation we can assume that this touching point is in $[-1/2,1/2)^d$. If the touching point is not in $\overline{B_\delta(0)}$ then the strong maximum principle Lemma B.2 implies that $u_\delta \equiv (x_d+s_\delta^-)_+$, contradicting (9.1). □

Remark 9.4. Since we have assumed that $\mu_\delta \neq 0$ the hypothesis of Lemma 9.3 is satisfied. So, for the remainder of the section we will assume that u_δ is properly translated so that (9.3) holds.

9.2. **Initial $o_\delta(1)$-flatness.** Our first result constraints the defect sites intersecting $\partial\{u_\delta > 0\}$ to only those $o_\delta(1)$-close to $\{x_d = 0\}$ – recalling that we have normalized by a lattice translation so that (9.3) holds.

Lemma 9.5. *Let u_δ be a sequence of strong Birkhoff solutions satisfying the hypotheses described in Section 9.1. Then $\mu_\delta \to 0$ and $s_\delta^\pm \to 0$ as $\delta \to 0$.*

Proof. Due to Lemma A.9, u_δ are uniformly Lipschitz continuous. Every subsequence of $\delta \to 0$ has a further subsequence (not re-labeled) so that u_δ converges locally uniformly to some $\mathcal{Z}$-periodic w, $\mu_\delta \to \mu_0$ as $\delta \to 0$, and also μ_δ has a fixed sign along this subsequence. To conclude, it is enough to show that $\mu_\delta, s_\delta^\pm \to 0$ along every such subsequence. To show this we claim:

$$w \text{ solves the homogeneous Bernoulli problem (2.2).} \tag{9.4}$$

Let us postpone the proof of (9.4) for now and complete the proof of the Lemma.

We will argue for the case $\mu_\delta > 0$ and $\mu_0 \geq 0$: the other case is similar. Taking the limit of (9.1) and (9.3)

$$(1+\mu_0)(x_d)_+ \leq w(x) \leq (1+\mu_0)(x_d+C)_+ \text{ and } 0 \in \partial\{w>0\}. \tag{9.5}$$

If $\mu_0 > 0$ then, by (9.5), the strict subsolution $(1+\mu_0)(x_d)_+$ touches w from below at 0, contradicting (9.4). Thus $\mu_0 = 0$. Then applying the strong maximum principle Lemma B.2 part (iii) we conclude that $w(x) \equiv (x_d)_+$. Since $\mu_\delta > 0$ we already know that $s_\delta^- \to 0$, thus it remains to show that $s_\delta^+ \to 0$. Recall that $(1+\mu_\delta)(x_d+s_+^\delta)_+$

touches u_δ from above at some $x_\delta \in \partial\Omega_\delta \cap B_C(0)$. Then x_δ is an outer-regular point, so non-degeneracy at outer regular free boundary points (Lemma A.10 (v)), and local uniform convergence of $u_\delta(x) \to (x_d)_+$ implies that $s^\delta_+ \to 0$.

We conclude that $\mu_\delta \to 0$ and $s^\pm_\delta \to 0$ along the full sequence.

Now we proceed to prove (9.4): we only need to check the free boundary condition. We argue for the supersolution condition: the other case is parallel. Consider a test function φ touching w strictly from below at $x_0 \in \partial\{w > 0\}$ with $\Delta\varphi(x_0) > 0$. The only interesting case is when $x_0 \in \mathbb{Z}^d$, so we may assume that $x_0 = 0$. Then there is a sequence $x_\delta \to 0$ and $c_\delta \to 0$ so that $\varphi(x) + c_\delta$ touches u_δ from below at x_δ. Since $\Delta\varphi(x_\delta) > 0$, $x_\delta \in \partial\Omega_\delta$ and $c_\delta = -\varphi(x_\delta)$. We can conclude now except when $x_\delta \in B_\delta(0)$ for all $\delta > 0$. In this case consider the blow-up sequences

$$\varphi_\delta(y) := \frac{1}{\delta}(\varphi(x_\delta + \delta y) - \varphi(x_\delta)) \ \text{ and } \ \tilde{w}_\delta(y) := \frac{1}{\delta}u_\delta(x_\delta + \delta y)$$

with φ_δ touching $\tilde{w}_\delta$ from below at $0 \in \partial\{\tilde{w}_\delta > 0\}$. As $\delta \to 0$, since φ is C^1,

$$\varphi_\delta(y) \to \nabla\varphi(0) \cdot y.$$

Along a further subsequence, we may assume that $x_\delta/\delta \to \tau \in B_1(0)$ and $\tilde{w}_\delta$ converges locally uniformly to $\tilde{w}$ solving the single-site problem

$$\Delta\tilde{w} = 0 \ \text{ in } \ \{\tilde{w} > 0\} \ \text{ with } \ |\nabla\tilde{w}| = Q(\tau + y).$$

Finally, since $\varphi_\delta \le \tilde{w}_\delta$, we find that

$$\tilde{w}_\delta(y) \ge \nabla\varphi(0) \cdot y.$$

If $|\nabla\varphi(x_0)| \le 1$ we are done, while if $|\nabla\varphi(x_0)| \ge 1$ then $\tilde{w}$ is one-sided flat in the sense of (2.4):

$$\tilde{w}(y) \ge (\nabla\varphi(x_0) \cdot y)_+ \ge (e \cdot y)_+ \ \text{ with } \ e := \frac{\nabla\varphi(x_0)}{|\nabla\varphi(x_0)|}.$$

Hence Proposition 2.2 applies and we know the blow-down is of the form

$$\lim_{r\to\infty} \frac{1}{r}\tilde{w}(ry) = (e \cdot y)_+ \ \text{ locally uniformly in } \ y \cdot e \ge 0.$$

but

$$(\nabla\varphi(x_0) \cdot y)_+ \le \lim_{r\to\infty} \frac{1}{r}\tilde{w}(ry) = (e \cdot y)_+ \ \text{ in } \ y \cdot e \ge 0$$

and so $|\nabla\varphi(x_0)| \le 1$.

□

Remark 9.6. From Theorem 7.3 part (b) we can derive at this point:

$$\lim_{\delta\to 0} Q^\delta_{\text{adv}} = \lim_{\delta\to 0} Q^\delta_{\text{rec}} = 1.$$

9.3. **Refined flatness and fundamental solution bounds in $d = 2$.** Next we improve the flatness in Lemma 9.5 to the optimal scale $O(|\mu_\delta| + \delta)$ and establish the asymptotic behavior of u_δ near the holes. The proof for $d = 2$ will be presented here, and the next section will address $d \ge 3$.

Proposition 9.7. *Let $d = 2$ and let u_δ and μ_δ be as given in Section 9.1. Then there is $s_\delta = O(|\mu_\delta| + \delta) \to 0$ so that if we define*

$$\omega_\delta(x) := \frac{1}{\delta}[(1 + \mu_\delta)(x_d + s_\delta) - u_\delta(x)],$$

then in the case $\mu_\delta > 0$

$$0 \leq \omega_\delta(x) \leq \kappa_{\mathrm{adv}}^{\frac{1}{2\delta}}(\log \tfrac{1}{|x|})_+ + C(1 + \tfrac{\mu_\delta}{\delta}) \quad \textit{for } x \in (\square_{\mathcal{Z}} \times \mathbb{R}) \cap \Omega_\delta,$$

and, in the case $\mu_\delta < 0$,

$$0 \geq \omega_\delta(x) \geq \kappa_{\mathrm{rec}}^{\frac{1}{2\delta}}(\log \tfrac{1}{|x|})_+ - C(1 + \tfrac{|\mu_\delta|}{\delta}) \quad \textit{for } x \in (\square_{\mathcal{Z}} \times \mathbb{R}) \cap \Omega_\delta.$$

Recall that $\kappa^R_{\mathrm{adv/rec}}$ *satisfies, by Theorem 5.4,* $\lim_{R\to\infty} \kappa^R_{\mathrm{adv/rec}} = k_{\mathrm{adv/rec}}$.

We start with a Lemma to establish $O(|\mu_\delta| + \delta)$ flatness away from the defects.

Lemma 9.8. *Under the hypotheses of Proposition 9.7, if* $\mu_\delta > 0$ *then*

$$0 \leq (1 + \mu_\delta)(x_d + s_\delta^+) - u_\delta(x) \leq C(\delta + \mu_\delta) \quad \textit{on} \quad \Omega_\delta \setminus \cup_{z\in\mathcal{Z}} B_{1/2}(z), \tag{9.6}$$

where $C > 1$ *is some constant. A symmetrical result holds if* $\mu_\delta < 0$ *with* s_δ^-.

Proof. Let's assume that $\mu_\delta > 0$, the case $\mu_\delta < 0$ is analogous. By Lemma 9.3,

$$(1 + \mu_\delta)(x_d - \delta) \leq u_\delta(x) \leq (1 + \mu_\delta)(x_d + s_\delta^+), \tag{9.7}$$

which, in particular, yields the lower bound in (9.6). By the discussion from Section 9.1 (see the paragraph above Lemma 9.3) there is $x^0 \in \partial\Omega_\delta \cap \boxdot^{\mathcal{Z}}$ (see notation defined in (7.4)) with $x^0_d + s^+_\delta = 0$. For the remainder of the proof we denote $s_\delta = s^+_\delta$.

