Homogenization of a model problem on contact angle dynamics

Inwon C. Kim

May 25, 2007

Abstract

In this paper we consider homogenization of oscillating free boundary velocities in periodic media, with general initial data. We prove that there is a unique and stable effective free boundary velocity in the homogenization limit.

0 Introduction

Consider a compact set $K \subset I\!R^n$ with smooth boundary ∂K. Suppose that a bounded domain Ω contains K and let $\Omega_0 = \Omega - K$ and $\Gamma_0 = \partial\Omega$. We also assume that $\overline{Int(\Omega)} = \bar{\Omega}$.

Note that $\partial\Omega_0 = \Gamma_0 \cup \partial K$. For a continuous function $f(x,t) : I\!R^n \times [0,\infty) \to (0,\infty)$, let u_0 satisfy

$$-\Delta u_0 = 0 \text{ in } \Omega_0, \quad u_0 = f \text{ on } K, \quad \text{ and } u_0 = 0 \text{ on } \Gamma_0.$$

(see Figure 1.)

Let us define $e_i \in I\!R^n, i = 1, ..., n$ such that

$$(0.2) \qquad e_1 = (1,0,..0), e_2 = (0,1,0,..,0), ..., \text{ and } e_n = (0,..,0,1).$$

Consider a Lipschitz continuous function

$$g : I\!R^n \to [a,b], \quad g(x+e_i) = g(x) \text{ for } i = 1, ..., n$$

with Lipschitz constant M. In this paper we consider the behavior, as $\epsilon \to 0$, of the nonnegative (viscosity) solutions $u^\epsilon \geq 0$ of the following problem

$$(P)_\epsilon \qquad \begin{cases} -\Delta u^\epsilon = 0 & \text{in } \{u^\epsilon > 0\}, \\ \\ u^\epsilon_t = |Du^\epsilon|(|Du^\epsilon| - g(x/\epsilon)) & \text{on } \partial\{u^\epsilon > 0\} \end{cases}$$

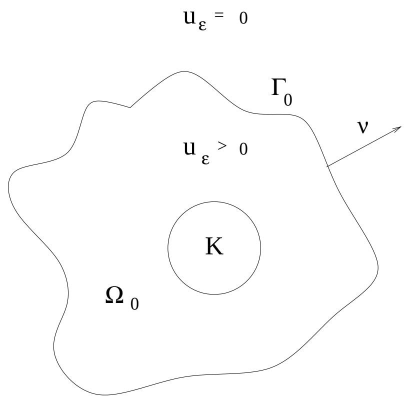

Figure 1: Initial setting of the problem

in $Q = (I\!R^n - K) \times (0, \infty)$ with initial data u_0 and smooth boundary data $f(x,t) > 0$ on $\partial K \times [0,\infty)$. Here Du denotes the spatial derivative of u. For simplicity in the analysis we will work with $a = 1$, $b = 2$ and $M = 10$ in the definition of g, but the method in this paper applies to general a, b and M.

We refer to $\Gamma_t(u^\epsilon) := \partial\{u^\epsilon(\cdot,t) > 0\} - \partial K$ as the *free boundary* of u^ϵ and to $\Omega_t(u^\epsilon) := \{u^\epsilon(\cdot,t) > 0\}$ as the *positive phase* of u^ϵ at time t. Note that if u^ϵ is smooth up to the free boundary, then the free boundary moves with outward normal velocity $V = \frac{u^\epsilon_t}{|Du^\epsilon|}$, and therefore the second equation in $(P)_\epsilon$ implies that

$$V = |Du^\epsilon| - g(\frac{x}{\epsilon}) = Du^\epsilon \cdot (-\nu) - g(\frac{x}{\epsilon})$$

where $\nu = \nu_{(x,t)}$ denotes the outward normal vector at $x \in \Gamma_t(u)$ with respect to $\Omega_t(u)$.

$(P)_\epsilon$ is a simplified model to describe contact line dynamics of liquid droplets on an irregular surface ([G].) Here $u(x,t)$ denotes the height of the droplet. Heterogeneities on the surface, represented by $g(\frac{x}{\epsilon})$ in $(P)_\epsilon$, result in contact lines with a fine scale structure that may lead to pinning of the interface and hysteresis of the overall fluid shape. We refer to [G] for numerical experiments and asymptotic analysis on smooth solutions, where the effective free boundary velocity is implicitly derived by solving an integro-differential system.

Below we state our main result (see Corollary 4.3 and Corollary 4.4):

Theorem 0.1 (main theorem)**.** *Let u^ϵ be a viscosity solution of $(P)_\epsilon$ with initial data u_0 and boundary data f. Then there exists a continuous function*

$$r(q) = I\!R^n - \{0\} \to [-2, \infty), r \text{ increases in } |q|$$

such that the following holds:

(a) If u_{ϵ_k} locally uniformly converges to u as $\epsilon_k \to 0$, then u is a viscosity solution of

$$(P) \qquad \begin{cases} -\Delta u = 0 & \text{in } \{u > 0\}, \\ \\ u_t = |Du| r(Du) & \text{on } \partial\{u > 0\} \end{cases}$$

in Q with initial data u_0 and boundary data f on ∂K.

(b) If u is the unique viscosity solution of (P) in Q with initial data u_0 and boundary data f on ∂K, then the whole sequence $\{u_\epsilon\}$ locally uniformly converges to u.

The uniqueness of u holds for example if $f = 1$ and both K and Ω_0 are star-shaped with respect to the origin, or if Ω_0 immediately expands or shrinks (Theorem 1.8.)

We refer to section 1 for definition of viscosity solutions. The notion of viscosity solutions for either $(P)_\epsilon$ and (P) is necessary since, even with smooth initial data, one cannot expect classical solutions to exist in global time. In fact solutions may develop singularities in finite time due to collision or pinch-off of free boundary parts.

In section 4 we will show that $r(q)$ may not strictly increase in $|q|$. In fact we will give an example where the *pinning interval*

$$I(\nu) := \{a : r(a\nu) = 0\} \text{ for a unit vector } \nu \in \mathbb{R}^n$$

has varying size depending on the normal direction ν (Lemma 3.15). On the other hand $r(q)$ strictly increases with respect to $|q|$ in $\{q : r(q) > 0\}$ (Lemma 3.16). Pinning intervals have been observed in physical and numerical experiments (See [G] and the references therein). The effect of the structure of g on the size of pinning interval, as well as on other features of $r(q)$, is an interesting open question.

There are extensive amount of literature on the subject of homogenization. For detailed survey on different approaches we refer to Caffarelli-Souganidis-Wang [CSW]. The very first paper on homogenization of parabolic equations is by Spagnolo [S]. Papanicolaou and Varadhan [PV] and Kozlov [Ko] were the first to consider the general problem of homogenizing linear, uniformly elliptic and parabolic operators. The first nonlinear result in the variational setting was obtained by Dal Maso and Modica [DM]. For fully nonlinear, uniformly elliptic and parabolic operators, Evans [E] and

Caffarelli [C] derived convergence results using maximum-principle type arguments.

Very little has been known for homogenization of free boundary problems due to the difficulties arising from the lower-dimensional nature of the interface: for example the periodicity of g in $(P)_\epsilon$ does not guarantee the interface $\Gamma_t(u)$ to be periodic in space. Caffarelli, Lee and Mellet ([CLM1]-[CLM2]) studied the homogenization of traveling front-type solutions of the flame-propagation type problem. Here the free boundary problem is investigated as the sharp interface limit of a phase-field Variational arguments have been used in [CM1]-[CM2] to study homogenization of stationary liquid drops given as energy minimizers.

In [K3] we studied the Stefan-type free boundary problem

$$(H)_\epsilon \qquad \begin{cases} (u^\epsilon_t) - \Delta u^\epsilon = 0 & \text{in } \{u^\epsilon > 0\}, \\ \\ u^\epsilon_t - g(\frac{x}{\epsilon})|Du^\epsilon|^2 = 0 & \text{on } \partial\{u^\epsilon > 0\} \end{cases}$$

and showed the existence of a unique motion law in the homogenization limit. The main idea in the analysis of [K3] is that, to describe the limiting problem, it is enough to decide whether a given 'test function' is either a subsolution or a supersolution of the problem. Such 'perturbed test-function method' has been previously taken first in [E], [C] and then more extensively in [CSW] for the homogenization of fully nonlinear equations in ergodic random media.

In this paper we extend the method introduced in [K3]. The challenge in our analysis is twofold. Besides the lower-dimensional structure of the free boundary. In particular the effective velocity depends on the normal direction of the interface. In stationary setting this is the main reason for the existence of the non-round drop (see [CL],[CM1]). The second, new challenge is that the free boundary $\Gamma_t(u^\epsilon)$ associated with $(P)^\epsilon$ does not always have positive velocity. This makes the problem considerably more unstable in the homogenization limit. The underlying intuition in [K3] is that the free boundary $\Gamma_t(u^\epsilon)$ associated with $(H)_\epsilon$ averages out in the limit $\epsilon \to 0$ since it propagates in the medium with strictly positive velocity $V = g(\frac{x}{\epsilon})|Du^\epsilon|$. In the case of $(P)_\epsilon$ this is no longer true. On the other hand the outline of the analysis performed in [K3] still applies to our case as long as the free boundary keeps moving, and we obtain a unique effective velocity $r(Du)$, either positive or negative. The pinned free boundaries, if they stay stalled as $\epsilon \to 0$, obviously have zero velocity in the homogenization limit. This observation suggests that the method in [K5] would apply to our case given that $\Gamma_t(u^\epsilon)$ evolves in a locally uniform manner.

Below we give the outline of the paper.

In section 1 we introduce the notion of viscosity solutions for $(P)_\epsilon$ and their properties.

In section 2, we study properties of maximal sub- and minimal supersolutions of $(P)_\epsilon$ with given obstacle $P_{q,r}$. An obstacle $P_{q,r}$ is a 'subsolution' for the limit problem if the maximal subsolution below $P_{q,r}$ converges to the obstacle as $\epsilon \to 0$, and similarly an obstacle $P_{q,r}$ is a 'supersolution' for the limit problem if the minimal supersolution above $P_{q,r}$ converges to the obstacle in the limit. The goal is to find a unique obstacle $P_{q,r}$ which serves for both sub- and supersolution of the limit problem, for each given $q \in I\!R^n$. We show the flatness of free boundary of the maximal sub- and minimal supersolution, with a 'good' obstacle.

In section 3, we prove that this is possible. In other words, we show that, for given $q \in I\!R^n$, there is a unique speed $r = r(q)$ such that both the maximal sub- and minimal supersolution of $(P)_\epsilon$ with obstacle $P_{q,r}$ converge to $P_{q,r}$ as $\epsilon \to 0$. This $r(q)$ will be our candidate for the function given in the effective free boundary velocity in (P). Proposition 3.8 and 3.11 are central in proving the uniqueness of $r(q)$.

In section 4, it is shown that $r(q)$ obtained in section 3 indeed yields the effective free boundary velocity in (P). The uniform convergence of $\{u^\epsilon\}$ then follows from the comparison principle (Theorem 1.7) for (P), as long as the uniqueness result holds for the initial data u_0.

1 Viscosity solutions and preliminary lemmas

Consider a space-time domain $\Sigma \subset I\!R^n \times [0, \infty)$ with smooth boundary. Let $\Sigma(s) := \Sigma \cap \{t = s\}$.

