# A free boundary problem arising in flame propagation

Inwon C. Kim*

Revised September 27, 2002

**Abstract**

We study a free boundary problem describing the propagation of laminar flames. The problem arises as the limit of a singular perturbation problem. We introduce the notion of viscosity solutions for the problem to show the maximum principle-type property of the solutions. Using this property we show the uniform convergence of the approximating solutions and the uniqueness of the viscosity solution under several geometric conditions on the initial data.

## 0 Introduction

In this paper we consider a free boundary problem for the heat equation. The classical formulation is as the following. Consider $\Omega_0$: an open subset of $I\!R^n$ and a nonnegative initial data $u_0 \in C(I\!R^n)$ with its nonempty bounded positivity set $\{u_0 > 0\} = \Omega_0$. The problem consists in finding a nonnegative continuous function $u$ in $Q = I\!R^n \times (0, \infty)$ such that

$$(P) \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } \{u > 0\}, \\ |Du| = 1 & \text{on } \partial\{u > 0\}, \\ u_0(x, 0) = u_0(x) \geq 0 & \end{cases}$$

We may also write the condition on the free boundary $\Gamma$ in the form $u_\nu = -1$, where $\nu$ denotes the derivative of $u$ with respect to the outward spatial normal $\nu$ to $\partial\{u > 0\}$. Formally the

*Department of Mathematics, MIT, Cambridge, MA. e-mail: ikim@math.mit.edu

velocity $V_n$ of the free boundary in the direction of $\nu$ is given by

$$V_n = u_t/|u_\nu| = u_t = \Delta u, \tag{0.1}$$

due to the boundary condition.

This problem $(P)$ occurs in combustion theory in the analysis of the propagation of curved flames, where $u$ denotes the *minus temperature* $\lambda(T_c - T)$, where $T_c$ is the flame temperature and $\lambda$ is a normalization factor. It is derived in the theory of equidiffusional premixed flames analyzed in the relevant limit of high activation energy, as developed for instance in Buckmaster and Ludford [BuL] (For further details see [CV1] and the survey paper [V].) After convenient simplifications the limit situation is reduced to solving the problem

$$(P^\epsilon) \qquad \begin{cases} u_t^\epsilon - \Delta u^\epsilon = -\beta^\epsilon(u^\epsilon), \\ \\ u^\epsilon(\cdot, 0) = u_{0,\epsilon} \end{cases}$$

as $\epsilon \to 0$.

Here $\beta^\epsilon(s) = 1/\epsilon\beta(s/\epsilon)$ with the following assumptions:

(i) $\beta$ is positive in the interval $I = \{0 < s < 1\}$ and 0 otherwise;

(ii) it is a $C^\infty$ function in $[0, \infty)$;

(iii) it is increasing for $0 \le s < 1/2$, decreasing for $1/2 < s \le 1$;

(iv) $\int_0^1 \beta(s)ds = 1/2$.

We also assume the following on the initial data of $u_0^\epsilon$:

(0.2) $u_{0,\epsilon}$ are nonnegative $C^\infty$ functions uniformly converging to $u_0$ with

$$d(\{u_{0,\epsilon} > 0\}, \{u_0 > 0\}) \to 0 \text{ as } \epsilon \to 0,$$

where $d(X, Y) = \inf\{|x - y| : x \in X, y \in Y\}$ for sets $X, Y \subset I\!R^n$.

Note that with above assumptions on the initial data $(P_\epsilon)$ admits a unique solution $u^\epsilon \in C^\infty(I\!R^n \times (0, \infty))$.

The existence and regularity of classical solutions of $(P)$ were proved in [GHV] when the initial data $u_0$ is radially symmetric, by using the elliptic-parabolic approach. When the initial data is smooth enough ($C^{3,\alpha}$), [BL] proved the short-time existence, uniqueness and regularity results for smooth solutions. The classical solutions of $(P)$ in special settings were constructed in [Me] and [AG].

Various concepts of generalized solutions have been introduced in the literature in order to justify the limit process and to obtain global time solutions of (P) for general data. In [CV1] it is proved that $u^\epsilon$ converges along subsequences to a function $u$ in $C^{1,1/2}_{loc}$). We call such functions $u$ as a *limit solution* of (P). Here a concept of *weak solution* is introduced to clarify the nature of the limit solutions with strictly superharmonic initial data. On the other hand the concept of *viscosity solution* for (P) was introduced by [LVW] with the same purpose. Assuming that $u_0$ is monotone in one direction, [LVW] shows that three concepts of solutions of (P), limit, viscosity and classical, agree with each other and yield a unique solution as long as classical solutions exist. [P] shows the uniqueness of the limit solutions as a minimal viscosity supersolution when $u_0$ is *starshaped* (see Corollary 2.6).

In general we should not expect any uniqueness result unless we impose some geometrical conditions on the initial data (see [V] for an example where non-uniqueness occures.) In this paper we introduce a notion of viscosity solution of $(P)$ to prove the global uniqueness and existence result for solutions of (P) for several classes of the initial data (see Corollary 2.6 and 3.5.) It follows that in these cases the whole sequence $(u^\epsilon)_\epsilon$ given above locally uniformly converges to the unique viscosity solution of (P). We point out that our notion of viscosity solutions is a class of viscosity solutions introduced in [LVW].

In section 1 we introduce a definition for viscosity solutions of $(P)$. By definition it follows that classical solutions of (P) are viscosity solutions of (P). In Theorem 1.3 we also show that limit solutions are viscosity solutions of (P).

In section 2, we show that a comparison principle holds for the viscosity solutions of $(P)$. As in the Hele-Shaw problem studied in [K], the difficulty comes from (a) the presence of a free boundary, (b) non-geometric nature of the problem and (c) lack of local classical solutions as test functions. Moreover, additional difficulty comes from the fact that the free boundary of solutions of $(P)$ might propagate with infinite speed. To overcome this technical difficulty we adopt a double sup/inf-convolution. (see section 2.) Based on the comparison principle, uniqueness results are proven for the class of initial data studied in [CV1] and [P]. In such cases it follows that $u_\epsilon$, the solutions of $(P^\epsilon)$ locally uniformly converge to the unique viscosity solution $u$ of $(P)$ as $\epsilon \to 0$.

In section 3, we study the problem $(P)$ in a domain $\Omega \times [0, \infty)$ with the Neumann boundary data on $\partial\Omega$. Assuming that $\partial\Omega$ is smooth, we locally transform $\Omega$ onto a half-plane by a local parameterization of the boundary. Then by reflection argument we can avoid dealing with the boundary condition. We also show a uniqueness and convergence result for the class of initial data studied by [LVW].

# 1 Definition of the viscosity solutions

Extending the notion in [K], we define viscosity solutions of $(P)$ as follows.

**Definition 1.1** *(1) A nonnegative continuous function $u$ defined in $Q$ is a viscosity subsolution of $(P)$ if (i) $\overline{\{u>0\}}\cap\{t=0\}=\overline{\{u_0>0\}}$ and (ii) for every $\phi\in C^{2,1}(Q)$ that has a local maximum of $u-\phi$ in $\overline{\{u>0\}}\cap\{t\le t_0\}$ at $(x_0,t_0)$,*

$$(a)\quad (\phi_t-\Delta\phi)(x_0,t_0)\le 0 \qquad \text{if } u(x_0,t_0)>0,$$

$$(b)\quad \min(\phi_t-\Delta\phi,1-|D\phi|)(x_0,t_0)\le 0 \qquad \text{if } (x_0,t_0)\in\partial\{u>0\} \text{ and } u(x_0,t_0)=0.$$

*(2) A nonnegative continuous function $v$ defined in $Q$ is a viscosity supersolution of $(P)$ if for every $\phi\in C^{2,1}(Q)$ that has a local minimum zero of $v-\phi$ in $\overline{\{v>0\}}\cap\{t\le t_0\}$ at $(x_0,t_0)$,*

$$(a)\quad (\phi_t-\Delta\phi)(x_0,t_0)\ge 0 \qquad \text{if } (x_0,t_0)\in\{v>0\},$$

$$(b)\quad \max(\phi_t-\Delta\phi,1-|D\phi|)(x_0,t_0)\ge 0 \qquad \text{if } (x_0,t_0)\in\partial\{v>0\} \text{ and if}$$

$$(1.1)\quad |D\phi|\neq 0 \text{ on } \{\phi=0\} \text{ and } \{\phi>0\}\cap\{v>0\}\cap B(x_0,t_0)\neq 0 \text{ for any ball } B(x_0,t_0).$$

We say that $u\in C(\bar{Q})$ is a *viscosity solution* of $(P)$ if it is both a viscosity sub- and supersolution.