Recall $\boxdot^{\mathcal{Z}}_r := r\boxdot^{\mathcal{Z}}$.

$$S := \max_{\boxdot^{\mathcal{Z}} \cap \Omega_\delta} (u_\delta(x) - x_d). \tag{9.8}$$

Since $x^0 \in \partial\Omega_\delta \cap \boxdot^{\mathcal{Z}}$ and $u_\delta(x^0) = 0$,

$$S \geq u_\delta(x^0) - x^0_d = -x^0_d = s_\delta. \tag{9.9}$$

Note that $(x_d+S)_+$ touches u_δ from above in $\boxdot^{\mathcal{Z}} \cap \Omega_\delta$ at x^1, where the maximum in (9.9) is achieved. Since $u_\delta(x)-x_d$ is harmonic in $\Omega_\delta \cap \boxdot^{\mathcal{Z}}$, either (i) $x^1 \in \boxdot^{\mathcal{Z}} \cap \partial\Omega_\delta$ or (ii) $x^1 \in \Omega_\delta \cap \partial\boxdot^{\mathcal{Z}}$.

First suppose (i). Then $(x_d + S)_+$ touches u_δ from above at x^1 in an open neighborhood. If $x^1 \notin \overline{B_\delta(0)}$ then Lemma B.2 implies $u_\delta(x) \equiv (x_d + S)_+$ which contradicts $\mu_\delta > 0$. If $x^1 \in B_\delta(0) \cap \partial\Omega_\delta$ then $S \leq \delta$. Due to (9.9) $s_\delta = -x^0_d \leq \delta$. Since(9.7) yields $(1 + \mu_\delta)x_d - u_\delta(x) \leq C\delta$ in Ω_δ, we conclude (9.6) in case of (i).

It remains now to consider the case (ii). We will aim to apply Harnack inequality, Corollary 2.8, in an annular domain $\boxdot^{\mathcal{Z}}_{3/2} \setminus B_{1/4}$. Note

$$\max_{\boxdot^{\mathcal{Z}}_{3/2} \cap \Omega_\delta} (u_\delta(x) - x_d) \leq \max_{\boxdot^{\mathcal{Z}}_{3/2} \cap \Omega_\delta} (u_\delta(x) - (1 + \mu_\delta)x_d) + \mu_\delta \max_{\boxdot^{\mathcal{Z}}_{3/2} \cap \Omega_\delta} x_d \leq s_\delta + C_0\mu_\delta,$$

So

$$(x_d - \delta)_+ \leq u_\delta(x) \leq (x_d + s_\delta + C_0\mu_\delta)_+ \quad \text{in} \quad \Omega_\delta \cap \boxdot^{\mathcal{Z}}_{3/2}.$$

Due to Lemma 9.5, $s_\delta + C\mu_\delta \to 0$ as $\delta \to 0$, and thus the Harnack inequality Corollary 2.8 applies for sufficiently small δ. Call $A := \boxdot^{\mathcal{Z}}_1 \setminus B_{1/2}$. Since $\boxdot^{\mathcal{Z}}_{3/2} \setminus B_{1/4} \supset\supset A$, Corollary 2.8 implies that

$$\max_{\Omega_\delta \cap A} (x_d + s_\delta + C_0\mu_\delta - u_\delta(x)) \leq C \min_{\Omega_\delta \cap A} (x_d + s_\delta + C_0\mu_\delta - u_\delta(x)). \tag{9.10}$$

Utilizing (ii) to plug in $x^1 \in \partial\boxdot^{\mathcal{Z}}_1 \subset A$ on the right hand side yields

$$\min_{\Omega_\delta \cap A} (x_d + s_\delta + C_0\mu_\delta - u_\delta(x)) \leq s_\delta + C_0\mu_\delta - S \leq C_0\mu_\delta,$$

using (9.9) for the last inequality. Plugging this back into (9.10) we get

$$\max_{\Omega_\delta \cap A}(x_d + s_\delta - u_\delta(x)) \leq C\mu_\delta. \tag{9.11}$$

Lastly we look for the upper bound in $\{x_d \geq 1\}$. Note that Lemma 9.5 ensures $\{x_d \geq 1\} \subset \Omega_\delta$ for $\delta << 1$. Since $\square_{\mathcal{Z}} \times \{1\} \subset A$, using $\mathcal{Z}$-periodicity of u_δ,

$$\begin{aligned}\max_{\{x_d=1\}}((1+\mu_\delta)(x_d+s_\delta) - u_\delta(x)) &= \max_{\square_{\mathcal{Z}}\times\{1\}}((1+\mu_\delta)(x_d+s_\delta) - u_\delta(x)) \\ &\leq \max_{\Omega_\delta\cap A}((1+\mu_\delta)(x_d+s_\delta) - u_\delta(x)) \\ &\leq C\mu_\delta,\end{aligned}$$

where the last line is due to (9.11). Since $(1+\mu_\delta)(x_d+s_\delta) - u_\delta(x)$ is bounded harmonic function in $\{x_d > 1\}$, the maximum principle in this domain implies that

$$\max_{\{x_d\geq 1\}}((1+\mu_\delta)(x_d+s_\delta) - u_\delta(x)) \leq C\mu_\delta.$$

Combining this with (9.11) gives the upper bound in $\Omega_\delta \setminus \cup_{z\in\mathcal{Z}} B_{1/2}(z)$. □

Proof of Proposition 9.7. We prove the case $\mu_\delta > 0$: the other case is symmetrical. By Lemma 9.8, $s_\delta = s_\delta^+$ satisfies

$$u_\delta(x) \geq x_d + s_\delta - C_0(\mu_\delta + \delta) \text{ on } \partial B_{1/2}(0) \cap \Omega_\delta.$$

Define

$$R_\delta := \frac{1}{2\delta} \text{ and } k_\delta := \frac{1}{\delta \log\frac{1}{2\delta}}[s_\delta - C_0(\mu_\delta + \delta)], \tag{9.12}$$

and

$$\varphi(x) := \delta u_{\text{adv}}^{k_\delta, R_\delta}(\tfrac{x}{\delta}),$$

where $u_{\text{adv}}^{k,R}$ is from Definition 5.1. Due to the definition of $u_{adv}^{k,R}$ and k_δ,

$$\varphi(x) = (x_d + \delta k_\delta \log\tfrac{|x|}{\delta})_+ = (x_d + s_\delta - C_0(\mu_\delta+\delta))_+ \text{ on } \partial B_{1/2}(0).$$

Since φ is the minimal supersolution of (7.5) in $B_{1/2}(0)$ with this boundary data, $\varphi(x) \leq u_\delta(x)$ in $B_{1/2}(0)$. Since $\{u_\delta = 0\} \cap \overline{B_\delta(0)} \neq \emptyset$ and so, since $\varphi \leq u_\delta$, also $\{\varphi = 0\} \cap \overline{B_\delta(0)} \neq \emptyset$. Therefore, by the definition of $\kappa_{\text{adv}}^{R_\delta}$ in Definition 5.1,

$$k_\delta \leq \kappa_{\text{adv}}^{R_\delta}. \tag{9.13}$$

By Lemma 5.9

$$\varphi(x) \geq x_d + k_\delta \delta \log\tfrac{|x|}{\delta} - C_1\delta \text{ in } \{\varphi > 0\}.$$

So we obtain the desired upper bound in $B_{1/2}(0) \cap \{\varphi > 0\}$:

$$\begin{aligned}\omega_\delta(x) &\leq \frac{1}{\delta}[(1+\mu_\delta)(x_d+s_\delta) - \varphi(x)] \\ &\leq \frac{1}{\delta}[(1+\mu_\delta)(x_d+s_\delta) - (x_d + k_\delta\delta\log|x| + k\delta\log\tfrac{1}{\delta} - C_1\delta)] \\ &= \frac{1}{\delta}[k_\delta\delta\log\tfrac{1}{|x|} + (s_\delta - k_\delta\delta\log\frac{1}{\delta}) + C_1\delta + \mu_\delta(x_d+s_\delta)] \\ &= \kappa_{\text{adv}}^{R_\delta}\log\tfrac{1}{|x|} + (C_0+C_1) + \delta^{-1}\mu_\delta(x_d+s_\delta).\end{aligned}$$

For the last equation we used (9.13) to bound k_δ in the first term, and the definition of k_δ in (9.12) for the middle term in parenthesis.

Combining this upper bound inside of $B_{1/2}(0)$ with periodicity and the bounds from Lemma 9.8 outside of $\cup_{z\in\mathcal{Z}} B_{1/2}(z)$, we obtain the result. $\square$

9.4. **Refined flatness and fundamental solution bounds in $d \geq 3$.**

Proposition 9.9. *Let $d \geq 3$, and let μ_δ and u_δ as given in Section 9.1. Then there is $s_\delta \to 0$ so that if we call*

$$\omega_\delta(x) := \frac{1}{\delta^{d-1}}[(1+\mu_\delta)(x_d + s_\delta) - u_\delta(x)],$$

then for $x \in (\square_{\mathcal{Z}} \times \mathbb{R}) \cap \Omega_\delta$,

$$k_{\rm rec}|x|^{2-d} - C(1 + \delta|x|^{1-d} + \tfrac{|\mu_\delta|}{\delta^{d-1}}) \leq \omega_\delta(x) \leq k_{\rm adv}|x|^{2-d} + C(1 + \delta|x|^{1-d} + \tfrac{|\mu_\delta|}{\delta^{d-1}}).$$

In particular, $s_\delta = O(\delta^{d-1} + |\mu_\delta|)$.

We expect that the maximum and minimum heights $s_\delta^\pm$ defined in (9.2) are at least $O(\delta)$, since this level of oscillation can be achieved just within the defect. However for $d \geq 3$ we expect the oscillations of u_δ away from the defect to be of lower order, δ^{d-1}. This requires more precise choice of height s_δ, in contrast to the $d = 2$ case where it was sufficient to choose the height as one of $s_\delta^\pm$.