For a nonnegative real valued function $u(x,t)$ defined for $(x,t) \in \Sigma$, define

$$\Omega(u) = \{(x,t) \in \Sigma : u(x,t) > 0\}, \quad \Omega_t(u) = \{x : (x,t) \in \Sigma : u(x,t) > 0\};$$

$$\Gamma(u) = \partial\Omega(u) - \partial\Sigma, \quad \Gamma_t(u) = \partial\Omega_t(u) - \partial\Sigma(t).$$

Let us consider a continuous function

$$F(q,y) : (I\!R^n - \{0\}) \times I\!R^n \to [-2, \infty)$$

which is increasing in $|q|$, $|q| - 2 \le F(q,y,\nu) \le |q| - 1$, and $F(q, y + e_k) = F(q,y)$ for $k = 1, ..., n$. We also assume that F is Lipschitz continuous in y with Lipschitz constant 10.

Consider the free boundary problem

$$(\tilde{P})_\epsilon \qquad \begin{cases} -\Delta u^\epsilon = 0 & \text{in } \{u^\epsilon > 0\}, \\ \\ u^\epsilon_t - |Du^\epsilon| F(Du^\epsilon, \frac{x}{\epsilon}) = 0 & \text{on } \partial\{u^\epsilon > 0\} \end{cases}$$

in Σ with appropriate boundary data. We prove existence and uniqueness of the solution in this generality to apply the results to both $(P)_\epsilon$ and (P), and to various local barriers constructed in the analysis.

We extend the notion of viscosity solutions of Hele-Shaw problem introduced in [K1]. Roughly speaking viscosity sub- and supersolutions are defined by comparison with local, smooth super and subsolutions (we call such functions *barriers*). Viscosity solutions were first introduced by Crandall and Lions for studying Hamilton-Jacobi equations (see [CIL]).

Definition 1.1. *A nonnegative upper semicontinuous function u defined in Σ is a* viscosity subsolution *of $(\tilde{P})_\epsilon$ if*

(a) for each $a < T < b$ the set $\overline{\Omega(u)} \cap \{t \leq T\} \cap \Sigma$ is bounded; and

(b) for every $\phi \in C^{2,1}(\Sigma)$ such that $u - \phi$ has a local maximum in $\overline{\Omega(u)} \cap \{t \leq t_0\} \cap \Sigma$ at (x_0, t_0) then

$$(i) \quad -\Delta\phi(x_0, t_0) \leq 0 \qquad \text{if } u(x_0, t_0) > 0.$$

$$(ii) \quad \text{if } (x_0, t_0) \in \Gamma(u), |D\phi|(x_0, t_0) \neq 0 \text{ and } -\Delta\varphi(x_0, t_0) > 0,$$

then

$$(\phi_t - |D\phi| F(D\phi, \frac{x_0}{\epsilon}))(x_0, t_0) \leq 0.$$

Note that because u is only upper semicontinuous there may be points of $\Gamma(u)$ at which u is positive.

Definition 1.2. *A nonnegative lower semicontinuous function v defined in Σ is a* viscosity supersolution *of $(\tilde{P})_\epsilon$ if for every $\phi \in C^{2,1}(\Sigma)$ such that $v - \phi$ has a local minimum in $\Sigma \cap \{t \leq t_0\}$ at (x_0, t_0), then*

$$(i) \quad -\Delta\phi(x_0, t_0) \geq 0 \qquad \text{if } v(x_0, t_0) > 0,$$

$$(ii) \quad \text{if } (x_0, t_0) \in \Gamma(v), |D\phi|(x_0, t_0) \neq 0 \text{ and } -\Delta\varphi(x_0, t_0) < 0,$$

then

$$(\phi_t - |D\phi|F(D\phi, \frac{x_0}{\epsilon}))(x_0, t_0) \geq 0.$$

Let $K, \Omega_0, \Gamma_0, f, u_0$ and Q be as given in the introduction.

Definition 1.3. *u is a viscosity subsolution of $(\tilde{P})_\epsilon$ in Q with initial data u_0 and fixed boundary data $f > 0$ if*

(a) u is a viscosity subsolution of $(\tilde{P})_\epsilon$ in Q,

(b) u is upper semicontinuous in $\bar{Q}$, $u = u_0$ at $t = 0$ and $u \leq f$ on ∂K.

(c) $\overline{\Omega(u)} \cap \{t = 0\} = \overline{\Omega(u_0)}$.

Definition 1.4. *u is a viscosity supersolution of $(\tilde{P})_\epsilon$ in Q with initial data u_0 and fixed boundary data f if u is a viscosity supersolution in Q, lower semicontinuous in $\bar{Q}$ with $u = u_0$ at $t = 0$ and $u \geq f$ on ∂K.*

For a nonnegative real valued function $u(x,t)$ in Σ we define

$$u^*(x,t) := \limsup_{(\xi,s)\in\Sigma\to(x,t)} u(\xi,s).$$

and

$$u_*(x,t) := \liminf_{(\xi,s)\in\Sigma\to(x,t)} u(\xi,s).$$

Note that u^* is upper semicontinuous and u_* is lower semicontinuous.

Definition 1.5. *u is a viscosity solution of $(\tilde{P})_\epsilon$ (in Q with boundary data u_0 and f) if u is a viscosity supersolution and u^* is a viscosity subsolution of $(\tilde{P})_\epsilon$ (in Q with boundary data u_0 and f.)*

Definition 1.6. *We say that a pair of functions $u_0, v_0 : \bar{D} \to [0,\infty)$ are (strictly) separated (denoted by $u_0 \prec v_0$) in $D \subset I\!R^n$ if*

(i) the support of u_0, $\mathrm{supp}(u_0) = \overline{\{u_0 > 0\}}$ restricted in $\bar{D}$ is compact and

(ii) in $\mathrm{supp}(u_0) \cap \bar{D}$ the functions are strictly ordered:

$$u_0(x) < v_0(x).$$

Theorem 1.7. *(Comparison principle) Let u, v be respectively viscosity sub- and supersolutions of $(\tilde{P})_\epsilon$ in Σ. If $u \prec v$ on the parabolic boundary of Σ, then $u(\cdot,t) \prec v(\cdot,t)$ in Σ.*

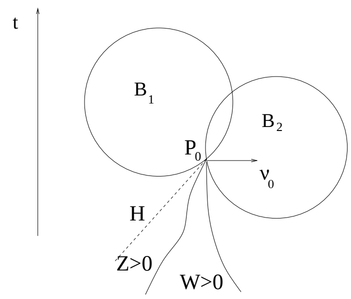

Figure 2: Geometry of positive phases at the contact time.

The proof is parallel to the proof of Theorem 2.2 in [K1]. We only sketch the outline of the proof below.

Sketch of the proof

1. For $r, \delta > 0$ and $0 < h << r$, define the sup-convolution of u

$$Z(x,t) := (1+\delta) \sup_{|(y,s)-(x,t)|<r} u(y,(1+\delta)s)$$

and the inf-convolution of v

$$W(x,t) := (1-\delta) \inf_{|(y,s)-(x,t)|<r-ht} v(y,(1-\delta)s)$$

in $\tilde{\Sigma} \cap \{r \leq t \leq r/h\}$, $\tilde{\Sigma} := \{(x,t) \in \Sigma : d(x, \partial\Sigma(t)) \geq r\}$.

By upper semi-continuity of $u-v$, $Z(\cdot,r) \prec W(\cdot,r)$ for sufficiently small $r, \delta > 0$. Moreover a parallel argument as in the proof of Lemma 1.3 in [K1] yields that if $r << \delta\epsilon$, then Z and W are respectively sub- and supersolutions of $(\tilde{P})_\epsilon$.

2. By our hypothesis and the upper semi-continuity of $u-v$, $Z \leq W$ on $\partial\tilde{\Sigma}$ and $Z < W$ on $\partial\tilde{\Sigma} \cap \bar{\Omega}(Z)$ for sufficiently small δ and r. If our theorem is not true for u and v, then Z crosses W from below at $P_0 := (x_0,t_0) \in \tilde{\Sigma} \cap [r,T]$. Due to the maximum principle of harmonic functions, $P_0 \in \Gamma(Z) \cap \Gamma(W)$. Note that by definition $\Omega(Z)$ and $\Omega(W)$ have respectively an interior ball B_1 and exterior ball B_2 at P_0 of radius r in space-time (see Figure 2.)

3. Let us call H the tangent hyperplane to the interior ball B_1 at P_0. Since $Z \leq W$ for $t \leq t_0$ and $P_0 \in \Gamma(Z) \cap \Gamma(W)$, it follows that

$$B_1 \cap \{t \leq t_0\} \subset \Omega(Z) \cap \Omega(W); \quad B_2 \cap \{t \leq t_0\} \subset \{Z=0\} \cap \{W=0\}$$

with $\bar{B}_1 \cap \bar{B}_2 \cap \{t \leq t_0\} = \{P_0\}$.

Moreover, since Z and W respectively satisfies the free boundary motion law

$$\frac{Z_t}{|DZ|}(x,t) \leq F(DZ, \frac{x}{\epsilon})(x,t) \text{ on } \Gamma(Z)$$

and

$$\frac{W_t}{|DW|}(x,t) \geq F(DW, \frac{x}{\epsilon})(x,t) + h \text{ on } \Gamma(W),$$

the arguments of Lemma 2.5 in [K1] applies for Z to yield that H is not horizontal. In particular $B_1 \cap \{t = t_0\}$ and $B_2 \cap \{t = t_0\}$ share the same normal vector ν_0, outward with respect to B_1, at P_0.

Formally speaking, it follows that

$$\frac{Z_t}{|DZ|}(P_0) \leq F(DZ, \frac{x_0}{\epsilon})(P_0) \leq F(DW, \frac{x_0}{\epsilon})(P_0) \leq \frac{W_t}{|DW|}(P_0) - h,$$

where the second inequality follows since both $DZ(P_0)$ and $DW(P_0)$ is parallel to $-\nu_0$, $F(q,y)$ in increases in $|q|$, and $Z(\cdot, t_0) \leq W(\cdot, t_0)$ in a neighborhood of x_0. Above inequality says that the free boundary speed of Z is strictly less than that of W at P_0, contradicting the fact that $\Gamma(Z)$ touches $\Gamma(W)$ from below at P_0.

For rigorous argument one can construct barrier functions based on the exterior and interior ball properties of Z and W at P_0. For details see the proof of Theorem 2.2 in [K1].

□

For $x \in I\!R^n$, we denote $B_r(x) := \{y \in I\!R^n : |y - x| < r\}$.

Theorem 1.8. *(a) There exists a viscosity solution u of $(\tilde{P})_\epsilon$ in Q with initial data u_0 and fixed boundary data f.*

(b) $u(\cdot, t)$ is harmonic in $\Omega_t(u)$, $u^*(\cdot, t)$ is harmonic in $\Omega_t(u^*)$, and $\Gamma(u^*) = \Gamma(u)$.

For $(c) - (d)$ we remove the space dependence in F, that is we assume $F = F(Du)$.

(c) If Ω and K are star-shaped with respect to the origin, then there is a unique viscosity solution u of $(\tilde{P})$ with boundary data u_0 and $f = 1$. Moreover in this case $\Omega_t(u)$ is star-shaped with respect to the origin for all $t > 0$.