**Remark.**

* Condition (i) in (1) is to control the behavior of $u$ at $t=0$. Without this condition the solution of the heat equation with the initial data $u_0$ is a viscosity solution and therefore the uniqueness does not hold.

* Condition (1.1) is to insure that the zero level set $\partial\{\phi>0\}$ is smooth and $\phi_+=\max(\phi,0)$ is nontrivial in $\{v>0\}$ near $(x_0,t_0)$.

A smooth function $u\in C^{2,1}(\{u>0;t>0\})\cap C^{1,0}(\overline{\{u>0\}})$ with $Du\in C(\overline{\{u>0\}})$ and initial data $|Du_0|=1$ on $\Gamma_0=\partial\{u_0>0\}$ is a *classical solution* of (P) if $u$ satisfies (P) in the classical sense. The following is clear from the definition.

**Corollary 1.2** *If $u$ is a classical solution of (P) for $t\le T$, then $u$ is a viscosity solution of (P) for $t\le T$.*

Throughout the paper, we assume that $u_0 \in C^1(\overline{\{u_0 > 0\}})$ with $|Du_0| = 1$ on $\partial\{u_0 > 0\}$. Let $u^\epsilon$ is the unique classical solution of $(P)_\epsilon$ with initial data $u_{0,\epsilon}$ as given in section 1. Then along a subsequence $(u^\epsilon)$ locally uniformly converge to a continuous limit solution $u$ (refer [CV1].) Note that such $u$ need not to be unique.

**Theorem 1.3** *$u$ is a viscosity solution of* $(P)$.

**Proof.**

1. First we show that $u$ is a viscosity subsolution of (P). The following test function used in [P] is adopted for barrier arguments. Let us consider the family of functions $\{\phi^\epsilon\}_\epsilon > 0$, $\phi^\epsilon \in C^2(I\!R)$ such that

$$\phi^\epsilon_{ss}(s) = \gamma_\epsilon(\phi^\epsilon(s)), \tag{1.2}$$

where

$$\gamma_\epsilon(s) = \begin{cases} c\beta_\epsilon(s) & \text{in } [a\epsilon, \epsilon), \\ 0 & \text{otherwise .} \end{cases}$$

Here the constant $c > 1$ will be chosen later and $a > 0$ is chosen such that

$$\int_{a\epsilon}^{\epsilon} \gamma^\epsilon(s)ds = \frac{1}{2}.$$

Let us normalize $\phi^\epsilon$ by $\phi^\epsilon(s) = a\epsilon$ for $s \le a\epsilon$. Observe that $\phi^\epsilon(s) \to s_+$ locally uniformly as $\epsilon \to 0$.

2. Suppose that there is $f \in C^{2,1}(Q)$ such that $u - f$ has its local maximum zero at $(x_0, t_0)$ in $\overline{\{u > 0\}}$. If $u(x_0, t_0) > 0$ one can easily check that $(f_t - \Delta f)(x_0, t_0) \le 0$ due to the stability property of viscosity solutions, and thus we may assume that $u(x_0, t_0) = 0$. Suppose that $f$ satisfies $\min(f_t - \Delta f, 1 - |Df|) > 0$ at $(x_0, t_0)$. By subtracting $\delta(x - x_0)^2 - \delta(t - t_0 + \delta)$ from $u - f$ if needed, we may assume that the maximum is strict and $u$ crosses $f$ from below at $t = t_\delta$, where $t_0 - \delta \le t_\delta \le t_0$. Then (along a subsequence) for small $\epsilon > 0$ $u^\epsilon - \phi^\epsilon(f)$ has its local maximum zero at $(x_\epsilon, t_\epsilon)$ with $(x_\epsilon, t_\epsilon) \to (x_\delta, t_\delta)$ as $\epsilon \to 0$. Since $u^\epsilon > 0$ is a subsolution of $(P^\epsilon)$, we have the following inequality at $(x_\epsilon, t_\epsilon)$:

$$\phi^\epsilon_s(f) \cdot (f_t - \Delta f) - \phi^\epsilon_{ss}(f)|Df|^2 \le -\beta^\epsilon(u^\epsilon).$$

Now if we choose $1 < c < 1/|Df|^2(x_0, t_0)$, then above inequality leads to a contradiction.

3. Finally consider a stationary supersolution $h \ge 0$ of (P). (For example consider a radially symmetric harmonic function $h$ in the set $D_r = \{x : |x - x_0| > r\}$ with $h > 0$ in $D_r$ and

$h = 0$ on $\partial D_r$. After a constant multiplication we may assume that $|Dh| = 1$ on $\partial D_r$ and thus $h(x,t) = h(x)$ is a classical supersolution of (P) with free boundary $D_r \times [0,\infty)$.) By comparing $u^\epsilon$ with $\phi^\epsilon(h)$ and by applying similar arguments to $u^\epsilon - \phi^\epsilon(h)$ as above, we can easily check that $\overline{\{u > 0\}} \cap \{t = 0\} = \overline{\{u_0 > 0\}}$. Thus we conclude that $u$ is a viscosity subsolution of $(P)$.

4. To prove that $u$ is a viscosity supersolution, suppose that there is a $C^{2,1}$ function $g$ such that $u - g$ has its local minimum zero at $(x_0, t_0)$ in $\overline{\{v > 0\}}$. As before we only have to check when $u(x_0, t_0) = 0$ and when $g$ satisfies $\max(g_t - \Delta g, 1 - |Dg|) < 0$ at $(x_0, t_0)$. Without loss of generality we may assume that the minimum is strict. Thus for small $\epsilon$ and for $\phi^\epsilon$ as given above, (along a subsequence) $u^\epsilon - \phi^\epsilon(g)$ has its strict minimum at $(x_\epsilon, t_\epsilon)$ which converges to $(x_0, t_0)$ as $\epsilon \to 0$. We have the following inequality at $(x_\epsilon, t_\epsilon)$:

$$\phi^\epsilon_s(g)(g_t - \Delta g) - \phi^\epsilon_{ss}(g)|Dg|^2 \geq -\beta^\epsilon(u^\epsilon).$$

This contradicts the definition of $\phi^\epsilon$ and the fact that $c > 1$.

□

# 2 Comparison principle

**Definition 2.1** *We say that a pair of functions $u_0, v_0$ is strictly ordered if*

*(i)* $\mathrm{supp}(u_0) = \overline{\{u_0 > 0\}}$ *is bounded, and it satisfies*

$$\mathrm{supp}(u_0(x)) \subset \mathrm{Int}(\mathrm{supp}(v_0(x))).$$

*(ii) inside* $\mathrm{supp}(u_0)$ *the functions are strictly ordered:*

$$u_0(x) < v_0(x).$$

We denote such ordering, or separation, by the symbol $u_0 \prec v_0$.