Proof. The argument will proceed in three steps. First we obtain the bounds in $\boxdot^{\mathcal{Z}}$ (defined in (7.4)), using the single-site barriers constructed in Theorem 6.1. Next we extend these to global bounds by using the asymptotic growth of $u_\delta(x) = (1+\mu_\delta)x_d + O(1)$ as $x_d \to +\infty$ and using the $\mathcal{Z}$-periodicity. Finally we use Harnack inequality and Lemma 9.5 to conclude.

Let $u_{\rm adv/rec}(x;s)$ be from Theorem 6.1. By Theorem 6.1

$$|u_{\rm adv/rec}(x;s) - (x_d + s - \kappa_{\rm adv/rec}(s)|x|^{2-d})| \leq C_1|x|^{1-d} \tag{9.14}$$

with $\sup_s |\kappa_{\rm adv/rec}(s)|$ and C_1 having universal bounds. Therefore we can choose a large universal constant C_0 so that

$$2 \max_{x\in\partial\boxdot^{\mathcal{Z}}} \left[\max\{k_{\rm adv}, |k_{\rm rec}|\}|x|^{2-d} + C_1\delta|x|^{1-d}\right] \leq C_0 \quad \text{for all} \quad 0 < \delta < 1/2.$$

Define the candidates for s_δ as

$$\begin{aligned} t_\delta^+ &:= \max_{(\boxdot^{\mathcal{Z}}\setminus B_{1/2})\cap\Omega_\delta} \{u_\delta(x) - x_d\} + C_0\delta^{d-1}, \quad \text{and} \\ t_\delta^- &:= \min_{(\boxdot^{\mathcal{Z}}\setminus B_{1/2})\cap\Omega_\delta} \{u_\delta(x) - x_d\} - C_0\delta^{d-1}. \end{aligned} \tag{9.15}$$

Now we focus on the bounds in $\boxdot^{\mathcal{Z}}$. We will argue only for the lower bound of ω_δ: the proof of the other bound is symmetric. Define the barrier

$$\varphi(x) := \delta u_{\rm rec}(\tfrac{x}{\delta}; \tfrac{t_\delta^+}{\delta}) \quad \text{and} \quad k_\delta := \kappa_{\rm rec}(\tfrac{t_\delta^+}{\delta}).$$

Evaluating on $\overline{\Omega_\delta} \cap \partial\boxdot^{\mathcal{Z}}$ and using (9.14),

$$\begin{aligned} \varphi(x) = \delta u_{\rm rec}(\tfrac{x}{\delta}; \tfrac{t_\delta^+}{\delta}) &\geq x_d + t_\delta^+ - k_\delta\delta^{d-1}|x|^{2-d} - C_1\delta^d|x|^{1-d} \\ &\geq x_d + \max_{\partial\boxdot^{\mathcal{Z}}\cap\Omega_\delta}\{u_\delta(x) - x_d\} + C_0\delta^{d-1} - \frac{1}{2}C_0\delta^{d-1} \\ &> x_d + \max_{\partial\boxdot^{\mathcal{Z}}\cap\Omega_\delta}\{u_\delta(x) - x_d\} \\ &\geq u_\delta(x). \end{aligned}$$

Thus $\varphi > u_\delta$ on $\overline{\Omega_\delta} \cap \partial \boxdot^{\mathcal{Z}}$. Since φ is a maximal subsolution of (1.1) in $\square^{\mathcal{Z}}$, we conclude that $u_\delta \leq \varphi$ in $\boxdot^{\mathcal{Z}}$ or

$$u_\delta(x) \leq x_d + t_\delta^+ + \delta^{d-1}[-k_\delta |x|^{2-d} + C_0 + C_1 \delta |x|^{1-d}] \ \text{ in } \boxdot^{\mathcal{Z}}. \tag{9.16}$$

Also by definition $-k_\delta \leq -\min_s \kappa_{\mathrm{rec}}(s) = k_{\mathrm{rec}}$.

Now we proceed to obtain global bounds. By the $\mathcal{Z}$-periodicity of u_δ and Lemma C.1,

$$\begin{aligned}\max_{\{x_d \geq 1\}} (u_\delta(x) - (1+\mu_\delta)x_d) &= \max_{\{x_d = 1\}} (u_\delta(x) - (1+\mu_\delta)x_d) \\ &= \sup_{\square_{\mathcal{Z}} \times \{1\}} (u_\delta(x) - (1+\mu_\delta)x_d) \leq t_\delta^+ + (-\mu_\delta)_+,\end{aligned}$$

Symmetrical arguments for the minimum imply that

$$t_\delta^- - (\mu_\delta)_+ \leq u_\delta(x) - (1+\mu_\delta)x_d \leq t_\delta^+ + (-\mu_\delta)_+ \ \text{ in } \{\, x_d \geq 1\}. \tag{9.17}$$

Thus, combining (9.16) (in $x_d \leq 1$) and (9.17) (in $x_d \geq 1$) yields the global bounds

$$u_\delta(x) \leq (1+\mu_\delta)(x_d + t_\delta^+) + \delta^{d-1}[-k_{\mathrm{rec}}|x|^{2-d} + C + C\delta|x|^{1-d}] + C|\mu_\delta| \ \text{ in } \Omega_\delta \cap [\square_{\mathcal{Z}} \times \mathbb{R}] \tag{9.18}$$

and

$$u_\delta(x) \geq (1+\mu_\delta)(x_d + t_\delta^-) + \delta^{d-1}[-k_{\mathrm{adv}}|x|^{2-d} - C - C\delta|x|^{1-d}] - C|\mu_\delta| \ \text{ in } \Omega_\delta \cap [\square_{\mathcal{Z}} \times \mathbb{R}]. \tag{9.19}$$

Finally, we aim to apply the Harnack inequality Corollary 2.8 to u_δ in the "annulus" domain $A := \boxdot^{\mathcal{Z}}_{3/2} \setminus \boxdot^{\mathcal{Z}}_{1/2}$. Note that u_δ solves the homogeneous Bernoulli problem (2.2) in A and satisfies, from (9.19) above,

$$u_\delta(x) \geq x_d + t_\delta^- - C(\delta^{d-1} + |\mu_\delta|) \ \text{ in } \ \Omega_\delta \cap A.$$

By Lemma 9.5, Corollary 2.8 applies to u_δ in A and we have

$$\sup_{\Omega_\delta \cap \partial \boxdot^{\mathcal{Z}}} (u_\delta(x) - (x_d + t_\delta^- - C(\delta^{d-1} + |\mu_\delta|))) \leq C \inf_{\Omega_\delta \cap \partial \boxdot^{\mathcal{Z}}} (u_\delta(x) - (x_d + t_\delta^- - C(\delta^{d-1} + |\mu_\delta|))).$$

Plugging in the point on $\partial \boxdot^{\mathcal{Z}}$ where $u_\delta(x) - x_d + C_0\delta^{d-1} = t_\delta^+$ is achieved on the left, and the point where $u_\delta(x) - x_d - C_0\delta^{d-1} = t_\delta^-$ is achieved on the right gives

$$t_\delta^+ - t_\delta^- \leq C(\delta^{d-1} + |\mu_\delta|).$$

Taking, for example, $s_\delta = t_\delta^-$ we conclude by the above estimate, (9.18), and (9.19). $\square$

9.5. **A rescaling limit leading to the perforated domain cell problem.** We are finally ready to prove Proposition 9.1 and Theorem 1.6 in this section.

Let ω_δ and s_δ be given from Propositions 9.7 and 9.9. At the moment we do not yet have an upper bound on μ_δ/δ^{d-1}, thus we will first consider the limit of

$$\tilde{\omega}_\delta(x) := \frac{\delta^{d-1}}{\mu_\delta}\omega_\delta(x) = \frac{1}{\mu_\delta}[(1+\mu_\delta)(x_d + s_\delta) - u_\delta(x)]. \tag{9.20}$$

The main technical result of this section is on the precompactness of the $\tilde{\omega}_\delta$ and the limiting PDE boundary value problem.

Lemma 9.10. *Consider a sequence $\delta \to 0$ such that $\mu_\delta \neq 0$ and $\frac{\mu_\delta}{\delta^{d-1}} \to \lambda \in [-\infty, \infty] \setminus \{0\}$. Then the $\tilde{\omega}_\delta$ are locally uniformly bounded and $C^{1,\alpha}$ in $\mathbb{R}^d_+ \setminus \mathcal{Z}$, and any subsequential limit of $\tilde{\omega}_\delta$, ω, is a $\mathcal{Z}$-periodic solution of*

$$\begin{cases} \Delta\omega = 0 & \text{in } \mathbb{R}^d_+ \\ \partial_{y_d}\omega = 1 & \text{on } \partial\mathbb{R}^d_+ \setminus \mathcal{Z}, \\ \langle \partial_{y_d}\omega(\cdot, y_d)\rangle' = 0 & \text{for } y_d > 0. \end{cases} \tag{9.21}$$

Moreover when $\lambda > 0$, with the notation $(\infty)^{-1} = 0$,

$$\begin{cases} \limsup_{x\to z} \dfrac{\omega(x)}{\Phi(x-z)} \leq k_{\text{adv}}\lambda^{-1} & \text{for } z \in \mathcal{Z}, \\ \liminf_{x\to z} \dfrac{\omega(x)}{\Phi(x-z)} \geq k_{\text{rec}}\lambda^{-1} & \text{for } z \in \mathcal{Z}. \end{cases} \tag{9.22}$$

When $\lambda < 0$, the above holds with the upper bound $k_{\text{rec}}|\lambda|^{-1}$ and the lower bound $k_{\text{adv}}|\lambda|^{-1}$.

For clarity of presentation, we postpone the proof of the Lemma until the end of the section to finish the proofs of Proposition 9.1 and Theorem 1.6.

Proof of Proposition 9.1 and Theorem 1.6. We will only argue for Q^δ_{adv}: the other case is similar.