(d) *If K is star-shaped with respect to the origin and $|Du_0| > 2$ or $|Du_0| < 1$ on Γ_0, then there is a unique viscosity solution u of $(\tilde{P})$ with boundary data u_0 and $f = 1$.*

Proof. 1. For (a), let us consider Ψ: the viscosity solution of $(\tilde{P})_\epsilon$ with $F(Du, y) \equiv |Du|$, with initial data u_0 and fixed boundary data f on ∂K. Such solution exists in Q and is unique due to [K1]. Note that Ψ is a supersolution of $(\tilde{P})_\epsilon$ in Q. Define

$$\mathcal{P} = \{z : z \text{ is a subsolution of } (\tilde{P})_\epsilon, z \leq f \text{ on } \partial K, \Gamma_0(z) = \Gamma_0, \text{ and } z \leq \Psi\},$$

Note that $\mathcal{P}$ is not empty. Let us define

$$\phi(x,t) = \inf_{|y-x|\leq 2t} u_0(y), \quad t \leq d(\Gamma_0, K)/2,$$

and let t_0 be the first time $\Gamma_t(\phi)$ hits K. Let $h(\cdot,t)$ be the harmonic function on $\Omega_t(\phi)$ with $h = 0$ on $\Gamma_t(\phi)$ and $h = f$ on K for $0 \leq t \leq t_0$, and $h(\cdot,t) \equiv 0$ for $t > t_0$. Then $z = h(x,t) \in \mathcal{P}$.

Next define

$$U(x,t) := \sup\{z(x,t) : z \in \mathcal{P}\}.$$

Arguing as in Theorem 4.7 in [K1] will yield that U_* is in fact a viscosity solution of $(\tilde{P})_\epsilon$ with boundary data Γ_0 and f on ∂K. We mention that the continuity of f and F is necessary for the argument.

2. For (b) parallel arguments as in the proof of Lemma 1.9 of [K2] applies. In particular

$$u(\cdot,t) = \inf\{\alpha(x) : -\Delta\alpha \geq 0 \text{ in } \Omega_t(u) - K, \alpha = f \text{ on } \partial K, \alpha \geq 0 \text{ on } \Gamma_t(u).\}$$

and

$$u^*(\cdot,t) = \sup\{\beta(x) : -\Delta\beta \leq 0 \text{ in } \Omega_t(u^*) - K, \beta = f \text{ in } \partial K, \beta \leq 0 \text{ in } \Gamma_t(u^*).\}$$

3. To prove (c), let u_1 and u_2 be two viscosity solutions of (P) with initial data u_0. By our hypothesis, for any $0 < \delta$,

$$u_1(x,0) \prec u_2((1+\delta)^{-1}x, 0),$$

Since $F(Du)$ is increasing with respect to $|Du|$,
$\tilde{u}(x,t) := u_2((1+\delta)^{-1}x, (1+\delta)^{-1}t)$ is a supersolution of $(\tilde{P}^\epsilon)$. Thus by Theorem 1.7

$$(u_1)^*(x,t) \prec \tilde{u}(x,t). \tag{1.3}$$

Since $\delta > 0$ is arbitrary, we obtain $u_1 \leq u_2$. Similarly $u_2 \leq u_1$, and thus $u_1 = u_2$. In particular (1.3) with $u_1 = u_2$ implies that $\Omega_{1/3C}(u_1)$ is star-shaped with respect to the origin.

4. To prove (d), first suppose $|Du_0| > 2$. Then $\Omega(u)$ immediately expands at $t = 0$ for any viscosity solution u. It follows that for any $0 < \delta << \epsilon$ and for any two viscosity solutions u_1 and u_2 of $(\tilde{P})_\epsilon$ with initial data u_0, there exists a constant $C > 0$ such that

$$u_1(x, 0) \prec u_2((1+\delta)^{-1}x, C\delta) \text{ in } I\!R^n - (1+\delta)K.$$

Hence by Theorem 1.7,

$$(u_1)^*(x,t) \prec u_2((1+\delta)^{-1}x, (1+\delta)^{-1}t + C\delta) \text{ for } t > 0.$$

We now send $\delta \to 0$ to obtain $u_1 \leq u_2$, and similarly $u_2 \leq u_1$, and thus $u_1 = u_2$, yielding uniqueness. □

For later use we state that the free boundary of a viscosity solution u of $(\tilde{P})_\epsilon$ in Q with initial data u_0 and and fixed boundary data f does not jump in time. The proof is parallel to that of Lemma 1 in [K3].

Lemma 1.9. $\Gamma(u)$ *does not jump in time, in the sense that for any point* $x_0 \in \Gamma_{t_0}(u^*)$ $(x_0 \in \Gamma_{t_0}(u))$ *there exists a sequence of points* $(x_n, t_n) \in \Gamma(u^*)$ $((x_n, t_n) \in \Gamma(u))$ *such that* $t_n < t_0$, $(x_n, t_n) \to (x_0, t_0)$.

2 Defining the limiting velocity

In this section we extend the notions introduced in [K3] to define the limiting free boundary velocity of the solutions of $(P)_\epsilon$ as $\epsilon \to 0$.

For given nonzero vector $q \in I\!R^n$ and $r \in [-2, \infty)$, we denote $\nu = \frac{q}{|q|}$ and define

$$P_{q,r}(x,t) := |q|(r(t-1) - x\cdot\nu)_+, \quad l_{q,r}(t) = \{x \in I\!R^n : r(t-1) = x\cdot\nu\}$$

Note that the free boundary of $P_{q,r}$, $\Gamma_t(P_{q,r}) := l_{q,r}(t)$, propagates with normal velocity r with its outward normal direction ν, and with
$l_{q,r}(1) = \{x \cdot \nu = 0\}$.

Next we construct a domain with which the obstacle problems will be defined. In $e_1 - e_n$ plane, consider a vector $\mu = e_n + \sqrt{3}e_1$. Let l to be the line which is parallel to μ and passes through $3e_1$. Rotate l with respect to e_n-axis and define $\mathcal{D}$ to be the region bounded by the rotated image and $\{x : -1 \leq x \cdot e_n \leq 6\}$ (see Figure 3). For any nonzero vector $q \in I\!R^n$, let us

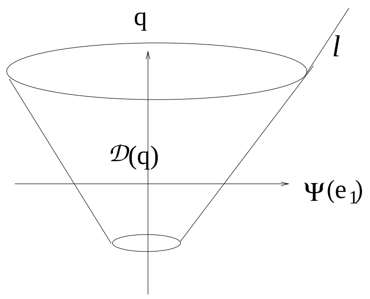

Figure 3: The spatial domain for test functions

define $\mathcal{D}(q) := \Psi(\mathcal{D})$, where Ψ is a rotation in $\mathbb{R}^n$ which maps e_n to $q/|q|$. Let us define

$$\mathcal{O} = \bigcup_{0 \leq t \leq 1} ((1+3t)\mathcal{D}(q) \times \{t\}).$$

Let us define the space-time domain $Q_1 := \mathcal{D}(q) \times [0,1]$ for $r \geq 0$, and $Q_1 := \mathcal{O}$ for $r < 0$.

Definition 2.1. *Let us define*

$$\bar{u}_{\epsilon;q,r} := (\sup\{u : \text{ a subsolution of } (P)_\epsilon \text{ qin } Q_1 \text{ with } u \leq P_{q,r}\})^*$$

$$\underline{u}_{\epsilon;q,r} := (\inf\{v : \text{ a supersolution of } (P)_\epsilon \text{ in } Q_1 \text{ with } u \geq P_{q,r}\})_*$$

Note that then $\bar{u}_{\epsilon;q,r}(\cdot,t)$ and $\underline{u}_{\epsilon;q,r}(\cdot,t)$ are both harmonic in their positive phases. The reason for defining rather complicated domain Q_1 is to guarantee that the free boundary of $\underline{u}_{\epsilon;q,r}$ and $\bar{u}_{\epsilon;q,r}$ does not detach too fast from $P_{q,r}$ as it gets away from the lateral boundary of Q_1. (see Lemma 2.4).

The following lemma is due to the fact that $1 \leq g \leq 2$.

Lemma 2.2. *For $r \geq |q|-1$, $P_{q,r} = \underline{u}_{\epsilon;q,r}$. For $r \leq |q|-2$, $P_{q,r} = \bar{u}_{\epsilon;q,r}$.*

Lemma 2.3. *For $r > |q|-1$, $\bar{u}_{\epsilon;q,r} \prec P_{q,r}$ in the interior of Q_1. For $r < |q|-2$, $P_{q,r} \prec \underline{u}_{\epsilon;q,r}$ in the interior of Q_1.*

Proof. For $r > |q|-1+\gamma$ for $\gamma > 0$, note that $P_{q,r}$ is a strict supersolution of $(P)_\epsilon$, i.e., the normal velocity $V = r$ of $l_{q,r}$ satisfies $V \geq |DP_{q,r}| - 1 + \gamma$. Thus by definition of viscosity subsolution, it follows that

$$\bar{u}_{\epsilon;q,r} \prec P_{q,r} \text{ in } Q_1.$$

For $r < |q|-2$ parallel argument applies. $\square$

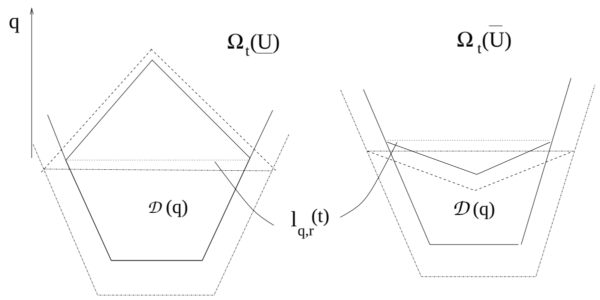

Figure 4: Barriers for lateral boundary control

For the rest of section 2 and section 3 we will restrict the analysis to the cases $r \in [-2, 2] \cap [|q| - 2, |q| - 1]$. For the remaining cases, $r \geq 2$, corresponding results can be proved by parallel, in fact easier, arguments.

In $e_1 - e_n$ plane, for each $0 \leq t \leq 1$ consider a line $l(t)$ which is parallel to the vector $e_1 + \sqrt{3}e_n$ and passes through $-e_1 + te_n$. Now rotate $l(t) \cap \{x \cdot e_1 \leq 0\}$ with respect to e_n-axis to obtain a hyper-surface $\underline{L}(t)$ in $I\!R^n$. Let $\mathcal{L}(t)$ be the region whose boundary is $\underline{L}(t)$ and contains $-e_n$. For a nonzero vector $q \in I\!R^n$ let us define $\mathcal{L}_q(t) = \Phi(\mathcal{L}(t))$ where Φ is the rotation map in $I\!R^n$ such that $\Phi(e_n) = \frac{q}{|q|}$. Let us define $\underline{U}_{q,r}$ to be the harmonic function in the region

$$\mathcal{L}_q(2\sqrt{3}rt) \cap \mathcal{D}(q), \tag{2.1}$$

with boundary data zero on $\partial\mathcal{L}_q(2\sqrt{3}rt) \cap \mathcal{D}(q)$ and $P_{q,r}$ on the rest of the boundary. To define $\bar{U}_{q,r}$ we replace $l(t)$ by $k(t)$, where $k(t)$ is parallel to the vector $2\sqrt{3}e_1 - e_n$ and passes through $-e_1 + te_n$ and replace $2\sqrt{3}$ in (2.1) by $1/2$ (See Figure 4).