**Theorem 2.2** *Let $u$ and $v$ be respectively a sub- and supersolution of the equation $(P)$ with strictly separated initial data, $u_0 \prec v_0$. Then the solutions remain ordered for all time:*

$$u(x,t) \prec v(x,t) \text{ for every } t > 0.$$

To obtain a finite propagation property, we apply inf- and sup-convolution twice, first in $D_r(x) = \{y : |x-y| \leq r\}$ and then $B_r(x,t) = \{(y,s) : |x-y|^2 + |t-s|^2 \leq r^2\}$. More precisely, in the domain $Q_r = I\!R^n \times [r, \infty)$ let us define functions $Z$ and $W$ as given below:

$$Z(x,t) = \sup_{B_r(x,t)} U(y,s) \quad \text{where } U(x,t) = \sup_{D_r(x)} u(y,t),$$

$$W(x,t) = \inf_{B_r(x,t)} V(y,s) \quad \text{where } V(x,t) = \inf_{D_r(x)} v(y,t).$$

Note that $Z, U$ and $W, V$ are respectively viscosity sub- and supersolution of $(P)$.

Since $u_0 \prec v_0$ and $\overline{\{u > 0\}} \cap \{t = 0\} = \overline{\{u_0 > 0\}}$, we can take $r$ small enough that $Z \prec W$ at $t = r$. For such $r > 0$ we claim that $Z \prec W$ for $t > r$, and thus $u \prec v$ for $t > 0$.

Suppose that the claim is not true. Then for $Z, W$ defined with $r > 0$ chosen small as above we have

$$0 < t_0 = \sup\{t \geq r : Z(x,\tau) \prec W(x,\tau) \quad \text{for } r \leq \tau < t\} < \infty.$$

By a barrier argument we can easily show that $\{Z > 0\} \cap \{t \leq t_0\}$ is bounded, and by continuity of $Z, W$ there is a point $P_0 = (x_0, t_0)$ where $Z - W$ attains its maximum zero in $\overline{\{Z > 0\}} \cap \{t \leq t_0\}$. If $Z(P_0) = W(P_0) > 0$, at $t = t_0$ the we get a contradiction by the maximum principle of heat equation. This implies that indeed $Z = W = 0$ and $P_0$ belongs to the set $\partial\{Z > 0\} \cap \partial\{W > 0\}$.

By definition the set $\{Z > 0\}$ has an interior space-time ball of radius $r$ at $P_0$, centered at $P_1 \in \partial\{U > 0\}$. On the other hand at $P_1$ the set $\{U > 0\}$ has an exterior space-time ball $B_1$ of radius $r$ centered at $P_0$, for $t \leq t_1$. (From now on we set $r = 1$ for simplicity.) By choosing appropriate origin and coordinates, we may assume that $P_0 = (0, t_0)$ and space projection of $\overline{P_0P_1} = d_1 e_1$, where $d_1 > 0$ and $e_1 = (1, 0, ..., 0)$. Similarly at $P_0$ by definition the set $\{W > 0\}$ has an exterior space-time ball, centered at $P_2 \in \partial\{V > 0\}$, while at $P_2$ the set $\{V > 0\}$ has a interior space-time ball $B_2$ for $t \leq t_0$, centered at $P_0$. Observe that the space projection of $\overline{P_2P_0} = d_2 e_1$, $d_2 > 0$.

Let $H$ be the tangent hyperplane to the interior ball of $Z$ at $P_0$.

**Lemma 2.3** *$H$ is not horizontal.*

**Proof.**

1. Suppose $H$ is horizontal. Then either $\overline{P_0P_1} = (0, ..., 0, 1)$ or $\overline{P_0P_1} = (0, ..., 0, -1)$. (Recall that we fixed $r = 1$ for simplicity.)

Figure 1.

2. We first consider the case $\overline{P_0P_1} = (0, ..., 0, 1)$. Then at $P_1 = (x_1, t_1)$ the set $\{U > 0\}$ has an exterior ball $B_1$ with horizontal tangency. By definition, $u = 0$ in the region $L_1 = \{(y, s) : |y - x| = 1, (x, s) \in B_1\}$. In particular at $p_1 \in \partial\{u > 0\}$, the set $\{u > 0\}$ has an exterior ball $B_1'$: a translate of $B_1$ with horizontal tangency and the set $\{u(\cdot, t_1) > 0\}$ has an exterior disk $D_1$ with center $P_1$. Let us set $P_1 - p_1 = e_1$. After comparing $u$ with a caloric function in the region $2D_1 \backslash D_1 \times [t_1 - \tau, t_1]$ with lateral boundary data zero on $\partial D_1$, $\max_{2D_1} u$ on $\partial(2D_1)$ and with a smooth nonnegative initial data, we can check that

$$\alpha = \limsup_{(x,t)\to p_1} \frac{u(x,t)}{|x|} < \infty.$$

3. Suppose $\alpha > 0$. Our goal here is to derive a contradiction by constructing a local supersolution $\phi$, which crosses $u$ from above at $p_1$ with $|D\phi|(p_1) < \alpha$.

For a technical reason we replace balls which are used in the definition of $Z$ by ellipsoids. In fact all our previous arguments work with $Z$ replaced by

$$Z'(x,t) = \sup_{E_r(x,t)} U(y,s),$$

where $E_r(x,t) = \{(y,s) : |y - x|^2 + k^2/2(s-t)^2 = k^2/2r^2\}$ with $k = 64(n-1)$. More precisely, for the sake of simplicity we are using the standard notation of $Z$ defined thru balls instead of $Z'$ except at this point, where the properties of ellipsoids are necessary. Therefore it should be pointed out that the function $Z$ we have been using and will use in the proofs is actually $Z'$ defined above thru ellipsoids.

4. Recall that we fix $r = 1$ for simplicity. By a change of coordinate we can set $p_1 = (0, 0)$. Now near $p_1$ we have $u = 0$ in the set $\Gamma = \{(x,t) : |x|^2 \leq -k^2 t\}$. Furthermore, $u = 0$ in the union of translates of $\Gamma$ by points in $D_1$. Therefore after scaling $u(x,t) \to \epsilon^{-1}u(\epsilon x, \epsilon^2 t)$, for any $\delta > 0$ we can construct a subsolution $\omega$ in a cylinder $C = \{|x| \leq k, -1 \leq t \leq 0\}$ such that

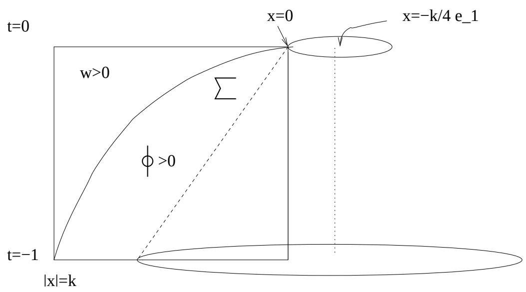


Figure 2.

$$\begin{cases} \omega = 0 & \text{at } \{t=-1\} \cap C, \\ \omega = 0 & \text{in } \Sigma = \{(x,t) : |x + k/4e_1|^2 \le k^2/16 - k^2 t\} \cap C, \text{ and} \\ \omega \le (\alpha+\delta)k & \text{on } |x| = k. \end{cases}$$

(see Figure 2.) Consider $\phi$: a smooth function given by

$$\text{(2.1)} \qquad \begin{cases} \phi(x,t) = f((1+1/2t)|x+k/4e_1|), \\ f(r) = 1/4\beta(r-k/4)^2 + (\beta-\delta)(r-k/4) \end{cases}$$

where $\beta = \alpha$. Observe that in the set $\Sigma$, $\phi$ satisfies

$$\phi_t - \Delta\phi \ge (1/2 - 4(n-1)/k)f' - f'' > 0 \quad \text{if } \delta << \alpha.$$

One can easily check that $\phi \ge 0$ on $\partial\Sigma$. Moreover, on $|x| = k$

$$\phi > 1/64\alpha k^2 + 3/4(\alpha-\delta)k > (\alpha+\delta)k$$

if $\delta << \alpha$ (for example if $\delta < 1/4\alpha$). Therefore $w \le \phi$ in the parabolic boundary of $\{w > 0\}$ in C, and by the maximal principle of the heat equation $u \le \phi$ in $C$. But then we have

$$\alpha < \limsup_{(x,t)\to p_1} \frac{\phi(x,t)}{|x|} = \alpha - \delta,$$

which is a contradiction.