First we prove Proposition 9.1. Let u_{δ_j} be a sequence of strong Birkhoff solutions with slopes $1 + \mu_{\delta_j}$ such that $\frac{\mu_{\delta_j}}{\delta_j^{d-1}} \to \lambda \in [0, +\infty]$. To prove Proposition 9.1 it is enough to show that

$$\lambda \leq \gamma_d|\xi|^{-1}k_{\text{adv}}. \tag{9.23}$$

If $\lambda = 0$ we conclude since $k_{\text{adv}} \geq 0$. If $k_{\text{adv}} = +\infty$ the claim is also trivial, so we can assume that $\lambda > 0$ and $k_{\text{adv}} < +\infty$ apply Lemma 9.10. Let ω be as given in as in Lemma 9.10 and define

$$A := k_{\text{rec}}\lambda^{-1} \quad \text{and} \quad B := k_{\text{adv}}\lambda^{-1}.$$

By (9.22), Lemma 7.4, and $\mathcal{Z}$-periodicity there is some $c_* \in [\gamma_d A, \gamma_d B]$ such that ω solves (7.9) and (7.8) and then Theorem 7.5 implies that

$$c_* = |\square_{\mathcal{Z}}| = |\xi|.$$

Therefore

$$|\xi| \leq \gamma_d B = \gamma_d k_{\text{adv}}\lambda^{-1} \tag{9.24}$$

yielding (9.23). We have now proved Proposition 9.1.

Note that, in particular, (9.24) implies that λ is finite, which allows us to consider the original error function ω_δ. Since the solution of (7.9) is unique modulo constants, it follows that the full sequence $\tilde{\omega}_{\delta_j}$ converges locally uniformly in $\overline{\mathbb{R}^d_+} \setminus \mathcal{Z}$ to ω (modulo constants). Thus

$$\omega_{\delta_j} = \frac{\mu_{\delta_j}}{\delta_j^{d-1}}\tilde{\omega}_{\delta_j} \to \lambda\omega \ \text{ locally uniformly (modulo constants) in } \ \mathbb{R}^d_+ \setminus \mathcal{Z}. \tag{9.25}$$

We now take as assumption that $0 < k_{\text{adv}} < +\infty$. Let us choose u_δ to be the minimal supersolutions with $\mu_\delta = Q^\delta_{\text{adv}} - 1$. Then, combining (9.23) with

Proposition 8.1 for the limit infimum, we conclude that

$$\lim_{\delta\to 0}\frac{\mu_\delta}{\delta^{d-1}} = \gamma_d|\xi|^{-1}k_{\rm adv}.$$

(9.25) now yields that

$$\omega_\delta \to \gamma_d|\xi|^{-1}k_{\rm adv}\omega \ \text{ locally uniformly (modulo constants) in } \ \mathbb{R}^d_+\setminus\mathcal{Z},$$

which concludes the proof of Theorem 1.6. □

Remark 9.11. The above proof is given for general μ^δ to also show the convergence of ω_δ for other sequences of strong Birkhoff plane-like solutions (not just with extremal slopes), which exist for every slope α_δ in the pinning interval $[Q^\delta_{\rm rec}, Q^\delta_{\rm adv}]$ by Theorem 7.3, as long as $\frac{\alpha_\delta-1}{\delta^{d-1}}$ converges to a nonzero value.

Finally we return to prove Lemma 9.10.

Proof of Lemma 9.10. First we show compactness and the asymptotic bounds near the holes. By hypothesis $\lambda_\delta := \frac{\mu_\delta}{\delta^{d-1}} \to \lambda \neq 0$ (but possibly equal to $\pm\infty$), so we may assume that μ_δ has a fixed sign and λ_δ^{-1} are bounded. We will continue in the case $\mu_\delta > 0$: the other case is similar.

By multiplying $\lambda_\delta^{-1} = \frac{\delta^{d-1}}{\mu_\delta} > 0$ on the bounds from Proposition 9.7 (in $d = 2$) or Proposition 9.9 (in $d \geq 3$), we have

$$|\tilde\omega_\delta| \leq C(1+\lambda_\delta^{-1}) \ \text{ in } \ \Omega_\delta\setminus\cup_{z\in\mathcal{Z}}B_{1/2}(z),$$

and the bounds near the defects, in $B_{1/2}(0)\cap\Omega_\delta$:

$$\begin{aligned}\tilde\omega_\delta(x) &\leq \lambda_\delta^{-1}k^\delta_{\rm adv}\Phi(x) + C + C\lambda_\delta^{-1}(1+\delta|x|^{1-d}),\\ \tilde\omega_\delta(x) &\geq \lambda_\delta^{-1}k^\delta_{\rm rec}\Phi(x) - C - C\lambda_\delta^{-1}(1+\delta|x|^{1-d}),\end{aligned} \tag{9.26}$$

where $\lim_{\delta\to 0}k^\delta_{\rm adv/rec} = k_{\rm adv/rec}$. By $\mathcal{Z}$-periodicity translated versions of the bounds hold in $B_{1/2}(z)$ for $z\in\mathcal{Z}$.

Let K be a compact subset of $\mathbb{R}^d\setminus\mathcal{Z}$. The uniform bounds above imply that the sequence $\tilde\omega_\delta$ is uniformly bounded on $K\cap\overline{\Omega}_\delta$, and Lemma 2.4 implies that ω_δ are uniformly $C^{1,\alpha}$ on $K\cap\overline{\Omega}_\delta$. Note that, by Lemma 9.5,

$$\{x_d + s^-_\delta > 0\}\subset\Omega_\delta\subset\{x_d+s^+_\delta>0\} \ \text{ with } \ s^\pm_\delta\to 0 \tag{9.27}$$

so Ω_δ is converging to $\mathbb{R}^d_+$ as $\delta\to 0$. Putting these together, (9.22) follows now easily from (9.26) by first fixing $z\in\mathcal{Z}$, taking the limit $\delta\to 0$ and then dividing by $\Phi(x-z)$ and taking the limit $x\to z$.

Next we establish the PDE conditions for ω. Combined with ω_δ being harmonic in Ω_δ and the convergence of Ω_δ from (9.27), we conclude that ω is harmonic in $\mathbb{R}^d_+$.

It remains to establish the Neumann condition away from the holes. The proof follows arguments from De Silva [38, Lemma 4.1]. Consider a smooth test function φ touching ω from above at some $x_0\in\partial\mathbb{R}^d_+\setminus\mathcal{Z}$ with $\Delta\varphi(x_0)<0$. Then there is $\overline{\Omega_\delta}\ni x_\delta\to x_0$ and $c_\delta\to 0$ so that $\varphi+c_\delta$ touches ω_δ from above at x_δ. Since ω_δ is harmonic in Ω_δ, it must be that $x_\delta\in\partial\Omega_\delta$. Recalling the definition of ω_δ,

$$\varphi(x)+c_\delta-\frac{1}{\mu_\delta}((1+\mu_\delta)(x_d+s_\delta)_+ - u_\delta(x)) \ \text{ has a local maximum 0 at } x_\delta,$$

and so if we define

$$\tilde\varphi(x) := (1+\mu_\delta)(x_d+s_1)_+ - \mu_\delta(\varphi(x)+c_\delta),$$

then $u_\delta - \tilde\varphi$ has a local minimum 0 at x_δ. Since $x_0 \notin \mathcal{Z}$, $x_\delta \notin \cup_{z\in\mathcal{Z}} B_\delta(z)$ for $\delta << 1$. Since $Q = 1$ in a local neighborhood of x_δ, from the viscosity supersolution property of u_δ it follows that

$$\begin{aligned} 1 \geq |\nabla\tilde\varphi(x_\delta)|^2 &= |(1+\mu_\delta)e_d - \mu_\delta \nabla\varphi(x_\delta)|^2 \\ &= (1+\mu_\delta)^2 - 2\mu_\delta \partial_d\varphi(x_\delta) + \mu_\delta^2|\nabla\varphi(x_\delta)|^2 \\ &= 1 + 2\mu_\delta(1-\partial_d\varphi(x_\delta)) + \mu_\delta^2|\nabla\varphi(x_\delta)|^2. \end{aligned}$$

Rearranging this gives

$$\partial_d\varphi(x_\delta) \geq 1 + \frac{1}{2}\mu_\delta|\nabla\varphi(x_\delta)|^2$$

and taking the limit as $\delta \to 0$

$$\partial_d\varphi(x_0) \geq 1.$$

Thus ω satisfies viscosity subsolution property for the Neumann condition $\partial_d\omega = 1$ on $\partial\mathbb{R}^d_+ \setminus \mathcal{Z}$. The viscosity supersolution property is proved similarly.

Lastly, note that $(1+\mu_\delta)x_d - u_\delta(x)$ is a bounded harmonic function in Ω_δ, which is periodic with respect to $\mathcal{Z}$ translations. Thus, by Lemma C.1, $\langle \partial_d u_\delta(\cdot, y_d)\rangle' = 1+\mu_\delta$ for any $y_d > 0$ large enough that $\{x \cdot e_d \geq y_d\} \subset \Omega_\delta$, where $\langle\cdot\rangle'$ is the tangential period cell average. Due to the Hausdorff convergence of $\overline{\Omega_\delta} \to \overline{\mathbb{R}^d_+}$, this implies that $\langle \partial_d\omega(\cdot, y_d)\rangle' = 0$ for all $y_d > 0$. □

10. Example of nonzero pinning force via linearization

In this section we will show that the capacities are nontrivial for a generic family of Q. As usual we will work on the case $e = e_d$, results for general normal direction $e \in S^{d-1}$ can be obtained by rotation. Our main tool is an asymptotic expansion of the capacity for defects of the form $Q = 1 + q$ with small q.