Lemma 2.4. *$\bar{u}_{\epsilon;q,r} = P_{q,r}$ and $\underline{u}_{\epsilon;q,r} = P_{q,r}$ on the parabolic boundary of $\bar{Q}_1$.*

Proof. 1. We will make use of the fact that

$$|D\bar{U}_{q,r}| > 2|q| \text{ and } |D\underline{U}_{q,r}| < \frac{1}{2}|q|$$

on their respective free boundaries. (Above inequalities follow from comparison with planar solutions at each time.)

We first prove the lemma for $r > 0$. Then we will find that, if $r > 0$, $\bar{U}_{q,r}(x,t)$ is a subsolution of $(P)_\epsilon$ since, on the free boundary, the normal velocity V of $\Gamma(\bar{U}_{q,r})$ satisfies

$$V = r/2 \leq |q|/2 - 1/2 \leq 2|q| - 2 \leq |D\bar{U}_{q,r}| - 2$$

if $|q| \geq 1$, which is our case since $r > 0$. Similarly for $r < 0$ $\underline{U}_{q,r}(x,t)$ is a supersolution of $(P)_\epsilon$ since

$$V = 2r \geq 2|q| - 4 \geq |q|/2 - 1 \geq |D\underline{U}_{q,r}| - 1$$

if $|q| \leq 2$. The lemma then follows from the comparison principle.

2. For $-2 < r < 0$ and $r \in [|q|-2, |q|-1]$, choose $a = a(r)$ such that $l_{q,r}(t)$ meets $\partial\mathcal{L}_q(a(r)t)$ on the lateral boundary of $(1+3t)\mathcal{D}(q)$. A straightforward computation then yields $a(r) > -1/2$. For $0 \leq t \leq 1$ define $\underline{V}_{q,r}(\cdot,t)$ to be the harmonic function in the region $\mathcal{L}_q(a(r)t) \cap (1+3t)\mathcal{D}(q)$ with boundary data zero on $\partial\mathcal{L}_q(a(r)t) \cap (1+3t)\mathcal{D}(q)$ and $P_{q,r}$ on the rest of the boundary of $\mathcal{O}$.

We claim that $\underline{V}_{q,r}(x,t)$ is a supersolution of $(P)_\epsilon$ in $Q_1 = \mathcal{O}$.

Indeed one can verify that, by comparing $\underline{V}_{q,r}(\cdot,t)$ with planar harmonic functions at each $t \in [0,1]$, $|D\underline{V}_{q,r}| \leq |q|/2$ on its free boundary. Since $r < 0$ and $|q| \leq 1$, we conclude that the normal velocity V of $\Gamma(\underline{V}_{q,r})$ satisfies

$$V \geq -1/2 \geq |q|/2 - 1 \geq |D\underline{V}_{q,r}| - g(\frac{x}{\epsilon}).$$

Similarly one can construct a subsolution $\bar{V}_{q,r}$ of $(P)_\epsilon$ in $\mathcal{O}$, by modifying the supersolution $\underline{U}_{q,r}$ constructed above. Now our conclusion follows by comparing $\underline{u}_{\epsilon;q,r}$ with $\underline{V}_{q,r}$, and $\bar{u}_{\epsilon;q,r}$ with $\bar{V}_{q,r}$.

□

Lemma 2.5. *For $q \in I\!R^n$ and $r \in [-2,2] \cap [|q|-2, |q|-1]$,*

(a) $\bar{u}_{\epsilon;q,r}$ is a subsolution of $(P)_\epsilon$ in Q_1 with $\bar{u}_{\epsilon;q,r} \leq P_{q,r}$ in $\bar{Q}_1$ and $\bar{u}_{\epsilon;q,r} = P_{q,r}$ on the parabolic boundary of $\bar{Q}_1$. Moreover $(\bar{u}_{\epsilon;q,r})_*$ is a solution of $(P)_\epsilon$ away from $\Gamma(\bar{u}_{\epsilon;q,r}) \cap l_{q,r}$.

(b) $\underline{u}_{\epsilon;q,r}$ is a supersolution of $(P)_\epsilon$ in Q_1 with $\underline{u}_{\epsilon;q,r} \geq P_{q,r}$ in $\bar{Q}_1$ and $\underline{u}_{\epsilon;q,r} = P_{q,r}$ on the parabolic boundary of $\bar{Q}_1$. Moreover $\underline{u}_{\epsilon;q,r}$ is a solution of $(P)_\epsilon$ away from $\Gamma(\underline{u}_{\epsilon;q,r}) \cap l_{q,r}$.

(c) $\bar{u}_{\epsilon;q,r}$ decreases in time if $r < 0$. $\underline{u}_{\epsilon;q,r}$ increases in time if $r > 0$.

Proof. 1. (a)-(b) of Lemma 2.5 can be proved as arguing in the proof of Lemma 4 in [K3], using Lemma 2.4.

2. To prove (c), note that by definition of $\bar{u}_{\epsilon;q,r}$ and $\underline{u}_{\epsilon;q,r}$ respectively as the maximal subsolution and the minimal supersolution in Q_1 with obstacle $P_{q,r}$,

$$\bar{u}_{\epsilon;q,r}(x,t+\tau) \leq \bar{u}_{\epsilon;q,r}(x,t)$$

for any $\tau > 0$ when $r < 0$, and

$$\underline{u}_{\epsilon;q,r}(x, t+\tau) \geq \underline{u}_{\epsilon;q,r}(x, t)$$

for any $\tau > 0$ when $r > 0$. This yields (c). □

The following corollary is due to Lemma 2.4 and by definition of $\bar{u}_{\epsilon;q,r}$ and $\underline{u}_{\epsilon;q,r}$.

Corollary 2.6. *For any given nonzero vector $q \in I\!\!R^n$, $\nu = \frac{q}{|q|}$ and for any $a \in [0,1]$, there is $\eta \in I\!\!R^n$ such that $a\nu + \eta \in \epsilon\mathbb{Z}^n$, $\eta \cdot \nu \geq \frac{1}{2}|\eta|$ and $\epsilon \leq |\eta| < 3\epsilon$. For this η the following holds:*

(a) For $r > 0$

$$\bar{u}_{\epsilon;q,r}(x + a\nu + \eta, t + \tau) \leq \bar{u}_{\epsilon;q,r}(x, t)$$

for $0 \leq \tau \leq r^{-1}(a + \eta \cdot \nu)$ and

$$\underline{u}_{\epsilon;q,r}(x + a\nu + \eta, t + \tau) \geq \underline{u}_{\epsilon;q,r}(x, t) \text{ in } Q_1.$$

for $\tau \geq r^{-1}(a + \eta \cdot \nu)$.

(b) For $r < 0$ the above inequalities are true with ν, η and r replaced by $-\nu$, $-\eta$, and $|r|$.

For a nonzero vector $q \in I\!\!R^n$ define

$$\underline{A}_{\epsilon;q,r}(t) = \Gamma_t(\underline{u}_{\epsilon;q,r}) \cap l_{q,r}(t) \cap B_{1/2}(0)$$

and

$$\bar{A}_{\epsilon;q,r}(t) = \Gamma_t(\bar{u}_{\epsilon;q,r}) \cap l_{q,r}(t) \cap B_{1/2}(0).$$

where $0 \leq t < \infty$. Also define the contact sets

$$\underline{A}_{\epsilon;q,r} := \bigcup_{1/2 \leq t \leq 1} \underline{A}_{\epsilon;q,r}(t), \quad \bar{A}_{\epsilon;q,r} := \bigcup_{1/2 \leq t \leq 1} \bar{A}_{\epsilon;q,r}(t).$$

Note that if $\bar{A}_{\epsilon;q,r}$ ($\underline{A}_{\epsilon;q,r}$) is empty, then $\bar{A}_{\epsilon;q,r}(t)$ ($\underline{A}_{\epsilon;q,r}(t)$) is empty for $t \geq 1$ due to Corollary 2.6.

Lastly define

$$\underline{r}(q) = \inf\{r : \underline{A}_{\epsilon;q,r} \neq \emptyset \text{ for } \epsilon \leq \epsilon_0 \text{ with some } \epsilon_0 > 0\},$$

$$\bar{r}(q) = \sup\{r : \bar{A}_{\epsilon;q,r} \neq \emptyset \text{ for } \epsilon \leq \epsilon_0 \text{ with some } \epsilon_0 > 0\}.$$

Note that by Lemma 2.3 $\bar{r}(q) \leq |q| - 1$ and $\underline{r}(q) \geq |q| - 2$. Below we show that the contact sets are empty or non-empty in a monotone fashion in r and ϵ.

Lemma 2.7. *(a) Let $r \neq 0$ and $\epsilon < |r|/8$. For $r < r_1$, if $\underline{A}_{2\epsilon;q,r_1}$ $(\bar{A}_{\epsilon;q,r})$is empty, then $\underline{A}_{\epsilon;q,r}$ $(\bar{A}_{\epsilon;q,r_1})$ is empty.*

(b) Suppose $r \neq 0$ and $\epsilon < |r|/8$. If $\bar{A}_{\epsilon_0;q,r}$ $(\underline{A}_{\epsilon_0;q,r})$ is empty, so is $\bar{A}_{\epsilon;q,r}$ $(\underline{A}_{\epsilon;q,r})$ for $0 < \epsilon < \epsilon_0/2$.

Proof. 1. Let $r < r_1$ and assume $\underline{A}_{2\epsilon;q,r_1} = \emptyset$. We compare $u_1 := \underline{u}_{\epsilon;q,r}$ with a minimal supersolution u_2 of $(P)_\epsilon$ in the domain Q_1 with obstacle $\tilde{P} := P_{q,r_1} - a\nu$, where $a > 0$. Note that for any $a > 0$ there exists $\xi \in \epsilon\mathbb{Z}^n$ such that $|\xi - a\nu| \leq 2\epsilon$. Let us choose $\tau \in [0, 2r^{-1}\epsilon]$ such that

$$l_{q,r}(\tau) - a\nu = l_{q,r}(0) - \xi \cdot \nu.$$

By comparison between u_2 and $\frac{1}{2}\underline{u}_{2\epsilon;q,r}(2(x+\xi), 2(t-\tau))$ in $\frac{1}{2}Q_1 \times [\tau, 1]$ and using the fact that $\underline{A}_{2\epsilon,q,r_1}$ is empty, it follows that $\Gamma(u_2)$ is away from l_{q,r_1} in $B_{3/4}(0) \times [1/4 + 2r^{-1}\epsilon, 1]$.

2. By definition of u_2, u_2 is a solution of $(P)_\epsilon$ away from $\tilde{P}$. Thus by Theorem 1.7, $u_2 \prec u_1$ in Q_1 as long as $P_{q,r_1}(\cdot + a\nu, t) \prec P_{q,r}(\cdot, t)$. Let $T(a)$ be the time at which $P_{q,r_1}(\cdot + a\nu, t) = P_{q,r}(\cdot, t)$. For each $t_0 \in [1/2, 1]$, one can choose a appropriately so that $T(a) = t_0$. From previous argument on u_2 and from the fact that $\epsilon < |r|/8$, it follows that $\underline{A}_{\epsilon;q,r}$ is empty.

3. The rest of (a) and (b) follows from parallel arguments.

□

Next proposition states that for $r > 0$ the free boundaries of $\bar{u}_{\epsilon;q,r}$ and $\underline{u}_{\epsilon;q,r}$ with "good" obstacles are relatively flat up to the order of ϵ.