5. Thus $\alpha = 0$. Now we can construct $\phi$ in (2.1) with $\beta = 1/4, \delta = 0$, which crosses $u$ from above at $p_1$ with $(\phi_t - \Delta\phi)(p_1) > 0, |D\phi|(p_1) = 1/4$. This contradicts the definition of $u$ as a viscosity subsolution of $(P)$.

6. Thus $\overline{P_0P_1} = (0, ..., -1)$ and we have $\overline{P_0P_2} = (0, ..., 0, 1)$. At $P_2 = (x_2, t_2)$ the set $\{V > 0\}$ has an interior ball $B_1$ with horizontal tangency. By definition, $v > 0$ in the region $L_2 = \{(y, s) : |y - x| = 1, (x, s) \in B_1\}$. In particular at $p_2 \in \partial\{v > 0\}$, the set $\{v > 0\}$ has an interior ball $B_1'$: a translate of $B_1$ with horizontal tangency and the set $\{u(\cdot, t_2) > 0\}$ has an interior disk $D_1$ with center $P_2$. A parallel argument as in the previous steps, by investigating the behavior of $v$ near the point $p_2$, leads to a contradiction.

$\square$.

Due to Lemma 2.3, if $H$ is the tangent hyperplane to the interior ball to $\{Z > 0\}$ at $P_0$, we can write $(e_1, m)$, $-\infty < m < \infty$ : the internal normal vector to $H$ with respect to $\{Z > 0\}$ at $P_0$. We call $m$ as *the advancing speed* of $H$ (with respect to $Z$) at $P_0$.

The following lemma is based on the corresponding result of [K] and [CV1] and thus we only present a brief version of the proof. For details, see [CV1]. For a constant $0 < c < 1$ we consider a *nontangential cone* $K$ defined as below:

$$K = \{x \in I\!R^n : \frac{x}{|x|} \cdot e_1 \geq c\}.$$

**Lemma 2.4** *In any nontangential cone $K$,*

$$\liminf_{x \to 0, x \in K} \frac{Z(x, t_0)}{(x_1)_+} \geq 1. \tag{2.2}$$

**Sketch of the proof.**

1. Suppose that (2.2) is not true. Then for a sequence of points $A_n = (x_{1n}, x_n')$ converging to $0 \in I\!R^n$ and lying in a nontangential cone $K$ we have

$$Z(Q_n) \leq (1 - \epsilon)(x_{1n})^+ \qquad \text{for some } \epsilon > 0 \tag{2.3}$$

with $Q_n = (A_n, t_0)$.

2. By arguments based on the definition of $Z$ (see [CV]) and taking $x_{1n} = \lambda > 0$ small in (2.3) we conclude that there is a set $L = L_\lambda$ in space-time as portrayed in Figure 3 where $U \leq \mu_\lambda = (1 - \epsilon)\lambda$. The boundary of $L_\lambda$ contains a concave part closer to the origin, which contains $P_1$ and exterior to $B_1$, and there we have $U = 0$. By straightforward computation it turns out to be that $L_\lambda$ is (in space) of depth $\lambda$ and width $O(\sqrt{\lambda})$. Let $L_\tau = L \cap \{t_1 - \tau \leq t \leq t_1\}$ for $\tau > 0$.

Figure 3.

3. Consider a space-time ball $\tilde{B}_1$ such that $\tilde{B}_1 \cap \{t = t_1\} = B_1 \cap \{t = t_1\}$ with the advancing speed of its tangent hyperplane at $P_1$ equal to $m - \epsilon$. (Here $m$ is the advancing speed of the tangent hyperplane of $B_1$ at $P_1$.) We compare $U$ in $L_\tau$ with a radially symmetric $C^{2,1}$ function $\phi$ such that

$$\begin{cases} \phi_t - \Delta\phi = 1 & \text{in } (2\tilde{B}_1 - \frac{1}{4}\tilde{B}_1) \cap \{t_1 - \tau < t \leq t_1\}, \\ \{\phi(\cdot, s) > 0\} = \tilde{B}_1^c \cap \{t = s\} & \text{for } t_1 - \tau < s \leq t_1, \\ |D\phi| = 1 - \epsilon/2 & \text{on } \partial\tilde{B}_1 \cap \{t = t_1\}. \end{cases}$$

(Such test function $\phi$ can be obtained from a slight modification from Appendix B in [K].) If we choose $\lambda$ small compared to $\epsilon$ and $\tau$, then it follows that $\phi$ is bigger than $(1-\epsilon)\lambda$ on $L_\tau \cap \{t = t_1 - \tau.\}$. Moreover by computations as in [K] it turns out that $U \leq \phi$ in the lateral boundary of $L_\tau$ for $\tau, \lambda$ small compared to $\epsilon$, and thus $U$ crosses $\phi$ from below at $P_1$. This leads to a contradiction since $\phi$ satisfies

$$\min(\phi_t - \Delta\phi, 1 - |D\phi|)(P_1) > 0.$$

$\square$

**Proof of Theorem 2.2**

1. By a change of coordinates, we set $P_0 = (x_0, t_0) = (0, 1)$. At $P_0$, the set $\{Z(x, t_0) > 0\}$ has an interior space-time ball $B$ with its inward normal vector $(e_1, m), |m| < \infty$. Note that since $|m| < \infty$ we have $B \cap \{t = 1 - \tau\} \neq \emptyset$ for $0 < \tau < \tau_0 = \tau_0(m)$.

Next we consider $h$: a $C^{2,1}$ solution of heat equation in the domain $B \cap \{1 - \tau \leq t \leq 1\}$ with $h(\cdot, 1 - \tau) \geq 0$ and $h = 0$ on $\partial B$. Since $Z < W$ at $t = 1 - \tau$ in $\{Z > 0\}$, there is $\epsilon > 0$ such that

$$Z + \epsilon h < W \qquad \text{in } \{Z > 0\} \cap \{t = 1 - \tau\}.$$

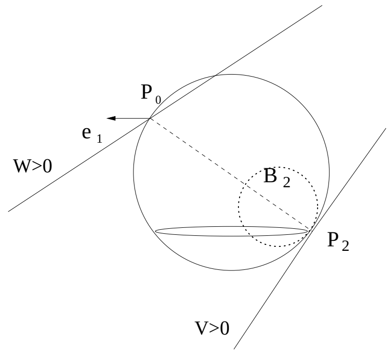


Figure 4.

Moreover, $Z + \epsilon h = 0 \leq W$ on $\partial B$. Thus by maximum principle of heat equations, $Z \leq W$ on $B \cap \{t = 1\}$ and for any nontangential cone $K$ we have

$$\liminf_{x \to 0, x \in K} \frac{W(x, t_0)}{(x_1)_+} \geq 1 + \epsilon Dh \cdot e_1 > 1 + \delta, \tag{2.4}$$

where $\delta > 0$ is a constant. The last inequality comes from the Hopf's formula.