Theorem 10.1. *Let $d \geq 2$. Let Q of the form $Q = 1 + q$ with $q = \sigma\tilde q$, $\tilde q \in C^{0,1}_c(B_1(0))$, and $0 < \sigma \leq \sigma_0$ sufficiently small depending on $\|\tilde q\|_{C^{0,1}}$. For every $s \in \mathbb{R}$ there is a solution u of (1.4) such that*

$$|u(x) - (x_d - s)| \leq C\sigma \quad \text{in} \quad \{u > 0\}$$

with capacity

$$k = \sigma \int_{\{x_d = s\}} \tilde q \; dS + O(\sigma^2).$$

As a consequence

$$[k_{\rm rec}, k_{\rm adv}] \supset \left[\sigma \min_s \int_{\{x_d=s\}} \tilde q \; dS + O(\sigma^2),\ \sigma \max_s \int_{\{x_d=s\}} \tilde q \; dS + O(\sigma^2)\right].$$

Theorem 10.1 is for small q. However, we can still use this information for large q by exploiting monotonicity and locality. Note that, for $0 < a < 1$, $\kappa_{\rm adv}(s)$ decreases if replace $q(x)$ with $aq(x)_+ - q(x)_-$. Similarly $\kappa_{\rm rec}(s)$ increases if replace $q(x)$ with $q(x)_+ - aq(x)_-$. Combining this fact with Theorem 10.1 we can deduce some information in the nonlinear regime:

Corollary 10.2. *Suppose that $Q = 1 + q$ with $q \in C^1_c(B_1(0))$.*

- *If there is some interval $I \subset \mathbb{R}$ such that $q|_{\{x_d \in I\}}$ is non-negative and not identically zero, then $k_{\rm adv} > 0$.*

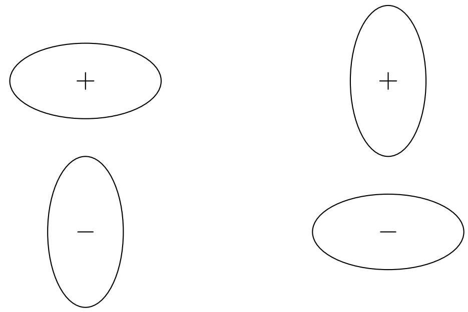

FIGURE 4. Defect q so that for every direction e there exist two planes $P_{\pm} = \{x \cdot e = s_{\pm}\}$ such that $\pm q \geq 0$ on $P_{\pm}$ and are nontrivial on $P_{\pm}$.

- *If there is some interval $I \subset \mathbb{R}$ such that $q|_{\{x_d \in I\}}$ is non-positive and not identically zero, then $k_{\mathrm{rec}} < 0$.*

We emphasize that the difference with Theorem 10.1 is that there is no requirement here that the defect q be small in magnitude. Based on the criteria in Corollary 10.2 is not so hard to arrange scenarios where $k_{\mathrm{adv}}(e) > 0 > k_{\mathrm{rec}}(e)$ for *all* directions $e \in S^{d-1}$ see Figure 4.

We proceed to prove Theorem 10.1. The desired solution will be constructed in hodograph coordinates (2.5), for linearization purposes. The corresponding PDE is

$$\begin{cases} \operatorname{tr}(A(\nabla_y v) D_y^2 v) = 0 & \text{in } \{y_d > 0\}, \\ (1 + \sigma \tilde{q}(y', v))(1 + \partial_{y_d} v) = \sqrt{1 + |\nabla_y' v|^2} & \text{on } \partial\{y_d > 0\}. \end{cases} \tag{10.1}$$

Proposition 10.3. *Let $0 < \sigma << 1$. Then for every $s \in \mathbb{R}$ there is a solution v_s of (10.1) of the form*

$$v_s(y) = s + \sigma w_s(y) + \sigma^2 r(y),$$

where $|r(y)| \leq C|\Phi(2 + |y|)|$ and

$$w_s(y) := \int_{\partial \mathbb{R}^d_+} \Phi(y - z') \tilde{q}(z', s) \, dS(z') \quad \text{for } y \in \mathbb{R}^d_+,$$

i.e. w_s solves the linearized problem with decay as $x_d \to \infty$:

$$\begin{cases} \Delta w_s = 0 & \text{in } \mathbb{R}^d_+, \\ \partial_{y_d} w_s = -\tilde{q}(y', s) & \text{on } \partial \mathbb{R}^d_+. \end{cases} \tag{10.2}$$

In particular the capacity k_s of the solution v_s, as in Theorem 3.2, satisfies

$$k_s = \sigma \int_{\partial \mathbb{R}^d_+} \tilde{q}(z', s) \, dS(z') + O(\sigma^2).$$

Theorem 10.1 is an immediate corollary of Proposition 10.3 by taking the inverse hodograph transform of the solution v_s which, as per the discussion in Section 2.5, produces a solution u_s of (1.4) with the same capacity.

Proof. Take $s = 0$ and we will omit the s subscripts, the argument is the same for other values of $s \in \mathbb{R}$. We will write the nonlinear PDE solved by $r = \sigma^{-2}(v(y) -$

$s - \sigma w(y))$, prove that it has a solution with the desired bounds, and then define $v(y) := s + \sigma w(y) + \sigma^2 r(y)$ to attain the desired solution of (10.1).

The nonlinear Neumann problem for r has the form

$$\begin{cases} \operatorname{tr}((I + \sigma B_\sigma(y, \nabla r))D^2 r) + \operatorname{tr}(B_\sigma(y, \nabla r)D^2 w) = 0 & \text{in } \{y_d > 0\} \\ \partial_{y_d} r = N_\sigma(y, r, \nabla' r) & \text{on } \partial\{y_d > 0\}. \end{cases} \tag{10.3}$$

The coefficients are defined so that if r solves (10.3) then $v(y) := s + \sigma w(y) + \sigma^2 r(y)$ solves (10.1). More specifically

$$B_\sigma(y, p) := \sigma^{-1}(A(\sigma \nabla w + \sigma^2 p) - I)$$

and

$$N_\sigma(y, r, p') := \frac{1}{\sigma^2}\left[\frac{\sqrt{1 + |\sigma \nabla_y' w + \sigma^2 p'|^2}}{1 + \sigma q(y', \sigma w + \sigma^2 r)} - 1 + \sigma \tilde{q}(y', 0)\right].$$

From now on we drop the σ subscript since that is fixed through the proof. Note that $B(y, r, p)$ and $N(y, r, p')$ satisfy the following bounds

$$\begin{aligned} |B| &\le C((2 + |y|)^{1-d} + \sigma|p|) \\ |\nabla_p B| &\le C\sigma \\ |N| &\le C((2 + |y|)^{2-2d} + \sigma^2 |p'|^2), \\ |\nabla_{p'} N| &\le C(\sigma(2 + |y|)^{1-d} + \sigma^2|p'|), \\ |\partial_r N| &\le C(1 + |p'|)\sigma \mathbf{1}_{B_1}(y) \end{aligned} \tag{10.4}$$

We will do a fixed point argument to deal with the non-proper r dependence in the nonlinear Neumann condition in (10.3). We define the functional space setting. First define C^1 norm weighted by the fundamental solution scaling

$$\|r\|_\Phi := \sup_{y \in \mathbb{R}^d_+}\left[\frac{|r(y)|}{|\Phi(2 + |y|)|} + \frac{|\nabla r(y)|}{|\nabla \Phi(2 + |y|)|}\right].$$

Then define

$$X := \left\{r \in C^1(\mathbb{R}^d_+ \cup \partial \mathbb{R}^d_+) : \|r\|_\Phi \le M\right\}.$$

This is a complete metric space under the $d_\Phi(r, r') := \|r - r'\|_\Phi$ metric. For each $r \in X$ we will show there exists a solution $\rho =: \mathbf{\Psi}[r]$ of

$$\begin{cases} \operatorname{tr}((I + \sigma B(y, \nabla r))D^2 \rho) + \operatorname{tr}(B(y, \nabla r)D^2 w) = 0 & \text{in } \{y_d > 0\} \\ \partial_{y_d} \rho = N(y, r, \nabla' r) & \text{on } \partial\{y_d > 0\} \end{cases} \tag{10.5}$$

given by the Neumann Green's kernel for the elliptic operator $\operatorname{tr}((I + \sigma B(y, \nabla r))\cdot)$ in $\mathbb{R}^d_+$. For M sufficiently large and $\sigma > 0$ sufficiently small we will show that $\mathbf{\Psi} : X \to X$ and is a contraction. In particular assume that $\sigma M \le 1$. We do not show all details from here, but just give a sketch.