Proposition 2.8. *Fix a nonzero vector $q \in \mathbb{R}^n$ and $r \in [|q| - 2, |q| - 1] \cap [-1/2, 2]$. Then there exists a dimensional constant $M > 0$ such that*

(a) If $\bar{A}_{\epsilon;q,r}$ is nonempty then

$$d(x, l_{q,r}(t)) \leq M\epsilon \text{ for } x \in \Gamma_t(\bar{u}_{2\epsilon;q,r}) \cap B_{1/2}(0), 0 \leq t \leq 1.$$

(b) If $\underline{A}_{\epsilon;q,r}$ is nonempty, then

$$d(x, l_{q,r}(t)) \leq M\epsilon \text{ for } x \in \Gamma_t(\underline{u}_{2\epsilon;q,r}) \cap B_{1/2}(0), 0 \leq t \leq 1.$$

Proof. 1. The proof of (b) is parallel to that of Lemma 7 in [K3].

2. To prove (a), Let $\nu = \frac{q}{|q|}$. For simplicity we drop q, r in the notation of $\bar{u}_{2\epsilon;q,r}$. First observe that, if $x_0 \in \Gamma_t(\bar{u}_{2\epsilon})$ with $d(x_0, l_{q,r}(t)) > \epsilon$, then

$$\bar{u}_{2\epsilon}(\cdot, t) < C\epsilon \text{ in } B_{2\epsilon}(x_0 - 3\epsilon\nu). \tag{2.2}$$

If not a barrier argument using Corollary 2.6(a) yields that $x_0 \in \Omega_t(\bar{u}_\epsilon)$, a contradiction.

3. Let y_0 be the furthest point of $\Gamma_{t_0}(\bar{u}_{2\epsilon})$ from $l_{q,r}(t_0)$ in $B_{1/2}(0)$. If (b) is not true, then

$$d(y_0, l_{q,r}(t_0)) = d_0 > M\epsilon \tag{2.3}$$

for some $t_0 \in [5\epsilon, 1]$. By definition of $\bar{u}_\epsilon$,

$$\frac{1}{2}\bar{u}_{2\epsilon}(2(x-\eta), 2(t-t_0)+t_0) \geq \bar{u}_\epsilon(x,t) \text{ in } \frac{1}{2}Q_1 + (\eta, t_0/2) \tag{2.4}$$

when $\eta \in \epsilon\mathbb{Z}^n$ satisfies $|\eta| \leq 1/2$ and $\eta \cdot \nu \geq r(t_0 - 1)$. It then follows from (2.2) and (2.4) that

$$\bar{u}_\epsilon(\cdot, t_0) \leq C\epsilon \text{ on } B_{3/4}(0) \cap (l_{q,r}(t_0) - (d_0 + 3\epsilon)\nu). \tag{2.4}$$

(2.5), (2.3) and the fact that $\bar{u}_\epsilon(\cdot, t)$ is subharmonic yields that, if M is chosen large enough,

$$\bar{u}_\epsilon(\cdot, t_0) \leq \epsilon/10 \text{ in } B_{3/4}(0) \cap (l_{q,r}(t_0) - 2\epsilon\nu). \tag{2.6}$$

Since $r \geq -1/2$, a barrier argument using (2.6) and Corollary 2.6(a) yields that $\Gamma_{t_0}(\bar{u}_{2\epsilon})$ is away from $l_{q,r}(t_0)$ for $t_0 \in [1/2, 1]$, a contradiction to our hypothesis. $\square$

For $r < -1/2$ the argument in [K3] no longer applies, due to the fact that (2.6) does not guarantee that $\Gamma(\bar{u}_{2\epsilon})$ recedes faster than $l_{q,r}$. Below we state a weaker result on the flatness of the free boundary.

For any $\gamma > 0$, define the *sup-convolution* of $\bar{u}_{\epsilon;q,r}$ on the spatial ball of size $\gamma\epsilon$,

$$\bar{v}_{\epsilon;q,r,\gamma}(x,t) := \sup_{y \in B_{\gamma\epsilon}(x)} \bar{u}_{\epsilon;q,r}(y,t).$$

Proposition 2.9. *Fix a nonzero vector $q \in \mathbb{R}^n$ and $r \in (-2, -1/2)$.*

(a) If $\bar{A}_{\epsilon;q,r}$ is nonempty, then for any $\gamma > 0$, there exists $M = M(\gamma) > 0$ such that, for any $\epsilon > 0$

$$d(x, l_{q,r}(t)) \leq M\epsilon \textit{ for } x \in \Gamma_{t_0}(\bar{v}_{2\epsilon;q,r,\gamma}) \cap B_{1/2}(0), 0 \leq t \leq 1.$$

(b) There exists a constant $M = M(|q|) > 0$ such that, if $\underline{A}_{\epsilon;q,r}$ is nonempty, then

$$d(x, l_{q,r}(t)) \leq M\epsilon \textit{ for } x \in \Gamma_t(\underline{u}_{2\epsilon;q,r}) \cap B_{1/2}(0), 0 \leq t \leq 1.$$

Proof of Proposition 2.9 (a):

1. Let $\nu = \frac{q}{|q|}$. For simplicity we drop q, r in the notation of $\bar{u}_{\epsilon;q,r}$, $\underline{u}_{\epsilon;q,r}$ and $\bar{v}_{\epsilon;q,r,\gamma}$ in the proof.

2. Let x_0 to be the furthest point of $\Gamma_{t_0}(\bar{v}_{2\epsilon,\gamma})$ from $l_{q,r}(t_0)$ in $B_{1/2}(0)$. We may assume that

$$d(x_0, l_{q,r}(t_0)) > M\epsilon,$$

where $M = M(\gamma) > 0$ is a constant to be determined. We claim that

$$\bar{u}_{2\epsilon}(\cdot, t_0) < C\epsilon \text{ in } B_{2\epsilon}(x_0 - 3\epsilon q). \tag{2.7}$$

where C is a dimensional constant to be chosen.

To show the claim, note that $\bar{u}_{2\epsilon}$ is strictly decreasing in time when $r < 0$ and in particular, due to Lemma 2.5,

$$\bar{u}_{2\epsilon}(x, t+\tau) \leq (\bar{u}_{2\epsilon})_*(x, t) \text{ for any } \tau > 0.$$

Hence if the claim is not true, then

$$(\bar{u}_{2\epsilon})_*(y_0, t) \geq C_0\epsilon \text{ for some } y_0 \in B_{2\epsilon}(x_0 - 3\epsilon q) \text{ and for } 0 \leq t < t_0$$

for sufficiently large $C_0 > 0$. Since $(\bar{u}_{2\epsilon})_*(\cdot, t)$ is lower semicontinuous, $\bar{u}_{2\epsilon}(\cdot, t_0 - \epsilon) > 0$ in $B_r(y_0)$ for some $0 < r < \epsilon$ for $0 \leq t < t_0$. Moreover, by Harnack inequality $\bar{u}_{2\epsilon}(\cdot, t_0 - \epsilon) \geq C_1\epsilon$ in $B_r(y_0)$ for a sufficiently large $C_1 > 0$.

Let us define

$$r(t) = \sqrt{r^2 + aC_1\epsilon(t - t_0 + \epsilon)}.$$

with a sufficiently small such that $x_0 \in B_{r(t_0)}(y_0)$ and $r(t_0) < 5\epsilon$.

Next construct a function ϕ in $I\!R^n \times [t_0 - \epsilon, t_0]$ such that

$$\begin{cases} -\Delta\phi(\cdot, t) = 0 & \text{in} \quad B_{2r(t)}(y_0) - B_{r(t)}(y_0); \\ \phi = C_1\epsilon & \text{in} \quad B_{r(t)}(y_0) \times [t_0 - \epsilon, t], \\ \phi(\cdot, t) = 0 & \text{in} \quad I\!R^n - B_{2r(t)}(x_0). \end{cases}$$

If a is sufficiently small and if C_1 is sufficiently large such that $|D\phi| > 3$ on $\Gamma(\phi)$, then

$$\frac{\phi_t}{|D\phi|} = r'(t) = \frac{aC_1\epsilon}{r(t)} \leq \frac{1}{2}|D\phi| \leq |D\phi| - 2.$$

Hence ϕ is a subsolution of $(P)_\epsilon$ in

$$\Sigma := \bigcup_{t_0 - \epsilon \leq t \leq t_0} (I\!R^n - B_{r(t)}(y_0)) \times t.$$

Now we compare $\bar{u}_{2\epsilon}$ and ϕ in Σ. First observe that $\phi \leq \bar{u}_{2\epsilon}$ in $\Sigma \cap \{t = t_0 - \epsilon\}$. Next observe that, if $\bar{u}_{2\epsilon}(\cdot, t)$ is positive in $B_{2r(t)}(y_0)$, by Harnack inequality applied to $\bar{u}_{2\epsilon}$

$$\bar{u}_{2\epsilon}(\cdot, t) \geq C_1\epsilon = \phi \text{ on } \partial B_{r(t)}(y_0).$$

On the other hand, as long as above inequality holds for $t_0 - \epsilon \leq t \leq s < t_0$,

$$\Omega_t(\phi) \subset \Omega_t(\bar{u}_{2\epsilon}) \text{ in } \Sigma \cap \{t_0 - \epsilon \leq t \leq s\}$$

due to Theorem 1.7. Hence it follows that $\phi \leq \bar{u}_{2\epsilon}$ in Σ. In particular, $x_0 \in \Omega_{t_0}(\bar{u}_{2\epsilon})$, yielding a contradiction.

3. Observe that, by definition of $\bar{u}_\epsilon$,

$$\frac{1}{2}\bar{u}_{2\epsilon}(2(x-\eta), 2(t-t_0)+t_0) \geq \bar{u}_\epsilon(x,t) \text{ in } \frac{1}{2}Q_1 + (\eta, t_0/2) \tag{2.8}$$

when $\eta \in \epsilon\mathbb{Z}^n$ satisfies $|\eta| \leq 1/2$ and $\eta \cdot \nu + \frac{1}{2} \geq |r|t_0$. It then follows from (2.7) and (2.8) that

$$\bar{u}_\epsilon(\cdot, t_0) < C_2\epsilon \text{ in } \{(x-x_0)\cdot\nu = -3\epsilon\} \cap B_{3/4}(0). \tag{2.9}$$

where C_2 is a dimensional constant.

4. Due to (2.8) for any ϵ-neighborhood of a point in

$$S = \{x : \frac{M}{4}\epsilon \leq r(t_0 - 1) - x\cdot\nu \leq \frac{M}{2}\epsilon\} \cap B_{3/4}(0),$$

there exists $z_0 \in \{\bar{v}_{\epsilon,\gamma}(\cdot, t_0) = 0\}$.

Due to (2.9) and the fact that $\bar{u}_\epsilon$ is subharmonic,

$$\bar{u}_\epsilon(\cdot, t_0) \leq \frac{\gamma\epsilon}{10} \text{ in } B_{2\epsilon}(z_0)$$

if $M = M(\gamma)$ is chosen sufficiently large. Moreover by definition of $v_{\epsilon;\gamma}$, $\bar{u}_\epsilon = 0$ in $B_{\gamma\epsilon}(z_0)$. Thus a barrier argument using the fact that $\bar{u}_\epsilon$ decreases in time would yield that $\bar{u}_\epsilon(\cdot, t_0 + 3\epsilon) = 0$ in $B_{2\epsilon}(z_0)$. In particular

$$\bar{u}_\epsilon(\cdot, t_0 + 3\epsilon) = 0 \text{ in } S. \tag{2.10}$$

6. (2.10) and Corollary 2.6(a) with $\tau = 0$ yields $\bar{A}_{\epsilon;q,r} = \emptyset$, contradicting our hypothesis.