2. At $P_2 = (x_2, t_2)$, due to the definition of $W$ the set $\{V > 0\}$ has an interior space-time ball $B_2$ with radius $0 < h \leq 1/2$ such that $B_2$ has a tangent hyperplane at $P_2$ with advancing speed equal to $m$ (When $h = 1/2$, $B_2$ has its center at $P_3 = (x_3, t_3) = (P_0 + P_2)/2$ and $P_0, P_2 \in \partial B_2$. See Figure 4.)

If $(x, s) \in B_2$ and if $d(x, \partial B_2 \cap \{t = s\}) = k$ then it follows that $d((x, s), P_0 + ke_1) \leq 1$ and by definition of $W$ we have $V(x, s) \geq W(P_0 + ke_1)$.

Thus if $h$ is small enough, due to (2.4) the following holds for $(x, s) \in \mathrm{int} B_2$:

$$(1 + \delta/2) d(x, \partial B_2 \cap \{t = s\}) < V(x, s). \tag{2.5}$$

( Observe that from the proof of Lemma 2.3 we have $B_2 \cap \{t = t_2\} \neq \emptyset$. )

3. Consider a smooth function $\varphi$ given by

$$\begin{cases} \varphi_t - \Delta \varphi = -1 & \text{in} \quad (2B_1 - \frac{1}{4}B_2) \cap \{t_2 - \tau < s <\leq t_2\}, \\ \{\varphi(\cdot, s) > 0\} = B_2 \cap \{t = s\} & \text{for } t_2 - \tau < s \leq t_2, \\ |D\varphi| = 1 + \delta/4 & \text{on } \partial B_2 \cap \{t = t_2\}. \end{cases}$$

( After translations in space and time, $\varphi = -c\phi$ where $\phi$ as in Lemma 2.4 and $c > 0$ is a constant.) It follows from (2.5) that $\phi \leq V$ and therefore $v - \phi$ has a local minimum zero at $P_2$ in $\overline{\{V > 0\}} \cap \{t \leq t_2\}$. But this leads to a contradiction since $\varphi$ satisfies

$$\max(\varphi_t - \Delta\varphi, 1 - |D\varphi|)(P_2) < 0.$$

$\square$

**Corollary 2.5** *For a given nonnegative initial data $u_0 \in C(I\!R^n)$ with bounded support, there exist the minimal and maximal viscosity solution of (P) with initial data $u_0$.*

**Proof.**

1. Suppose that $(u_0^\epsilon)\epsilon > 0$ satisfy (0.2) with $u_0^\epsilon \prec u_0$. (This could be done, for example by taking $u_0^\epsilon$: a smooth, nonnegative and close enough approximation of $u_0 - \epsilon$.) Fix $\epsilon > 0$ and $u^\delta$ be the smooth solution of $(P^\delta)$ with initial data $u_0^\epsilon$. Due to [CV1] and Theorem 1.2, along a subsequence $u\delta$ uniformly converges to a viscosity solution $u(x,t;\epsilon)$ of $(P)$ with initial data $u_0^\epsilon$. Therefore for any $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that $u^\delta \leq u(x,t;\epsilon) + \epsilon$.

2. Again Due to [CV1] and Theorem 1.3, along a subsequence $(u^{\delta(\epsilon)})_\epsilon$ uniformly converges as $\epsilon \to 0$ to a viscosity solution $U$ of $(P)$ with initial data $u_0$. By above arguments, if $v$ is any other viscosity solution of $(P)$ with initial data $u_0$, then $u(x,t,\epsilon) \leq v$ from Theorem 2.2, and thus $U \leq v$. Thus $U$ constructed as above is the minimal viscosity solution of $(P)$ with initial data $u_0$. The maximal solution can be constructed in a parallel way.

$\square$.

The uniqueness of the viscosity solution of $(P)$, as mentioned earlier, does not hold in general. However under certain conditions on the structure of the initial data, one can show the uniqueness results by using Theorem 2.2 and scaling arguments. Below we consider the cases which were studied in [P] (case a), [CV1] (case b). For these cases we first show that the comparison principle holds between solutions with the same initial data. As we explain later, this result immediately leads to the uniqueness of the viscosity solution.

**Corollary 2.6** *Suppose that $u_0 \in C(I\!R^n)$, $u_0 \geq 0$ with bounded support and $|Du_0|$=1 on the set $\partial\{u_0 = 0\}$. In addition, suppose that one of the following properties holds for $u_0$:*

*(a) $u_0$ is starshaped, i.e. $u_0(\alpha x) \prec \alpha u_0(x)$ for $\alpha > 1$.*

*(b) $u_0$ is $C^2(\{u_0 > 0\})$ and $u_0$ is strictly superharmonic, i.e. $\inf_{\{u_0>0\}} -\Delta u_0 > 0$.*

*Suppose that $u, v$ are respectively viscosity sub- and supersolution of (P) with initial data $u_0$ and $v_0$. Then the following property holds:*

$$\textit{If } u_0(x) \leq v_0(x), \textit{ then } u(x,t) \leq v(x,t) \quad \textit{for every } t > 0.$$

**Proof.**

1. Let us first assume that (a) holds for $u_0$. We observe that if $u$ is a viscosity subsolution of (P), then the version of (P) solved by $\bar{u}(x,t;\epsilon) = (1+\epsilon)^{-1}u((1+\epsilon)x, (1+\epsilon)^2 t)$ has $\bar{u}(x,0;\epsilon)$ as initial data for any $\epsilon > 0$. Moreover, by (a) $\bar{u}(x,0;\epsilon) \prec v_0(x)$. From Theorem 2.2 we get $\bar{u}(x,t;\epsilon) \leq v$. Since $u$ is continuous, we can send $\epsilon \to 0$ to conclude that

$$u(x,t) = \lim_{\epsilon \to 0} \bar{u}(x,t;\epsilon) \leq v(x,t).$$

2. Next we assume that (b) holds. It is clear that $u_t < 0$ at $t = 0$ in the positive set $\{u > 0\}$. We will show that in fact the free boundary of $u$ strictly shrinks at $t = 0$. Since the boundary $\Gamma = \partial\{u_0 > 0\}$ is a $C^2$-hypersurface, at each point $x_0 \in \Gamma$ there is an exterior space ball $B$ such that $\Gamma \cap B = \{x_0\}$. In the domain $\Omega = I\!R^n - B$, we consider a radially symmetric function $\varphi \in C^2(\overline{\Omega})$ such that

$$\begin{cases} -\Delta\varphi = c/2 = \inf_{\{u_0>0\}} -\Delta u_0 > 0 & \text{in } \Omega, \\ \\ |D\varphi| = 1, \varphi = 0 & \text{on } \partial B. \end{cases}$$

Note that

$$\begin{aligned} -\Delta\varphi(x_0) &= -\kappa(\varphi)(x_0) - \varphi_{nn}(x_0) \quad \text{and} \\ \\ -\Delta u_0(x_0) &= -\kappa(u_0)(x_0) - (u_0)_{nn}(x_0), \end{aligned}$$

where $\kappa(f,x)$ denotes the mean curvature of the set $\{f = 0\}$ at $x$. ($\kappa > 0$ for convex free boundary), and $n$ denotes the inward normal vector of the free boundary. Since $B$ is exterior to $\{u_0 > 0\}$, we have $-\kappa(\varphi) \geq -\kappa(u_0)$. Furthermore $-\Delta\varphi(x_0) = c < -\Delta u_0(x_0)$. Therefore we have $(u_0)_{nn}(x_0) < \varphi_{nn}(x_0)$ and thus $\varphi \geq u_0$ in a neighborhood of $B$. Since $\varphi$ is radially symmetric superharmonic function, there is a classical solution of $(P)$ with initial data $\varphi$ in short time with strictly shrinking free boundary, (see [V]) and thus we conclude that the free boundary of $u$ strictly shrinks at $t = 0$. Now consider $\bar{u}(x,t;\epsilon) = u(x, t+\epsilon)$. Since $\bar{u}(x,0;\epsilon) \prec u_0(x)$, $\bar{u} \leq v$ and sending $\epsilon \to 0$ implies that $u \leq v$.