To show that $\Psi(X) \subset X$ note that, applying the bounds (10.4), $\rho = \Psi[r]$ solves

$$\begin{cases} |\operatorname{tr}((I + \sigma B(y, \nabla r))D^2 \rho)| \le C(1 + \sigma M)(2 + |y|)^{1-2d} & \text{in } \{y_d > 0\} \\ |\partial_{y_d} \rho| \le C(1 + \sigma^2 M^2)(2 + |y|)^{2-2d)} & \text{on } \partial\{y_d > 0\}. \end{cases} \tag{10.6}$$

Recall that, given M, we will choose σ small so that $\sigma M \leq 1$. To show the contraction property: let $r, r' \in X$ and define $\nu = \mathbf{\Psi}[r] - \mathbf{\Psi}[r']$. Then ν solves

$$\begin{cases} |\mathrm{tr}((I + \sigma B(y, \nabla r)) D^2 \nu)| \leq C\sigma (2 + |y|)^{1-2d} \|r - r'\|_\Phi & \text{in } \{y_d > 0\} \\ |\partial_{y_d} \nu| \leq C\sigma (2 + |y|)^{2-2d} \|r - r'\|_\Phi & \text{on } \partial\{y_d > 0\}. \end{cases} \tag{10.7}$$

By standard Neumann kernel bounds applied in (10.6)

$$|\rho| \leq C|\Phi(2 + |y|)| \quad \text{and} \quad |\nabla \rho| \leq C|\nabla \Phi(2 + |y|)|$$

so we choose M sufficiently large depending on the dimensional constants in the previous line so that $\rho = \Psi[r] \in X$. Then applying Neumann kernel bounds again in (10.5)

$$|\nu| \leq C\sigma \|r - r'\|_\Phi |\Phi(2 + |y|)| \quad \text{and} \quad |\nabla \nu| \leq C\sigma \|r - r'\|_\Phi |\nabla \Phi(2 + |y|)|.$$

So $\|\nu\|_\Phi \leq C\sigma \|r - r'\|_\Phi$ and for σ sufficiently small this makes $\mathbf{\Psi}$ a contraction mapping of X.

Finally the computation of the expansion for the capacity just follows by computing $k = \lim_{|y|\to\infty} \Phi(y)^{-1} v(y)$, using the bound $|r(y)| \leq C|\Phi(2+|y|)|$, and observing that $\Phi(y)^{-1}\Phi(y - z') \to 1$ as $|y| \to \infty$ uniformly on z' in the support of $\tilde{q}(z', 0)$.

□

Appendix A. Viscosity solution properties

In this appendix we review the theory of viscosity solutions of the Bernoulli free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \cap U \\ |\nabla u| = Q(x) & \text{on } \partial\{u > 0\} \cap U. \end{cases} \tag{A.1}$$

Viscosity solutions and minimal supersolutions have been used and studied quite a bit in the literature on Bernoulli free boundary problems, see for example [9]. On the other hand, the notion of maximal subsolution has some additional serious challenges. In particular uniform non-degeneracy of maximal subsolutions is not known in dimension $d \geq 3$. As a particular consequence, stability of the viscosity subsolution property under uniform convergence is unknown without taking uniform non-degeneracy as an additional hypothesis. In this appendix we provide a (seemingly) new argument, in Lemma A.12, to show uniform stability of subsolutions under a very minor assumption.

A.1. Viscosity solutions.

We define viscosity solutions of (A.1) in a, more or less, standard way.

Definition A.1. Say that $u \in C(U)$ non-negative is a supersolution of (A.1) if whenever $\varphi \in C^\infty(U)$ touches u from below at $x \in U$ either

- $\Delta \varphi(x) \leq 0$ or
- $\varphi(x) = 0$ and $|\nabla \varphi(x)| \leq Q(x)$.

If E is a closed set say that u is a supersolution of (A.1) in E if u is a supersolution in some open neighborhood of E.

Definition A.2. Say that $u \in C(U)$ non-negative is a subsolution of (A.1) whenever $\varphi \in C^\infty(U)$ and φ_+ touches u from above at $x \in \overline{\{u > 0\}} \cap U$ either

- $\Delta \varphi(x) \geq 0$ or

- $\varphi(x) = 0$ and $|\nabla \varphi(x)| \geq Q(x)$.

If E is a closed set say that u is a subsolution of (A.1) in E if u is a subsolution in some open neighborhood of E.

Remark A.3. Note that subsolutions of (A.1) in U are subharmonic in U. If $\varphi \in C^\infty(U)$ touches u from above at $x \in U$, if $u(x) > 0$ then $\Delta\varphi(x) \geq 0$, and if $u(x) = 0$ then φ also touches the 0 function from above at x and so $\Delta\varphi(x) \geq 0$.

Next we introduce notions of minimal supersolution and maximal subsolution.

Definition A.4. Say that $u \in C(U)$ non-negative is a *minimal supersolution* of (A.1) if

- u is a supersolution of (A.1) in U
- for all balls $B \subset\subset U$ and $v \in C(\overline{B})$ a supersolution of (A.1) in B with $v > u$ on $\overline{\{u > 0\}} \cap \partial B$ then $v \geq u$ in B.

Definition A.5. Say that $u \in C(U)$ non-negative is a *maximal subsolution* of (A.1) if

- u is a subsolution of (A.1) in U
- for all balls $B \subset\subset U$ and $v \in C(\overline{B})$ a subsolution of (A.1) in B with $v < u$ on $\overline{\{v > 0\}} \cap \partial B$ then $v \leq u$ in B.

Note that these notions are somewhat weaker than the solutions which are typically obtained from Perron's method, due to the strict boundary ordering required, but the notions behave better under uniform limits. We also introduce the stronger notion of minimality and maximality which typically arise for Perron's method solutions.

Definition A.6. Say that $u \in C(U)$ non-negative is a *strongly minimal supersolution* (resp. *strongly maximal subsolution*) of (A.1) if

- u is a supersolution (resp. subsolution) of (A.1) in U
- for all balls $B \subset\subset U$ and $v \in C(U)$ supersolutions (resp. subsolutions) of (A.1) in U with $v = u$ on $U \setminus B$ then $v \geq u$ (resp. $v \leq u$).

Remark A.7. It is important, in the definition of strongly maximal subsolution, that v be a subsolution in a neighborhood of $\overline{B}$. Otherwise we could take v to be the harmonic lift in B to find a larger subsolution in B, and the only strongly maximal subsolutions would be harmonic in the entire U.

Lemma A.8. *The following two notions of strongly minimal supersolution, with respect to compact perturbation, of (A.1) on $\mathbb{R}^d$ are equivalent:*

1. *For every ball $B \subset\subset \mathbb{R}^d$ and $w \in C(\overline{B})$ a supersolution of (A.1) in B with $w = u$ on ∂B, $w \geq u$.*
2. *For every $w \in C(\mathbb{R}^d)$ a supersolution of (A.1) in $\mathbb{R}^d$ with $\{w \neq u\} \subset\subset U$, $w \geq u$.*

Proof. Similar to classical Perron's method extension is a supersolution. □

A.2. **Basic regularity properties.** Here we present the Lipschitz and non-degeneracy estimates. First the Lipschitz estimate, which only requires the viscosity solution property.

Lemma A.9 (Lipschitz estimate)**.** *There is $C(d) \geq 1$ so that if u is a viscosity solution of*

$$\Delta u = 0 \quad \textit{in} \quad \{u > 0\} \cap B_2 \quad \textit{and} \quad |\nabla u| \leq \Lambda \quad \textit{on} \quad \partial\{u > 0\} \cap B_2$$

then

$$\|\nabla u\|_{L^\infty(B_1)} \leq C(\Lambda + \operatorname*{osc}_{B_2} u).$$

The situation with non-degeneracy is much more complicated, general viscosity solutions may fail non-degeneracy due to, for example, the two-plane solutions $a(x_d)_+ + b(x_d)_-$ which are viscosity solutions as long as $0 \leq a, b \leq 1$.

Here we summarize several known results, see [24, Lemma A.2] and [22, Lemma 2.13] for further details and citations on the origins of these results.

Lemma A.10. *Let $u \in C(B_1)$ non-negative with $0 \in \partial\{u > 0\}$ and satisfying one of the following:*

- *(i) u is an inward energy minimizer of $\mathcal{J}$ in B_1.*
- *(ii) u is a minimal supersolution of (A.1) in B_1.*
- *(iii) $d = 2$ and u is a maximal subsolution in B_1 of (A.1) in B_1.*
- *(iv) $\partial\{u > 0\}$ is an L-Lipschitz graph in B_1 and u is a subsolution of (A.1) in B_1*
- *(v) u is a subsolution of (A.1) in B_1 and $\{u > 0\}$ has an exterior touching ball of radius 1 at 0.*

Then there is c_0 depending on n and $\min_{B_1} Q$ (and on L if in the case of (iv)) such that

$$\sup_{x \in B_r(0)} u(x) \geq c_0 r \quad \textit{for all } 0 < r < 1.$$

A.3. **Stability of viscosity solution properties with respect to limits.** The essential useful property of the viscosity solution notion is its (typical) stability property with respect to local uniform limits. This is actually a somewhat nontrivial issue in the case of the one-phase Bernoulli problem, mainly due to the subsolution notion which deals with touching from above in $\overline{\{u > 0\}}$ instead of in the entire domain. Usually this issue is resolved using non-degeneracy. However, we do not know whether uniform non-degeneracy estimates hold for all of the types of solutions that we are interested to consider, in particular maximal subsolutions.

First we show that the supersolution property is stable with respect to uniform limits, this is the relatively easy / standard direction.

Lemma A.11. *Suppose that $u_k \in C(U)$ is a sequence of supersolutions to (A.1) so that $u_k \to u$ locally uniformly in U. Then u is a supersolution of (A.1). Furthermore, if u_k are weakly minimal then u is weakly minimal.*

Proof. The first part is standard and can be found, for example, in [21, Lemma 3.1]. Consider the case when u_k are weakly minimal supersolutions. By uniform non-degeneracy of weakly minimal supersolutions Lemma A.10 $\overline{\{u_k > 0\}} \to \overline{\{u > 0\}}$ is Hausdorff distance on compact subsets of U. Let $B \subset\subset U$ and $v \in C(\overline{B})$ a supersolution of (A.1) in B with $v > u$ on $\overline{\{u > 0\}} \cap \partial B$. By continuity $v - u > 0$ in a neighborhood of $\overline{\{u > 0\}} \cap \partial B$, since $\overline{\{u_k > 0\}} \cap \partial B \to \overline{\{u > 0\}} \cap \partial B$ in Hausdorff distance $v > u_k$ on $\overline{\{u_k > 0\}} \cap \partial B$ for k sufficiently large. Thus, by weak minimality of u_k, $v \geq u_k$ in B and therefore, passing to the limit, $v \geq u$ in B. Thus v is weakly minimal. □

Next we prove a new result on the local uniform limit of subsolutions, without assuming non-degeneracy. Instead we introduce a very weak hypothesis that the limiting positivity set does not cover the entire domain, this seems much easier to verify in practical situations, and we have made use of it in the paper above.