□

We proceed to prove (b).

Proof of Proposition 2.9 (b):

1. Let $\nu = \frac{q}{|q|}$. Let us define $\tilde{l}_{q,r} = \tilde{l}_{q,r}(t)$ such that

$$\tilde{l}_{q,r}(t) := \{x : d(x, l_{q,r}(t)) = 2\epsilon \text{ and } x\cdot\nu \le r(t-1)\}.$$

Let $(x_0, t_0) \in \underline{A}_{\epsilon;q,r}$. We claim that

$$\sup_{y\in B_{2\epsilon}(x_0)} \underline{u}_{\epsilon;q,r}(y,t) \le C\epsilon, \tag{2.11}$$

where C is a dimensional constant. Otherwise a barrier argument using Corollary 2.6 will yield a contradiction to the fact that $\underline{u}_{\epsilon;q,r} \ge P_{q,r}$ and $(x_0, t_0) \in \underline{A}_{\epsilon;q,r}$. By comparison with translated versions of $2\underline{u}_{\epsilon;q,r}(x/2, t/2)$ and as in (2.8), it then follows that

$$\underline{u}_{2\epsilon;q,r}(\cdot, t) \le C_0\epsilon \text{ on } \tilde{l}_{q,r}(t_0) \cap B_{1/2}(0). \tag{2.12}$$

where $C_0 = C_0(n)$, for $0 \le t \le 1$. On the other hand,

$$\underline{u}_{2\epsilon;q,r}(x,t) \ge P_{q,r}(x,t) \ge d|q| \text{ on } \tilde{l}_{q,r} - d\nu. \tag{2.13}$$

2. Now we take

$$u_1(x,t) := 4 \sup_{y\in B_{(t-t_0)_+}(x)} \underline{u}_{2\epsilon;q,r}(y, 4(t-t_0)+t_0).$$

Then for $t \ge t_0$, u_1 is a subsolution of $(P)_\epsilon$ with normal velocity

$$V \le |Du_1| - g(\frac{x}{\epsilon}) - 1 \tag{2.14}$$

away from l_1, where

$$l_1(t_0+\tau) := l_1 \cap \{t = t_0 + \tau\} = l_{q,r}(t_0+\tau) + t\tau\nu.$$

Due to (2.12) and (2.13),

$$u_2(x,t) := u_1(x-\xi, t) \le u_3(x,t) := \inf_{y\in B_\epsilon(x)} \underline{u}_{2\epsilon;q,r}(y,t) \tag{2.15}$$

on $l_2 \cap \{t_0 \le t \le t_0 + 2\epsilon\}$, where $l_2 = l_1 + \xi$ and $\xi \in \epsilon\mathbb{Z}^n$ such that

$$\frac{M\epsilon}{2} \le |\xi| \le M\epsilon, M = M(|q|) = 4C_0|q|^{-1} \text{ and } |\xi - (\xi\cdot\nu)\nu| \le \epsilon.$$

Let

$$l_2^+ := \{(x,t) : x\cdot\nu \ge y\cdot\nu\}, \text{ where } y \text{ is the projection of } x \text{ on } l_2(t).$$

By Theorem 1.7 applied in the domain

$$\Sigma := Q_1 \cap (l_2^+ \times \{t_0 \leq t \leq t_0 + \epsilon\})$$

using (2.14) and (2.15), we obtain that

$$u_2(x,t) \leq u_3(x,t) \text{ in } \Sigma.$$

3. Note that, due to comparison with translated versions of $2\underline{u}_{\epsilon;q,r}(x/2, t/2)$ using the fact $\underline{A}_{\epsilon;q,r} \neq \emptyset$, for every point (x,t) in the zero set of $P_{q,r}$ for Q_1, there is a free boundary point of $\Gamma(\underline{u}_{2\epsilon;q,r})$ in $B_{2\epsilon}(x)$. Hence $u_3(x,t) = 0$ in $\{P_{q,r} = 0\} \cap \Sigma$.

Hence it follows that the free boundary of u_2 at $t = t_0$ is $M\epsilon$-flat. Since $0 \leq t_0 \leq 1$ is arbitrary, our conclusion follows. □

3 Uniqueness of the limiting velocity

Suppose q is a nonzero vector in $I\!R^n$.

Lemma 3.1. *Suppose* $-2 < r < 2$.

(a) Suppose $0 < \underline{r}(q) \leq r$. *Then* $\underline{u}_{\epsilon;q,r}$ *has its free boundary velocity bigger than* $\frac{r\epsilon}{10}$.

(b) Suppose $r \leq \underline{r}(q) < 0$. *Then* $\bar{u}_{\epsilon;q,r}$ *has its free boundary velocity less than* $\frac{r\epsilon}{10}$.

Proof. 1. Note that $\underline{u}_{\epsilon;q,r}$ increases in time for $r > 0$. In particular, formally $|D\underline{u}_{\epsilon;q,r}| \geq 1$ on the free boundary. This and the fact that the Lipschitz constant of g is less than 10 yield that

$$u_2(x,t) := (1-h)\underline{u}_{\epsilon;r,q}(x + \frac{h}{10}\epsilon\nu, t + r^{-1}h)$$

is a supersolution of $(P)_\epsilon$ with $P_{q,r}(\cdot,t) \prec u_2(\cdot,t)$ for any small $h > 0$. Hence due to Theorem 1.7 $\underline{u}_{\epsilon;q,r}(\cdot,t) \leq u_2$ in Q_1, which yields (a).

2. Similarly, $\bar{u}_{\epsilon;q,r}$ decreases in time for $r < 0$, yielding $|D\bar{u}_{\epsilon;q,r}| \leq 2$ on the free boundary. Parallel arguments as above then yield the inequality

$$(3.1) \qquad (1+h) \sup_{y \in B_{h\epsilon/10}(x)} \bar{u}_{\epsilon;r,q}(y, t + r^{-1}h) \leq \bar{u}_{\epsilon;r,q}(x,t)$$

in Q_1 for any small $h > 0$, from which (b) follows.

□

Corollary 3.2. *For any* $\gamma > 0$,

$$\tilde{u}_1(x,t) = (1 + r\gamma\epsilon) \inf_{y \in B_{r\gamma\epsilon^2}(x)} \underline{u}_{\epsilon;q,r}(y, (1+20\gamma)t).$$

is a supersolution for $0 < r < 2$ *and*

$$\tilde{u}_2(x,t) = (1 - r\gamma\epsilon) \sup_{y \in B_{r\gamma\epsilon^2}(x)} \bar{u}_{\epsilon;q,r}(y, (1+20\gamma)t)$$

is a subsolution of $(P)_\epsilon$ *for* $-2 < r < 0$.

For $n \in \mathbb{N}$, let us define corresponding maximal subsolution $\bar{w}^n_{\epsilon;q,r}$ and minimal supersolution $\underline{w}^n_{\epsilon;q,r}$ of $(P)_\epsilon$ in the "strip" domain

$$Q_n := nQ_1 \cap \{-2 \leq x \cdot \nu \leq 2\}.$$

with boundary data $P_{q,r}$. Parallel arguments as above then yields the following:

Corollary 3.3. *Lemma 3.1 also holds for* $\bar{w}^n_{\epsilon;q,r}$ *and* $\underline{w}^n_{\epsilon;q,r}$.

Next let us define, for $r \leq \bar{r}(q)$,

$$\bar{u}^\infty_{\epsilon;q,r} := (\limsup_{n\to\infty} \bar{u}^n_{\epsilon;q,r})^*$$

where

$$\bar{u}^n_{\epsilon;q,r}(x,t) := n\bar{u}_{\frac{\epsilon}{n};q,r}(\frac{x}{n}, \frac{t-1}{n} + 1).$$

and for $\gamma > 0$

$$\bar{v}^\infty_{\epsilon;q,r,\gamma}(x,t) := \sup_{y \in B_{\gamma\epsilon}(x)} \bar{u}^\infty_{\epsilon;q,r}(y,t).$$

Let us also define, for $r \geq \underline{r}(q)$,

$$\underline{u}^\infty_{\epsilon;q,r} := (\liminf_{n\to\infty} \underline{u}^n_\epsilon)_*$$

where

$$\underline{u}^n_{\epsilon;q,r}(x,t) := n\underline{u}_{\frac{\epsilon}{n};q,r}(\frac{x}{n}, \frac{t-1}{n} + 1).$$

Let $\nu = \nu(q) = \frac{q}{|q|}$, and let $M(\gamma)$ and $M(|q|)$ be the constants given respectively in Proposition 2.9 (a) and (b).

Lemma 3.4. *Suppose* $r < 0$. *Then* *(a)* $\bar{u}^{\infty}_{\epsilon;q,r}$ *is a subsolution of* $(P)_\epsilon$ *and*

$$P_{q,r}(x+M\epsilon\nu,t)\leq \bar{v}^{\infty}_{\epsilon;q,r,\gamma}(x,t)\leq (1+\gamma\epsilon)P_{q,r}(x-\gamma\epsilon\nu,t),$$

with $M=M(\gamma)$, *in* $\mathbb{R}^n\times[0,\infty)$.

(b) $\underline{u}^{\infty}_{\epsilon;q,r}$ *is a supersolution of* $(P)_\epsilon$ *such that*

$$P_{q,r}(x,t)\leq \underline{u}^{\infty}_{\epsilon;q,r}(x,t)\leq P_{q,r}(x-M\epsilon\nu,t)$$

with $M=M(|q|)$, *in* $\mathbb{R}^n\times[0,\infty)$.

(c) For $r>0$, *(a)-(b) holds with* M *as given in Proposition 2.8.*

(d)

$$\bar{u}^{\infty}_{\epsilon;q,r}(x+\mu,t)=\bar{u}^{\infty}_{\epsilon;q,r}(x,t)$$

for any lattice vector μ *orthogonal to* q.

(The same equality holds for $\underline{u}_{\epsilon;q,r}$.*)*

(e) for any $\mu\in\mathbb{Z}^n$ *such that* $\mu\cdot\nu\geq 0$,

$$\bar{u}^{\infty}_{\epsilon;q,r}(x+\epsilon\mu,t+r^{-1}\epsilon\mu\cdot q)\leq \bar{u}^{\infty}_{\epsilon;q,r}(x,t)$$

and

$$\underline{u}^{\infty}_{\epsilon;q,r}(x+\epsilon\mu,t+r^{-1}\epsilon\mu\cdot\nu)\geq \underline{u}^{\infty}_{\epsilon;q,r}(x,t).$$

Proof. 1. We will only prove the lemma for $\bar{u}^{\infty}_{\epsilon;q,r}$ with $r<0$ and $r\leq\bar{r}(q)$, since parallel argument holds for the rest of the cases.