$\square$.

By Theorem 1.3 there exists a viscosity solution of (P) given as a uniform limit of $(u^\epsilon)_\epsilon$ as $\epsilon \to 0$. It follows from Corollary 2.6 that such $u$ is indeed the unique viscosity solution of (P).

**Corollary 2.7** *Assuming that one of the conditions (a)-(b) holds for $u_0$, there is a unique viscosity solution $u$ of (P) with initial data $u_0$. Moreover for these cases the entire sequence of $(u^\epsilon)_\epsilon$ converges locally uniformly to $u$ as $\epsilon \to 0$, where $u^\epsilon$ is a solution of $(P^\epsilon)$ with initial data prescribed as in (0.2).*

# 3 Neumann boundary problem

In this section we study $(P)$ in a domain with the Neumann boundary condition. For given domain $\Omega \subset I\!R^n$, we denote $n = n(x)$: the outward unit normal vector w.r.t. $\Omega$ at $x \in \partial\Omega$. The problem is to find a solution $u \geq 0$ in $\Pi = \overline{\Omega} \times (0, \infty)$ such that

$$(P_2) \qquad \begin{cases} u_t - \Delta u = 0 & \text{in } \{u > 0\}, \\ |Du| = 1 & \text{on } \partial\{u > 0\}, \\ \partial u(x,t)/\partial n(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

**Definition 3.1** *(1) A nonnegative continuous function $u$ in $\Pi$ is a viscosity subsolution of $(P_2)$ if (i) $\overline{\{u > 0\}} \cap \{t = 0\} = \overline{\{u_0 > 0\}}$ and (ii) for every $\phi \in C^{2,1}(\Pi)$ that has a local maximum of $u - \phi$ in $\overline{\{u > 0\}} \cap \{t \leq t_0\}$ at $(x_0, t_0)$,*

$$\begin{cases} (\phi_t - \Delta\phi)(x_0, t_0) \leq 0 & \text{if } u(x_0,t_0) > 0, x_0 \in \Omega, \\ \min(\phi_t - \Delta\phi, 1 - |D\phi|)(x_0, t_0) \leq 0 & \text{if } u(x_0,t_0) = 0, x_0 \in \Omega, \\ \min(\phi_t - \Delta\phi, \partial\phi/\partial n)(x_0, t_0) \leq 0 & \text{if } u(x_0,t_0) > 0, x_0 \in \partial\Omega, \\ \min(\phi_t - \Delta\phi, 1 - |D\phi|, \partial\phi/\partial n)(x_0, t_0) \leq 0 & \text{if } u(x_0,t_0) = 0, x_0 \in \partial\Omega. \end{cases}$$

*(2) A nonnegative continuous function $v$ in $\Pi$ is a viscosity supersolution of $(P_2)$ if for every $\varphi \in C^{2,1}(\Pi)$ that has a local minimum of $v - \varphi$ in $\overline{\{v > 0\}} \cap \{t \leq t_0\}$ at $(x_0, t_0)$ (and with (1.1) if $v(x_0, t_0) = 0$,)*

$$\begin{cases} (\varphi_t - \Delta\varphi)(x_0, t_0) \geq 0 & \textit{if } v(x_0, t_0) > 0, x_0 \in \Omega, \\ \max(\varphi_t - \Delta\varphi, 1 - |D\varphi|)(x_0, t_0) \geq 0 & \textit{if } v(x_0, t_0) = 0, x_0 \in \Omega, \\ \max(\varphi_t - \Delta\varphi, \partial\varphi/\partial n)(x_0, t_0) \geq 0 & \textit{if } v(x_0, t_0) > 0, x_0 \in \partial\Omega, \\ \max(\varphi_t - \Delta\varphi, 1 - |D\varphi|, \partial\varphi/\partial n)(x_0, t_0) \geq 0 & \textit{if } v(x_0, t_0) = 0, x_0 \in \partial\Omega. \end{cases}$$

$u$ is a *viscosity solution* of $(P_2)$ if it is both viscosity sub- and supersolution of $(P_2)$.

**Theorem 3.2** *Assume that $\partial\Omega$ is $C^3$ and $\Omega$ is bounded. Let $u$ and $v$ be respectively a sub- and supersolution of $(P_2)$ with strictly separated initial data, $u_0 \prec v_0$. Then the solutions remain ordered for all time:*

$$u(x,t) \prec v(x,t) \textit{ for every } t > 0.$$

Let $R > 0$ be small enough that in $B_R(x_0) \cap \partial\Omega$, there exists a regular $C^3$ parameterization $\sigma$ in the variables $y' = (y_1, ..., y_{n-1})$ in a neighborhood $\mathcal{N}$ of origin in $I\!R^{n-1}$. Let

$$x = \sigma(y') + y_n \nu(\sigma(y')) := h^{-1}(y).$$

Then $h$ is $C^2(B_R(x_0), \mathcal{N})$ with nonvanishing Jacobian $Dh$ in $B_R(x_0)$ and $h(\Omega \cap B_R(x_0)) = \mathcal{N} \cap \{y_n > 0\}$.

For a function $v$ in $\Omega$, Let us define $v' = v(h^{-1}(y), t)$ in $y \in \mathcal{N} \cap \{y_N > 0\}$ and

$$\bar{v}(y,t) = \begin{cases} v'(y,t) & \text{if } y \in \mathcal{N} \cap \{y_n > 0\} \\ v'(y', -y_n, t) & \text{if } y \in \mathcal{N} \cap \{y_n \leq 0\}. \end{cases} \tag{3.1}$$

If $v$ is a viscosity sub- or supersolution of $(P_2)$ in $\Pi$, then it is easy to check that, with a parallel definition with Definition 1.1, $\bar{v}$ is a viscosity sub- or supersolution to the following problem in $\mathcal{D} = \mathcal{N} \times (0, \infty)$:

$$(P)' \qquad \begin{cases} (*) \quad \bar{v}_t - F(D^2\bar{v}, D\bar{v}, y, t) & \\ \quad = \bar{v}_t - \Sigma_{i,j}(a_{ij}(y)\bar{v}_{y_j}(y,t))_{y_i} = 0 & \text{in } \{\bar{v} > 0\}, \\ |Dh(h^{-1}(y)) \cdot D\bar{v}(y)| = 1 & \text{on } \partial\{\bar{v} > 0\}, \end{cases}$$

where $a_{ij}(y) = \nabla h_i(x) \cdot \nabla h_j(x), h(x) = y$ in $\mathcal{N}$.

( For example, consider the case in which $v$ is a viscosity subsolution of $(P_2)$ in $\Pi$. We claim that $\bar{v}$ is a viscosity subsolution of $(P)'$ in $\mathcal{D} = \mathcal{N} \times (0, \infty)$.