Lemma A.12. *Suppose that $u_k \in C(U)$ is a sequence of subsolutions to (A.1) so that $u_k \to u$ locally uniformly in U and*

$$U \setminus \left(\limsup_{k\to\infty} \overline{\{u_k > 0\}} \right) \neq \emptyset.$$

Then, in fact,

$$\limsup_{k\to\infty} \overline{\{u_k > 0\}} \subset \overline{\{u > 0\}}$$

and u is a subsolution of (A.1). In particular, if $x_k \in \partial\{u_k > 0\}$ and $x_k \to x \in U$ then $x \in \partial\{u > 0\}$. If u_k are weakly maximal then u is weakly maximal.

As far as we are aware previous analogous results in the literature have assumed that the u_k are uniformly non-degenerate in order to establish Hausdorff convergence of the positivity sets. We provide a proof that works without uniform non-degeneracy.

Proof. First let us first assume that

$$E^* := \limsup_{k\to\infty} \overline{\{u_k > 0\}} \subset \overline{\{u > 0\}} \tag{A.2}$$

and show that u is a subsolution. The sub-harmonic property in $\{u > 0\}$ is standard. Suppose that φ_+ touches u from above at $x_0 \in \partial\{u > 0\}$ with $\Delta\varphi < 0$. We can assume, by adding a parabola, that $\varphi - u$ has a strict local minimum in $\overline{\{u > 0\}}$ at x_0. By (A.2) and continuity of φ there exists $c_k \to 0$ so that $(\varphi + c_k)_+$ touches u_k from above at $\overline{\{u_k > 0\}} \ni x_k \to x_0$. Since φ is strictly superharmonic by the definition of subsolution of (A.1) we must have $x_k \in \partial\{u_k > 0\}$ and $|\nabla\varphi(x_k)| \geq Q(x_k)$. Taking the limit in this inequality we get $|\nabla\varphi(x_0)| \geq Q(x_0)$.

Now we aim to show that (A.2) holds. Suppose that E^* is a strict superset of $\overline{\{u > 0\}}$. By hypothesis $U \setminus E^*$ nonempty. It is also open since $E^* \cap U$ is relatively closed in U. Thus there must exist $x_0 \in U \setminus E^*$ with

$$0 < d(x_0, E^*) < \min\{d(x_0, \overline{\{u > 0\}}), d(x_0, \partial U)\}.$$

Then $B_r(x_0) \subset\subset U$ touches E^* from the outside at some y_0 with $r = d(x_0, E^*)$. We can adjust x_0 inward along the ray from x_0 to y_0, if necessary, and adjust r accordingly so that the touching is strict $\partial B_r(x_0) \cap \partial E_* = \{y_0\}$. By definition of E^* there exists a sequence of points $y_k \to y_0$ and radii $r_k \to r$ so that $B_{r_k}(x_*)$ touches $\overline{\{u_k > 0\}}$ from the exterior at $y_k \in \partial\{u_k > 0\}$. By non-degeneracy of viscosity subsolutions at outer regular points Lemma A.10 part (v)

$$\sup_{x \in B_s(y_0)} u_k(x) \geq cs \ \text{ for all } \ 2|y_k - y_0| \leq s \leq r.$$

By the uniform limit

$$\sup_{x \in B_s(y_0)} u(x) \geq cs \ \text{ for all } \ 0 < s \leq r$$

implying that $y_0 \in \partial\{u > 0\}$ which is a contradiction.

Finally consider the case when u_k are weakly maximal. Let $B \subset\subset U$ and $v \in C(U)$ a subsolution of (A.1) in B with $v < u$ on $\overline{\{v > 0\}} \cap \partial B$. By local uniform convergence and compactness of $\overline{\{v>0\}}\cap\partial B$ we have $v<u_k$ for sufficiently large k. Thus, by weak maximality of u_k, $v\le u_k$ in B and therefore, passing to the limit, $v\le u$ in B. Thus v is weakly maximal.

□

Next we show a scenario where we can establish (strong) minimality or maximality of the limit.

Lemma A.13. *Suppose that $u_k\in C(\overline{U})$ is a sequence of minimal supersolutions (resp. maximal subsolutions) of (A.1) in U such that $u_k\nearrow u$ locally uniformly in U (respectively $u_k\searrow u$). Then u is a minimal supersolution (resp. maximal subsolution) of (A.1) in U.*

Proof. We just do the maximal subsolution case to show how to establish the additional hypothesis in Lemma A.12.

Let $B\subset\subset U$ and $v\in C(\overline{B})$ a subsolution of (A.1) in B with $v\ge u$ in B and $v=u$ on ∂B. The pointwise maximum $w_k(x):=\max\{u_k(x),v(x)\}$ is a maximum of subsolutions so it is a subsolution in B. It is also continuous in $\overline{B}$ and, since $u_k\ge u$ by hypothesis, also $w_k=u_k$ on ∂B. Thus the maximal subsolution property of u_k implies that $w_k\le u_k$ and so, taking the limit on both sides, $\max\{u,v\}\le u$ and so $v\le u$.

Next we aim to show the subsolution property of u. Call

$$E^*:=\bigcap_k\overline{\{u_k>0\}}\supset\overline{\{u>0\}}.$$

If E^* were equal to $\overline{\{u>0\}}$ we would be done by Lemma A.12. If $E^*\cap U=U$ then $\overline{\{u_k>0\}}\cap U=U$ for all k and then Lemma A.14, see below, implies that $\{u_k>0\}=U$ in which case u is harmonic in U and therefore it is indeed a maximal subsolution of (A.1). Otherwise $U\setminus E^*$ is nonempty and the hypothesis of Lemma A.12 holds so u is a subsolution.

□

We conclude by showing a weak regularity property of the positivity set for maximal subsolutions which we used in the previous proof.

Lemma A.14. *Let u be a maximal subsolution in U then* $\operatorname{int}(\overline{\{u>0\}})=\{u>0\}$.

Proof. Suppose that $x_0\in\operatorname{int}(\overline{\{u>0\}})$ so there is some ball B centered at x_0 with $\overline{B}\subset\operatorname{int}(\overline{\{u>0\}})$. Let v be the harmonic lift of u in B, i.e.

$$\Delta v=0\ \text{ in }\ B\ \text{ with }\ v=u\text{ in }\ U\setminus B.$$

Since $u>0$ somewhere on ∂B, $v>0$ in B. Since u is subharmonic in U, see Remark A.3, $v\ge u$ and also v is subharmonic in U, by the same proof as the classical Perron's method. Also notice that $\overline{\{v>0\}}=\overline{\{u>0\}}$, since they agree on $U\setminus B$ and $\overline{B}$ is contained in both sets.

We next check that v is a subsolution of (A.1) in U. Suppose φ smooth and φ_+ touches v from above at $x\in\overline{\{v>0\}}\cap U=\overline{\{u>0\}}\cap U$ with $\Delta\varphi(x)<0$. Then $x\notin B$ since v is harmonic and positive there. Thus $x\in\overline{\{u>0\}}\setminus B$ and φ_+ touches u from above at x and so the subsolution condition of u implies that $\varphi(x)=0$ and $|\nabla\varphi(x)|\ge Q(x)$.

Thus the maximal subsolution property of u implies that $v = u$ and so $u(x_0) = v(x_0) > 0$. Since x_0 was arbitrary we showed that $\mathrm{int}(\overline{\{u > 0\}}) \subset \{u > 0\}$ completing the proof. $\square$

A.4. **Perron's method for maximal subsolutions.** In this section we recall the Perron's method for existence of maximal subsolutions. The existence of minimal supersolutions via Perron's method can be found in [9, Theorem 6.1].

Theorem A.15. *Let U be an outer regular domain and g be a non-negative continuous function on ∂U. Suppose $\overline{v}$ is an outer regular, continuous, R-supersolution in $\overline{U}$ with $g \prec \overline{v}$ on ∂U. Let $\mathcal{S}$ be the class of subsolutions*

$$\mathcal{S} := \{w \in C(\overline{U}) \text{ is a subsolution of (A.1) in } U,\ w \leq g \text{ on } \partial U, \text{ and } w \leq \overline{v} \text{ in } \overline{U}\}$$

then the maximal subsolution

$$u(x) := \sup\{w(x) : w \in \mathcal{S}\}$$

is a viscosity solution of the free boundary problem (A.1) in U with $u \in C(\overline{U})$ and $u = g$ on ∂U.

Appendix B. Blow-downs of one-sided flat solutions

In this appendix we will prove Proposition 2.2. Before proceeding to the proof, which is in Section B.1, we present a few Lemmas, which are independently useful and are applied in a few places in the paper.

The first result shows that certain isolated singularities of the Bernoulli problem are removable.