2. Note that $\bar{u}^{n}_{\epsilon;q,r}$ is the maximal subsolution which is smaller than $P_{q,r}$ in $Q_n:=nQ_1$ with boundary data $P_{q,r}$. Therefore $\bar{u}^{n}_{\epsilon;q,r}$ is decreasing in n and thus converges to $\bar{u}^{\infty}_{\epsilon;q,r}$. Moreover due to Propositions 2.8 (a) and 2.9 (a)

$$(1-\frac{M(\gamma)\epsilon}{n})P_{q,r}(x+M(\gamma)\epsilon\nu,t)\leq \sup_{y\in B_{\gamma\epsilon}(x)}\bar{u}^{n}_{\epsilon;q,r}(y,t),$$

and thus (a) holds.

3. We claim that

$$\Gamma(\bar{u}^{\infty}_{\epsilon;q,r})\subset\limsup_{n\to\infty}\Gamma(\bar{u}^{n}_{\epsilon;q,r}). \tag{3.2}$$

If (3.2) is false, then there exists $(x,t) \in \Gamma(\bar{u}^{\infty}_{\epsilon;q,r})$ and $h>0$ such that

$$B_h(x) \times [t-h, t+h] \in \Omega(\bar{u}^n_{\epsilon;q,r}) \text{ for all } n>0.$$

Choose $(y,s) \in B_{h/2}(x) \times [t-h, t+h]$ such that $c_0 = \bar{u}^{\infty}_{\epsilon;q,r}(y,s) > 0$. Due to Harnack's inequality it follows that

$$\bar{u}^n_{\epsilon;q,r}(\cdot, s) > Cc_0 \text{ in } B_{h/2}(x) \text{ for any } n,$$

where C is a dimensional constant. This contradicts the fact that $(x,t) \in \Gamma(\bar{u}^{\infty}_{\epsilon;q,r}$.

Now standard viscosity solutions argument will prove that $\bar{u}^{\infty}_{\epsilon;q,r}$ is a viscosity subsolution of $(P)_\epsilon$.

4. Suppose $\mu \in \mathbb{Z}^n$ with $\mu \cdot \nu = 0$. Observe that, for any n such that $\epsilon|\mu| \leq N \leq n$,

$$\bar{u}^{n+N}_{\epsilon;q,r}(x+\epsilon\mu, t) \leq \bar{u}^n_{\epsilon;q,r}(x,t) \leq \bar{u}^{n-N}_{\epsilon;q,r}(x+\epsilon\mu, t) \text{ in } Q_n,$$

Hence taking $n \to \infty$ it follows that

$$\bar{u}^{\infty}_{\epsilon;q,r}(x+\epsilon\mu, t) = \bar{u}^{\infty}_{\epsilon;q,r}(x,t).$$

5. (e) follows from the fact that, for any $\mu \in \mathbb{Z}^n$ such that $\mu \cdot \nu \geq 0$,

$$\bar{u}^{n+N}_{\epsilon;q,r}(x+\epsilon\mu, t + r^{-1}\epsilon\mu\cdot\nu) \leq \bar{u}^n_{\epsilon;q,r}(x,t)$$

if $N \geq |\mu|$.

$\square$

Let $\bar{w}^n_{\epsilon;q,r}$ and $\underline{w}^n_{\epsilon;q,r}$ as given in Corollary 3.3 and define

$$\underline{w}^{\infty}_{\epsilon;q,r} := (\lim_{n\to\infty} \underline{w}^n_{\epsilon;q,r})_*$$

and

$$\bar{w}^{\infty}_{\epsilon;q,r} := (\lim_{n\to\infty} \bar{w}^n_{\epsilon;q,r})^*$$

Note that the limit exists since $\bar{w}^n_{\epsilon;q,r}$ is decreasing and $\underline{w}^n_{\epsilon;q,r}$ is increasing in n. Also note that

$$\bar{u}^n_{\epsilon;q,r} \leq \bar{w}^n_{\epsilon;q,r}, \quad \underline{w}^n_{\epsilon;q,r} \leq \underline{u}^n_{\epsilon;q,r}.$$

Above inequality and parallel arguments as above then yields the following:

Corollary 3.5. *(a)-(e) holds for* $\bar{w}^{\infty}_{\epsilon;q,r}$ *and* $\underline{w}^{\infty}_{\epsilon;q,r}$.

Lemma 3.6. *For* $r = \underline{r}(q) \neq 0$ *and for* $0 < \epsilon < 1$, *there exists* $U_{\epsilon;q,r}$, *a subsolution of* $(P)_\epsilon$ *in* $I\!R^n \times [0,1]$ *with the following properties:*

(a) $U_{\epsilon;q,r} \geq P_{q,r}$.

(b) $U_{\epsilon;q,r}(x+\mu,t) = U_{\epsilon;q,r}(x,t)$ *for any* $\mu \in \epsilon\mathbb{Z}^n$ *orthogonal to* q.

(c) for any $\mu \in \epsilon\mathbb{Z}^n$ *with* $\mu\cdot\nu \geq 0$,

$$U_{\epsilon;q,r}(x+\mu, t + r^{-1}\mu\cdot\nu) \geq U_{\epsilon;q,r}(x,t).$$

(d) $d(\Gamma_t(U_{\epsilon;q,r}), l_{q,r}(t)) \leq M\epsilon$ *for* $0 \leq t \leq 1$,

where M *is the constant given in Propositions 2.8 and 2.9 (b).*

Proof. 1. Take $r(\delta) = \underline{r}(q) - \delta$ for any small $\delta > 0$. Then by definition of $\underline{r}(q)$, there exists $\epsilon_0 > 0$ such that

$$d(\Gamma_t(\underline{u}_{\epsilon_0;q,r_\delta}) \cap B_{1/2}(0), l_{q,r}(t)) > 0 \text{ for } \frac{1}{2} \leq t \leq 1. \tag{3.3}$$

Take the supremum of ϵ_0 satisfying (3.3) which is less than $|r|/8$ and denote it by $\epsilon(\delta)$. From Lemma 2.7 (b) it follows that for $0 < \epsilon \leq \frac{\epsilon(\delta)}{2}$, $\underline{u}_{\epsilon;q,r(\delta)}$ is a solution of $(P)_\epsilon$ in $B_{1/2}(0)\times[1/2,1]$.

2. Suppose $\epsilon_k := \frac{\epsilon(\delta_k)}{2} \to \epsilon_1 > 0$ along a subsequence $\delta_k \to 0$. One can check that

$$\underline{u}_{\epsilon_k;q,r_k} \text{ locally uniformly converges to } \underline{u}_{\epsilon_1;q,r}$$

as $k\to\infty$, where $r_k := r(\delta_k)$. It follows that $\underline{u}_{\epsilon;q,r}$ with $\epsilon \leq \epsilon_1$ is a solution of $(P)_\epsilon$ in $B_{1/2}(0)\times[1/2,1]$. Moreover due to the definition of $\underline{r}(q)$ and Propositions 2.8 and 2.9, $\Gamma(\underline{u}_{\epsilon;q,r})$ stays in $M\epsilon$-neighborhood of $l_{q,r}$ in $B_{1/2}(0)\times[1/2,1]$. Using above properties of $\underline{u}_{\epsilon;q,r}$, similar arguments as in the proof of Lemma 3.4 yields that

$$U_{\epsilon;q,r} := (\lim_{n\to\infty} n\underline{u}_{\frac{\epsilon}{n};q,r}(\frac{x}{n}, \frac{t-1}{n}+1))^*$$

satisfies (a)-(d) in above lemma (Note that the limit exists since the sequence is increasing in n.)

3. When $\epsilon(\delta)\to 0$ as $\delta \to 0$, observe that at $\epsilon = 2\epsilon(\delta)$ with sufficiently small δ, the free boundary $\Gamma(\underline{u}_{\epsilon;q,r(\delta)})$ has a contact point with $l_{q,r(\delta)}$ in $B_{1/2}(0)\times[1,2]$. Due to Propositions 2.8 and 2.9, $\Gamma(\underline{u}_{\epsilon;q,r(\delta)})$ then stays

within $M\epsilon$-neighborhood of $l_{q,r(\delta)}$ in $B_{1/2}(0)\times[1/2,1]$. Choose a sequence $\delta_k\to 0$ such that ϵ_k decreases in k and define

$$v_k(x,t)=\alpha_k \underline{u}_{2\epsilon_k;q,r_k}(\frac{x}{\alpha_k},\frac{t-1}{\alpha_k}+1),\quad \alpha_k=\frac{\epsilon}{2\epsilon_k}.$$

We define

$$U_{\epsilon;q,r}(x,t):=(\lim_{k\to\infty}v_k(x,t))^*.$$

Above limit exists since for any v_k, there exists N such that $v_k\leq v_{k+l}$ in Q_{α_k} if $l\geq N$, due to the fact that r_k and α_k increases in k and $\Gamma(v_k)$ stays in $M\epsilon$-neighborhood of l_{q,r_k}.

Parallel arguments in the proof of Lemma 3.4 then yield that $U_{\epsilon;q,r}$ satisfies (a)-(d) in our lemma.

□

Let $q\in I\!R^n$, $|q|\neq 0$. We call q a *rational* vector if

$$q=m(a_1e_1+...a_1e_n),\quad m\in I\!R \text{ and } a_i\in\mathbb{Q}.$$

Lemma 3.7. *$\bar{r}(q)=\underline{r}(q)$ for rational vector $q\in I\!R^n$.*

Proof. 1. First we show that $\bar{r}(q)\leq\underline{r}(q)$. If $0>r=\bar{r}(q)>(1-20\gamma)\underline{r}(q)$ for some $\gamma>0$, we compare $\underline{u}^{\infty}_{\epsilon;q,r}$ and

$$v_1(x,t):=(1-r\gamma\epsilon)\sup_{|y-x|<r\gamma\epsilon}\bar{w}^{\infty}_{\epsilon;q,r}(y,(1+20\gamma)t),$$

using Corollaries 3.3 and 3.5 and argue as in the proof of Lemma 10 in [K3] to draw a contradiction. Similar argument applies to yield a contradiction for the case $0<\underline{r}(q)<\bar{r}(q)$.

2. Suppose $r_1=\bar{r}(q)<r_2=\underline{r}(q)$. Then for any $\epsilon>0$ there is a global subsolution $U_{\epsilon;q,r_2}$ of $(P)_\epsilon$ given in Lemma 3.6. In particular $U_{\epsilon;q,r_2}$ is periodic with respect to a direction perpendicular to q, according to Lemma 3.6 (b). On the other hand at $r_3=(r_1+r_2)/2$ there is $\epsilon_0>0$ for which $\bar{u}_{\epsilon_0;q,r}$ is a solution in $B_{1/2}(0)\times[1/2,1]$. Now we compare a translation of $U_{\epsilon;q,r_2}$ and $\bar{u}_{\epsilon_0;q,r}$ in Q_1 such that $U_{\epsilon;q,r_2}$ crosses from $\bar{u}_{\epsilon_0;q,r}$ at $t\in[1/2,1]$. From the periodicity of $U_{\epsilon;q,r_2}$ and the boundary data of $\bar{u}_{\epsilon_0;q,r}$, it follows that the first contact point is at the intersection of the free boundary points of $U_{\epsilon;q,r_2}$ and $\bar{u}_{\epsilon_0;q,r}$ in $B_{1/2}(0)\times[1/2,1]$. This contradictions Theorem 1.7. □

We will next prove that, for a nonzero vector $q_0\in I\!R^n$ and $r\neq 0$, if $r>\underline{r}(q_0)$ and if $q=aq_0$ with $a<1$ then for sufficiently small ϵ the free boundary of $\bar{u}_{\epsilon;q,r}$ falls behind $l_{q_0,r}$ by a positive distance after a positive amount of time. (Corresponding result for $\underline{u}_{\epsilon;q,r}$ will be proved in Proposition 3.11.)