Suppose there is a smooth function $\phi : \mathcal{D} \to I\!R$ such that $\bar{v} - \phi$ has a local maximum at $(y_0, t_0) \in \mathcal{D}$. Without loss of generality we may assume that the maximum of $\bar{v} - \phi$ is strict. We show that if $y_0 \in \{y : y_n = 0\}$ and $\bar{v}(y_0, t_0) > 0$ then

$$\phi_t - F(D^2\phi, D\phi, y, t) \leq 0. \tag{3.2}$$

Let us define $\varphi = \phi(y, t) + \phi(y', -y_n, t) + \epsilon y_n$ for $y \in \mathcal{N}$ and $\Phi(x, t) = \varphi(h(x), t)$ for $x \in h^{-1}(\mathcal{N})$. Then $\Phi$ is smooth in a small neighborhood $\mathcal{M}$ of $(\bar{x}, t_0), \bar{x} = h^{-1}(y_0)$. Moreover, since $v(\bar{x}, t_0) > 0$, if $\epsilon > 0$ is small enough then $v - \Phi$ has a local maximum at $(x_\epsilon, t_\epsilon) \in \mathcal{M} \cap \Pi$ with $v(x_\epsilon, t_\epsilon) > 0$. Since

$$\partial\Phi(x, t)/\partial n = \partial\varphi/\partial(y_n) = \epsilon > 0 \quad \text{for } x \in \partial\Omega,$$

the inequality (3.2) follows from Definition 3.1, (1). Other cases can be also shown by parallel arguments as above, and thus our claim follows.)

Since $a_{ij}(0) = \nabla h_i(x_0) \cdot \nabla h_j(x_0) = \delta_{ij}$, we can choose $R > 0$ small enough so that in $\mathcal{N} = h(B_R(x_0))$ we have

$$1/2|\xi|^2 \leq \Sigma_{i,j} a_{ij} \xi_i \xi_j \leq 2|\xi|^2$$

and

$$1/2|\xi| < |Dh(h^{-1}(y)) \cdot \xi| < 2|\xi| \quad \text{for } \xi \in I\!R^n.$$

Note that with above conditions solutions of $(*)$ with smooth boundary data are $C^{2,1}$ up to the boundary. (For example see [CC]).

Now we consider $u$ and $v$ as given in Theorem 3.2. For $r > 0$, we define $Z$ and $W$ as below:

$$Z(x, t; r, Q) = \sup_{B_r(x,t) \cap \Pi} U(y, s) \quad \text{where } U(x, t) = \sup_{D_r(x) \cap \Pi} u(y, t),$$
$$W(x, t; r, Q) = \inf_{B_r(x,t) \cap \Pi} V(y, s) \quad \text{where } V(x, t) = \inf_{D_r(x) \cap \Pi} v(y, t).$$

Note that $Z, U$ and $W, V$ are respectively viscosity sub- and supersolution of $(P_2)$. Suppose the theorem does not hold. Then for $r > 0$ so small that $Z \prec W$ at $t = r$ there is $0 \leq T < \infty$ such that

$$T = \sup\{t \geq r : Z(x,\tau) \prec W(x,\tau) \quad \text{for } 0 \leq \tau < t\}.$$

Therefore $Z - W$ has its maximum zero in $\overline{\{Z(\cdot,t) > 0; t \leq T\}}$ at $P = (x_0, T)$. If $Z(P) = W(P) > 0$, a standard viscosity argument leads to a contradiction. (For example see [GS]). If $P \in \Omega$, then we could argue as in section 2 to lead to a contradiction. Thus $Z(P) = W(P) = 0$ and $P = (x_0, T) \in \partial\Omega$.

Defined by (3.1) with $h(x_0) = 0$, the functions $\bar{Z}$ and $\bar{W}$ has a local maximum at $(0, T)$ in $\overline{\{\bar{Z} > 0\}} \cap \{t \leq T\} \cap \mathcal{N}$. We are going to apply the sup/inf- convolutions again in the new neighborhood $\mathcal{N}$, and therefore to avoid confusion we denote $\bar{Z}$ by $\tilde{u}$ and $\bar{W}$ by $\tilde{v}$. Let $\tilde{Z} = \tilde{Z}(y,t;\delta,\mathcal{N})$ and $\tilde{W} = \tilde{W}(y,t;\delta,\mathcal{N})$ be the corresponding sup- and inf-convolution of $\tilde{u}$ and $\tilde{v}$ as above. Then for small $\delta > 0$, $\tilde{Z} - \tilde{W}$ has its maximum zero at $P_0 = (y_0, t_0)$ in the domain $\overline{\{\tilde{Z} > 0\}} \cap \{t \leq t_0\} \cap \mathcal{N}$ at $t = t_0$, where

$$t_0 = \sup\{t \leq r : \tilde{Z}(y,\tau) \prec \tilde{W}(y,\tau), 0 < \tau < t\}$$

By definition of $\tilde{Z}$, at $P_0$ there is a interior ball $B_1$ to the set $\{\tilde{Z} > 0\}$ with center $P_1 \in \partial\{\tilde{U} > 0\}$, where $\tilde{U} = \sup_{D_r(x)\cap\Pi} \tilde{u}(y,t)$. Let $H$ be the tangent hyperplane to $B_1$ at $P_0$.

**Lemma 3.3** *$H$ is not horizontal.*

**Proof.**

1. $H$ is horizontal when $\overline{P_0P_1}$ is either $(0, ..., r)$ or $(0, ..., -r)$. We only prove that $\overline{P_0P_1} \neq (0, ..., r)$. The other part can be shown similarly.

2. Suppose $\overline{P_0P_1} = (0, ..., r)$. By definition there is a point $p_1 = (x_1, t_1) \in \partial\{\tilde{u} > 0\}$ where the set $\{\tilde{u} > 0\}$ has an exterior ball $B_1$ with horizontal tangency and the set $\{\tilde{u}(\cdot, t_1) > 0\}$ has an exterior disk $D_1$ with center $P_1$. A parallel argument as in Lemma 2.4 implies that $\tilde{u}(P_1) = 0$. Let us set $P_1 - p_1 = e_1$. After comparing $\tilde{u}$ with a solution of $(*)$ in the region $(2D_1 - D_1) \times [t_1 - \tau, t_1]$ with boundary data zero on $\partial D_1$ and $\max_{2D_1}\tilde{u}$ on $\partial(2D_1)$, we can easily check that

$$\alpha = \limsup{}^*_{(x,t_1)\to p_1} \frac{\tilde{u}(x,t)}{|x|} < \infty.$$

3. Assume that $\alpha > 0$. Observe that for $x \in \mathcal{N}$ and $\epsilon > 0$ $\tilde{u}_\epsilon = \epsilon^{-1}\tilde{u}(\epsilon x, \epsilon^2 t)$ solves

$$(P)'_\epsilon \qquad \begin{cases} (*)_\epsilon \quad u_t - F^\epsilon(D^2u, Du, y, t) \\ \\ \quad = u_t - \Sigma_{i,j}(a_{ij}(\epsilon y)u_{y_j}(y,t))_{y_i} = 0. \quad \text{in } \{u > 0\}, \\ \\ |Dh(h^{-1}(\epsilon y)) \cdot Du(y,t)| = 1 \qquad\qquad \text{on } \partial\{u > 0\}. \end{cases}$$

Consider $\phi$ given by (2.1) with $\beta = \alpha$ and $\delta << \alpha$. since $\phi_t - \Delta\phi > 0$ and $a_{ij}(0) = \delta_{ij}$, $\phi$ is indeed a strict supersolution of $(P)'_\epsilon$ for small $\epsilon > 0$ with $|D\phi|(p_1) = \beta - \delta$. Thus a parallel argument using $\phi$ as a barrier function leads to a contradiction.

4. Thus $\alpha = 0$. Since $2|D\tilde{u}| > |Dh(h^{-1}(y)) \cdot D\tilde{u}| > 1$, we can conclude by comparing $\tilde{u}$ with $\phi$ in (2.1) with $\beta = 1/4, \delta = 0$.