Lemma B.1. *If $u \in \mathrm{Lip}(B_1)$ is a viscosity solution (2.2) in $B_1 \setminus \{0\}$, then u is a viscosity solution of (2.2) in B_1.*

Proof. If $0 \in \{u > 0\}$ then u is harmonic in a neighborhood of 0 by the standard removable singularity result for bounded harmonic functions. Suppose that φ smooth touches u from below at $0 \in \partial\{u > 0\}$ with $\Delta\varphi > 0$. Write $\nabla\varphi(0) = \alpha e$ with $|e| = 1$. Then since u is Lipschitz we can choose a sequence $r_k \to 0$ so that the blow up limit

$$w(x) = \lim_{r_k \to 0} \frac{u(r_k x)}{r_k} \ \text{ locally uniformly in } \mathbb{R}^d.$$

Furthermore, by [9, Lemma 11.17] using Lipschitz regularity to rule out the superlinear blow-up, the limit is linear in $\{x \cdot e > 0\}$

$$w(x) = \lim_{r \to 0} \frac{u(rx)}{r} = \alpha(x \cdot e) \ \text{ in } \ \{x \cdot e > 0\}.$$

Since w is a local uniform limit $r_k^{-1}u(r_k x)$, which are viscosity solutions of (2.2) in $B_{1/r_k} \setminus \{0\}$, then w is a supersolution of (2.2) in $\mathbb{R}^d \setminus \{0\}$ by Lemma A.11. Since $\alpha(x \cdot e)$ is a smooth test function touching w from below at any point $x_0 \in \{x \cdot e = 0\} \setminus \{0\}$ we find that $\alpha \leq 1$.

For the subsolution case the argument is similar, using Lemma A.12. Note that the test function touching from above guarantees the hypothesis of Lemma A.12. $\square$

Lemma B.2 (A strong maximum principle)**.** *Suppose that $u \in C(\overline{B_1})$ solves (2.2) in B_1 and $0 \in \partial\{u > 0\}$ then:*

(i) If $u(x) \leq (x_d)_+$ then $u(x) \equiv (x_d)_+$ in B_1
(ii) If $u(x) \geq (x_d)_+$ then $u(x) \equiv x_d$ in $\{x_d > 0\} \cap B_1(0)$.
(iii) There is $1/4 \geq \eta_0(d) > 0$ sufficiently small so that if $(x_d)_+ \leq u(x) \leq (x_d + \eta_0)_+$ in $B_1(0)$ then $u(x) \equiv (x_d)_+$ in $B_{1/2}(0)$.

Note that the result of part (ii) of the previous lemma cannot be extended to $x_d < 0$ without further hypothesis, problematic examples include the two-plane solution $u(x) = |x_d|$ or families near the two-plane solution like $u(x) = (x_d)_+ + (-s - x_d)_+$ for $s \geq 0$. These example are ruled out by the flatness hypothesis in part (iii).

Proof. In the case $u(x) \geq (x_d)_+$, note that $w(x) = u(x) - x_d$ is non-negative and harmonic in $x_d > 0$ with $w(0) = 0$. The argument of Hopf Lemma implies that either $u(x) \equiv x_d$ in $x_d > 0$ or

$$u_\infty(x) - x_d \geq c x_d \quad \text{near} \quad x = 0.$$

In the second case $(1 + c)x_d$ touches u_∞ from below at $0 \in \partial\{u > 0\}$ which contradicts that u is a supersolution of (2.2). Thus $u \equiv x_d$ in $x_d > 0$.

In the case of part (iii), when u is also flat from above, we can apply the Harnack inequality Corollary 2.8 to derive that $u(x) \equiv (x_d)_+$.

In the case $u(x) \leq (x_d)_+$. Suppose $u \not\equiv (x_d)_+$ in B_1, so there is some $0 < r < 1$ so that $u(x) < (x_d)_+$ somewhere on ∂B_r. Let ψ be the harmonic function in $B_r^+ := B_r \cap \mathbb{R}^d_+$ with $\psi(x) = (x_d)_+ - u(x)$ on $\partial(B_r^+)$. By the choice of r then $\psi > 0$ in B_r^+. By Hopf Lemma $\partial_{x_d}\psi(0) > 0$. So $x_d - \psi$ touches u from above in $\overline{\{u > 0\}}$ at 0 with slope < 1 contradicting the viscosity subsolution property of u. □

Next we prove an approximate version of the previous strong maximum principle in the case of a Bernoulli solution with a localized defect. This result plays an important role in the proof of Proposition 5.10.

Lemma B.3 (Defect version of strong maximum principle)**.** *Let $L \geq 1$ and $1/4 > \eta_0(d) > 0$ from Lemma B.2. For all $\delta > 0$ there exists $1 > r_0(d, \delta, L) > 0$ sufficiently small so that if $u \in \mathrm{Lip}(B_1)$ is a viscosity solution (2.2) in $B_1 \setminus B_{r_0}$ with*

$$u(0) = 0, \ (x_d)_+ \leq u(x) \leq (x_d + \eta_0)_+ \ \text{ in } B_1, \quad \text{and} \quad \|\nabla u\|_{L^\infty(B_1)} \leq L, \tag{B.1}$$

then

$$u(x) \leq (x_d + \delta)_+ \ \text{ in } \ B_{1/2}.$$

Proof. The proof is by a compactness argument. Suppose that there exists sequence of $r_k \to 0$, solutions u_k of (2.2) in $B_1 \setminus B_{r_k}$ all satisfying (B.1), such that

$$\max_{\overline{B_{1/2} \cap \{u_k > 0\}}} (u_k(x) - x_d) \geq \delta_0. \tag{B.2}$$

for some $\delta_0 > 0$. Up to a subsequence we can assume that the u_k converge locally uniformly in B_1 to some $u \in \mathrm{Lip}(\overline{B_1})$. Note that all properties in (B.1) are stable with respect to the locally uniform limit so u satisfies (B.1) as well. By Lemma A.11 and Lemma A.12 u solves (2.2) in $B_1 \setminus \{0\}$. Note that the hypothesis of Lemma A.12 is satisfied since $u_k(x) \leq (x_d + \eta_0)_+$ for all k. Then Lemma B.1 implies that u solves (2.2) in the entire B_1.

Finally we want to establish (B.2) also holds for u. Note that Lemma A.12 also implies that

$$\limsup_{k \to \infty} \overline{\{u_k > 0\}} \subset \overline{\{u > 0\}}. \tag{B.3}$$

By (B.2) there exists $x^k \in \overline{B_{1/2} \cap \{u_k > 0\}}$ such that $u_k(x) = (x^k)_d + \delta_0$, up to a subsequence the x^k converge to some $x^\infty \in \overline{B_{1/2}}$ and by (B.3) also $x^\infty \in \overline{\{u > 0\}}$. So, also using uniform convergence, we conclude that u satisfies (B.2) as well.

Now to conclude the contradiction, we apply Lemma B.2 which implies that $u(x) \equiv (x_d)_+$ in $B_{1/2}$ but this contradicts that u satisfies (B.2). □

B.1. **Proof of Proposition 2.2.** The family of functions $\{\frac{1}{r}u(rx)\}_{r>0}$ is uniformly Lipschitz due to Lemma A.9, and any sequence of $r \to +\infty$ has a subsequence r_k along which $u_k(x) := \frac{1}{r_k}u(r_k x)$ locally uniformly converges. Let us fix one such convergent sequence and call its limit $u_\infty(x)$.

First assume that u was flat from above, (2.3), then

$$u_k(x) \leq (x_d - t/r_k)_+ \quad \text{and so} \quad u_\infty(x) \leq (x_d)_+.$$

Using the same upper bound again, the hypothesis of Lemma A.12 holds, and so u_∞ is a viscosity solution of (2.2) in $\mathbb{R}^d \setminus \{0\}$. Then Lemma B.1 implies that u_∞ is actually a viscosity solution of (2.2) in $\mathbb{R}^d$. Since u was not identically zero there is $x_0 \in \partial\{u > 0\}$ and then $x_k := x_0/r_k \in \partial\{u_k > 0\}$ converge to 0 and Lemma A.12 again implies that $0 \in \partial\{u_\infty > 0\}$. Finally Lemma B.2 implies that $u_\infty(x) \equiv (x_d)_+$.

We next suppose that u satisfies the lower bound (2.4). Then

$$u_\infty(x) \geq (x_d)_+$$

and so $0 \in \overline{\{u_\infty > 0\}}$. Further $u_\infty(0) = \lim_{r_k \to \infty} \frac{1}{r_k}u(0) = 0$. So $0 \in \partial\{u_\infty > 0\}$. By Lemma A.11 u_∞ is a supersolution of (2.2). Then Lemma B.2 implies that $u_\infty(x) \equiv x_d$ in $\{x_d \geq 0\}$.

□

Appendix C. More technical lemmas

Lemma C.1. *Suppose that u solves*

$$\Delta u = 0 \quad \text{in} \quad \{x_d > 0\}$$

and u is bounded, and $\mathcal{Z}$-periodic for some integer lattice $\mathcal{Z}$ with $\operatorname{span}(\mathcal{Z}) = \mathbb{R}^{d-1} \times \{0\}$*. Then $s := \lim_{x_d \to +\infty} u(x)$ exists,*

$$|u(x) - s| \leq Ce^{-cx_d} \quad \text{and} \quad |D^k u| \leq C_k e^{-cx_d}.$$

Furthermore

$$\langle \partial_{x_d} u(\cdot, x_d)\rangle' = 0 \quad \text{for all} \quad x_d > 0$$

where $\langle\cdot\rangle'$ denotes the average over the fundamental domain of the lattice $\mathcal{Z}$.

Proof. The exponential convergence follows from periodicity, maximum principle (here is where boundedness is used), and interior elliptic estimates. See, for example, [26, Lemma 5.2] The formula for the average of the normal derivative follows from the integration by parts identities:

$$\frac{d}{dx_d}\langle \partial_{x_d} u(\cdot, x_d)\rangle' = \langle \partial_{x_d}^2 u(\cdot, x_d)\rangle' = -\langle \Delta' u(\cdot, x_d)\rangle' = 0.$$

The limit $\partial_{x_d} u(\cdot, x_d) \to 0$ as $x_d \to \infty$ fixes the constant of integration.

□

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