□

To prove Theorem 3.2, it is enough to construct a classical, (strict) sub- and supersolution near

$$\begin{cases} (3.3) \quad \phi_t - F(D^2\phi, D\phi, y, t) = c > 0 & \text{in } (2B_1 - B_1) \cap \{t_1 - \tau < s \leq t_1\}, \\ \{\phi(\cdot, s) > 0\} = B_1^c \cap \{t = s\} & \text{for } t_1 - \tau < s < t_1, \\ |D\phi| = m - \lambda/2 & \text{on } \partial B_1 \cap \{t = t_1\}. \end{cases}$$

Such test function can be obtained by the following steps:

a. First solve $\varphi$ for (3.3) = 1 in $(2B_1 - B_1)$ with zero boundary data on $\partial B_1$ , a constant positive data $M$ on $\partial(2B_1)$, and a smooth nonnegative initial data at $t = t_1 - \tau$. Then $\varphi$ is $C^{2,1}$ up to the lateral boundary.

b. So far $\phi$ satisfies the first two conditions. Note that $|D\phi| > 0$ on $\partial B_1$ by the Hopf's formula, and thus last condition can be obtained by simply letting $\phi = c\varphi$, where $c$ is a proper constant.

Note that $(e_1, L)$ is a normal vector to $B_1$ at $P_1$ and therefore

$$|D\phi|(P_1) = D\phi(P_1) \cdot e_1, \quad |Dh(h^{-1}(y_0)) \cdot D\phi|(P_1) = m^{-1}|D\phi|(P_1) < 1.$$

Now arguments as in the proof of Lemma 2.3 leads to a contradiction since $\phi$ satisfies

$$\min(\phi_t - F(D^2\phi, D\phi, y, t), 1 - |Dh(h^{-1}(y_0)) \cdot D\phi|)(P_1) > 0.$$

□

**Proof of Theorem 3.2**

1. By changing coordinates, we set $P_0 = (y_0, t_0) = (0, 1)$. At $P_0$, the set $\{\tilde{Z}(x, t_0) > 0\}$ has an interior space-time ball $B$ with its inward normal vector $(e_1, L), |L| < \infty$. Note that since $|L| < \infty$ we have $B_2 \cap \{t = 1 - \tau\} \neq \emptyset$ for $0 < \tau < \tau_0 = \tau_0(L)$.

Next we consider $h$, a $C^{2,1}$ solution of $(*)$ in the domain $B\cap\{1-\tau\le t\le 1\}$ with $h(\cdot,1-\tau)\ge 0$ and $h=0$ on $\partial B$. Since $\tilde Z<\tilde W$ at $t=1-\tau$ in $\{\tilde Z>0\}$, there is $\epsilon>0$ such that

$$\tilde Z+\epsilon h<\tilde W \qquad \text{in } \{\tilde Z>0\}\cap\{t=1-\tau\}.$$

Moreover, $\tilde Z+\epsilon h=0\le\tilde W$ on $\partial B$. Thus by the maximal principle of the equation $(*)$, $\tilde Z\le\tilde W$ on $B_2\cap\{t=1\}$ and for any nontangential cone $K$ we have

$$\liminf_{x\to 0, x\in K}\frac{\tilde W(x,t_0)}{(x_1)_+}\ge m+\epsilon Dh\cdot e_1>m+\delta, \tag{3.4}$$

where $\delta>0$ is a constant. The last inequality comes from the Hopf's formula and the uniform ellipticity of $F$.

2. Let $P_2\in\partial\{\tilde v>0\}$ be the point where the value of $\tilde W(P_0)$ is obtained. At $P_2=(x_2,t_2)$, due to the definition of $\tilde W$ the set $\{\tilde v>0\}$ has an interior space-time ball $B_2$ with small radius $h>0$. If $h$ is small enough, due to (3.4) the following holds for $(x,s)\in \text{int}B_2$:

$$(m+\delta/2)d(x,\partial B_2\cap\{t=s\})<\tilde v(x,s). \tag{3.5}$$

3. As before, for small $\tau>0$ we can construct a $C^{2,1}$ function $\varphi$ such that

$$\begin{cases}\varphi_t-F(D^2\varphi,D\varphi,y,t)=-c<0 & \text{in} \quad (B_2-\frac{1}{4}B_2)\cap\{t_1-\tau<s\le t_1\},\\ \{\varphi(\cdot,s)>0\}=B_2\cap\{t=s\} & \text{for } t_1-h<s\le t_1,\\ |D\varphi|=m+\delta/4 & \text{on } \partial B_2\cap\{t=t_2\}.\end{cases}$$

It follows from (3.5) that $\phi\le\tilde v$ and therefore $\tilde v-\varphi$ has a local minimum zero at $P_2$ in $\overline{\{\tilde v>0\}}\cap\{t\le t_2\}$, which leads to a contradiction. We proved that $Z\le W$ for any $r>0$, and this leads to $u\le v$ as $r\to 0$.

□

**Corollary 3.4** *Suppose that $\Omega=I\!R\times\Sigma$, where $\Sigma$ is a bounded domain with $\partial\Sigma: C^3$. In addition assume that $u_0\in C^1(\Omega)$ with $(u_0)_{x_1}>0$ in $\{u_0>0\}$. If $u,v$ are viscosity sub- and supersolutions of $(P_2)$ with $u_0\le v_0)$, then $u\le v$ for all $t\ge 0$.*

**Proof.**

1. The assumption of $\Omega$ being bounded in Theorem 3.2 is only to guarantee that the set $\partial\{u > 0\}$ is bounded locally in time, and therefore to guarantee the contact point of $u$ and $v$ to exist. By comparing to a traveling wave solution $w(x,t)) = 1/c(1 - e^{-cx_1}), c > 0$ with $u_0(x) \prec w(x,0)$, we can easily prove that for each $T > 0$ the set $\partial\{u > 0\} \cap \{t \leq T\}$ is bounded if $u$ is a subsolution of $(P_2)$ with the initial data $u_0$. Thus Theorem 3.2 still holds for our case.

2. Next we observe that if $u$ is a viscosity subsolution of $(P_2)$ with the initial data $u_0$, then the same holds for $\bar{u}(x,t;\epsilon) = u(x_1 - \epsilon, x_2, ...x_n, t)$ with $\bar{u}(x,0) \prec u_0(x)$. This and Theorem 3.2 implies that $\bar{u}(x,t;\epsilon) \leq v$ for $t > 0$. Now we can conclude by sending $\epsilon \to 0$. $\square$

Next we prove a uniqueness and convergence result for solutions in a cylindrical domain with a monotonicity condition on the initial data. For $\epsilon > 0$ and $\beta^\epsilon$ given as in $(P^\epsilon)$ we consider the approximating equation:

$$(P_2^\epsilon) \qquad \begin{cases} u_t^\epsilon - \Delta u^\epsilon = -\frac{1}{\epsilon^2} u \exp(-\frac{1}{\epsilon} u) = -\beta^\epsilon(u), & \text{in } Q, \\ \partial u(x,t)/\partial n = 0 & \text{for } x \in \partial\Omega \end{cases}$$

Let $u^\epsilon$ be a solution of $(P_2^\epsilon)$ with initial data $u_0^\epsilon$ satisfying (0.2). For discussions on the convergence properties of $u^\epsilon$ with additional assumptions on $u^\epsilon(x,0)$ we refer to [LVW]. Here we only state that if a limit solution of $(P_2^\epsilon)$ exists then it is the unique viscosity solution of $(P_2)$.

**Corollary 3.5** *Let $\Omega$ and $u_0$ satisfy conditions in* Corollary 3.4. *Suppose that along a subsequence $(u^\epsilon)_\epsilon$ given above locally uniformly converges to $u$ as $\epsilon \to 0$. Then such $u$ is unique. Moreover $u$ is the unique viscosity solution $u$ with $u_0$ as initial data.*

**Proof** A parallel argument as in Theorem 1.3 implies that $u$ is a viscosity solution of $(P_2)$ with initial data $u_0$. Now we can conclude from Corollary 3.4.

$\square$.

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