Later we will prove $r(q) := \bar{r}(q) = \underline{r}(q)$ in Lemma 3.12. In this case Proposition 3.8 and 3.11 suggests a "robust" uniqueness for the effective free boundary speed $r(q)$, as long as $r(q) \neq 0$: that is, with other choices of r and with a slight perturbation on the size of q, the free boundary of u^ϵ moves significantly slower or faster than r, detaching itself from the obstacle $l_{q,r}$. For $r(q) = 0$, such uniqueness is no longer true (see Lemma 3.15).

Proposition 3.8. *Suppose that q is a nonzero vector in $\mathbb{R}^n$. Then there exists a dimensional constant $C(n) > 0$ such that for sufficiently small $\gamma > 0$, $r_1 = (1 - C(n)\gamma)r$ and $q_1 = (1 - C(n)\gamma)q$ the following holds:*

(a) Suppose $r \geq \underline{r}(q) > 0$. Then for $\frac{\epsilon_0}{100} < \epsilon < \epsilon_0 = \frac{r\gamma^{10}}{8nM}$,

$$d(\Gamma_t(\bar{u}_{\epsilon;q_1,r_1}), l_{q,r_1}(t) \cap B_{1/4}(0)) > M\epsilon_0$$

for $\frac{M\epsilon_0}{r\gamma^2} \leq t \leq 1$, where M is the constant given in Proposition 2.8.

(b) Suppose $\underline{r}(q) \leq r < 0$. Then for ϵ_0 as above and for $\frac{\epsilon_0}{100} < \epsilon < \epsilon_0$,

$$d(\Gamma_t(\bar{u}_{\epsilon;q_1,r_1}), l_{q,r_1}(t) \cap B_{1/4}(0)) > M\epsilon_0$$

for $\frac{M\epsilon_0}{|r|\gamma^2} \leq t \leq 1$, where $M = M(|q|)$ is the constant given in Proposition 2.9 (b).

Proof. Let us denote $N = \gamma^{-8}$. Then there exists $\xi \in \mathbb{Z}^n$ depending on ν such that

$$0 \leq |\xi| \leq N, \quad -\xi \cdot \nu = m \in [1, 2].$$

(See Figure 5.)

Proof of Proposition 3.8 (a).

1. Consider the domain

$$\Pi := \{(x,t) : |x| \leq 1/2 + (n+1)Nt, 0 \leq t \leq \frac{2M\epsilon_0}{r\gamma^2}\}.$$

Observe that $\Pi \subset Q_1$ by definition on ϵ_0. Let $C(n) > 0$ be a dimensional constant and define

$$q_1 = (1 - C(n)\gamma)q \text{ and } u_1 := \breve{u}_{\epsilon;q_1,r_1}, \tag{3.5}$$

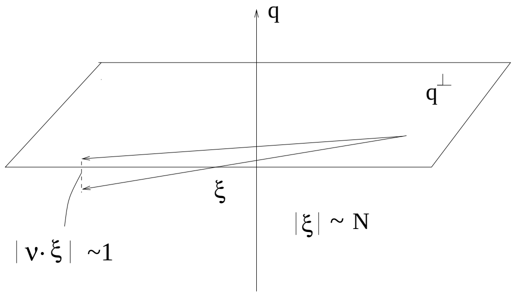

Figure 5: A slightly downward translation by a lattice vector

where $\breve{u}_{\epsilon;q_1,r_1}$ is the maximal subsolution below P_{q_1,r_1}, defined the same as $\bar{u}_{\epsilon;q_1,r_1}$, in the domain Π instead of Q_1. A parallel argument as in Lemma 2.4 yields that $u_1 = P_{q,r}$ on the parabolic boundary of Π. Note that $\bar{u}_{\epsilon;q,r_1} \leq u_1$ since $\Pi \subset Q_1$.

It follows from (3.4), the definition of Π, u_1, and Theorem 1.7 that

$$u_1(x+\epsilon\xi, t+m(r_1)^{-1}\epsilon) \leq u_1(x,t), \tag{3.6}$$

in $B_{1/4}(0)$, where μ is any lattice vector orthogonal to q such that $|\mu| \leq nN$.

Let us choose $\alpha \in [\frac{C(n)}{2}\gamma, C(n)\gamma]$ such that

$$(1+\alpha)r_1 - (1-\alpha)r = r\gamma^2.$$

Next we define

$$u_2(x,t) := (1-2\alpha)\underline{u}^{\infty}_{\epsilon;q,r}(x,(1-\alpha)t + r^{-1}(M+C_1\gamma)\epsilon_0) \tag{3.7}$$

where $C_1 > 0$ is a dimensional constant to be chosen later.

Parallel argument as in the case of u_1 yields that

$$u_2(x+\epsilon\xi, t+m(1-\alpha)^{-1}r^{-1}\epsilon) \geq u_2(x,t). \tag{3.8}$$

in $B_{1/4}(0)$, where μ is as given in (3.6).

3. Finally, set

$$\tilde{u}_1(x,t) := (1+2\alpha)\sup_{y\in B_{\gamma\epsilon}(x)} u_1(y,(1+\alpha)t); \qquad \tilde{u}_2(x,t) := \inf_{y\in B_{C_1\gamma\epsilon}(x)} u_2(y,t).$$

Note that $\tilde{u}_1$ and $\tilde{u}_2$ are respectively a sub- and supersolution of $(P)_\epsilon$ if $C(n)$ is large with respect to C_1. Our goal is to prove that

$$\Omega(\tilde{u}_1) \subset \Omega(\tilde{u}_2) \text{ in } \Sigma := \bar{B}_{1/4}(0) \times [0, \frac{2M\epsilon_0}{r}\gamma^2] \tag{3.9}$$

if C_1 and $C(n)$ is sufficiently large.

Due to Lemma 3.4, $\Gamma(u_2)$ stays within the $M\epsilon$-strip of $l_{q,r}(t)$. This and the fact that

$$(1+\alpha)r_1 - (1-\alpha)r = r\gamma^2 \text{ and } \bar{u}_{\epsilon;q,r} \leq u_1$$

yields our theorem for $\frac{M\epsilon_0}{r\gamma^2} \leq t \leq \frac{2M\epsilon_0}{r\gamma^2}$ once (3.9) is proved. For $\frac{M\epsilon_0}{r\gamma^2} \leq t \leq 1$, the theorem holds due to Corollary 2.6, (a) for $\bar{u}_{\epsilon;q,r}$.

4. Suppose that $\Gamma(\tilde{u}_1)$ contacts $\Gamma(\tilde{u}_2)$ from below at (x_0, t_0) for the first time in Σ. By definition of u_2, $t_0 > M\epsilon_0$. Let us define

$$\mathcal{S} := \{y \in B_{1/2}(0) : |(y - x_0)\cdot v| \leq N\epsilon \text{ for any } v \text{ orthogonal to } q.\}$$

Due to (3.6) and (3.8) we have

$$\Omega_{t-2\gamma^2\epsilon}(\tilde{u}_1) \subset \Omega_t(\tilde{u}_2) \text{ in } \mathcal{S}\times[t_0 - r^{-1}M\epsilon, t_0 + M\epsilon]. \tag{3.10}$$

To see this, let $\Phi(u)$ be the characteristic function of the support of u. Then

$$\begin{aligned}
\Phi(\tilde{u}_1)(x, t-2\gamma^2\epsilon) &\leq \Phi(\tilde{u}_1)(x+\epsilon\xi, t-2\gamma^2\epsilon - m\epsilon(1+\alpha)^{-1}(r_1)^{-1}) \\
&\leq \Phi(\tilde{u}_2)(x+\epsilon\xi, t-2\gamma^2\epsilon - m\epsilon(1+\alpha)^{-1}(r_1)^{-1}) \\
&\leq \Phi(\tilde{u}_2)(x, t-2\gamma^2\epsilon + m\epsilon((1-\alpha)^{-1}r^{-1} - m(1+\alpha)^{-1}(r_1)^{-1}) \\
&\leq \Phi(\tilde{u}_2)(x,t),
\end{aligned}$$

where the first inequality is due to (3.6), the second inequality due to the fact $\Omega_t(\tilde{u}_1) \leq \Omega_t(\tilde{u}_2)$ in $B_{1/4}(0)\times[0,t_0]$, the third inequality due to (3.8), and the last inequality holds due to (3.4) and the fact that $\tilde{u}_2$ increases in time.

5.

Lemma 3.9. *If $C_1 = C_1(n)$ in (3.7) is sufficiently large, then*

$$\tilde{u}_1(x,t) \leq \inf_{y\in B_{2\gamma\epsilon}(x)} u_2(y,t) \tag{3.11}$$

on $\Gamma(\tilde{u}_1)\cap(\mathcal{S}\times[t_0 - r^{-1}M\epsilon_0, t_0])$ and

$$\tilde{u}_1(x,t) \leq \inf_{y\in B_{\gamma\epsilon}(x)} u_2(y,t) \tag{3.12}$$

in $\Omega(\tilde{u}_1)\cap(B_{3M\epsilon}(x_0)\times[t_0 - r^{-1}M\epsilon_0, t_0])$.

Proof. Let $u_3 := \inf_{y \in B_{\gamma\epsilon}(x)} u_2(y,t)$. (3.11) holds due to (3.10) and the fact that

$$\tilde{u}_2(x, t + 2\gamma^2\epsilon) \leq \inf_{y \in B_{2\gamma\epsilon}(x)} u_2(y,t),$$

if C_1 is sufficiently large dimensional constant, which can be proven as in the proof of Lemma 12 in [K3]. In fact, since $\underline{u}^{\infty}_{\epsilon;q,r}$ increases in time, formally $|Du_2| \geq 1$ on $\Gamma(u_2)$. Thus (3.11) yields

$$\tilde{u}_1 \leq u_3 + \gamma\epsilon \text{ on } \Gamma(\tilde{u}_1) \cap (\mathcal{S} \times [t_0 - r^{-1}M\epsilon_0, t_0]).$$

Note that by definition of u_1 and u_2,

$$\tilde{u}_1 \leq u_3 \text{ on } (l_{q,r} - t_0\nu) \cap \{0 \leq t \leq t_0\}.$$

Let $h(x)$ be the harmonic function in

$$D = \mathcal{S} \cap \{-t_0 \leq (x - x_0) \cdot \nu \leq 2M\epsilon_0\}$$

with $h = 0$ on $\{(x - x_0) \cdot \nu = -t_0\}$, $h = \gamma\epsilon$ on $\{(x - x_0) \cdot \nu = 2M\epsilon_0\}$ and $h = -M|q|\epsilon_0$ on $\partial\mathcal{S}$. Since $\Gamma(u_2)$ is $M\epsilon$-flat and $u_1 - u_3$ is subharmonic,

$$u_1 - u_3 \leq h \text{ in } D \times [t_0 - r^{-1}M\epsilon_0, t_0].$$

Due to the fact that the width of $\mathcal{S}$ is $N\epsilon$ with $N \geq (r^2\gamma^8)^{-1}1$ and $\epsilon \geq \epsilon_0/100$, $h \geq 0$ in $3M\epsilon_0$-neighborhood of x_0. Hence (3.12) follows if $0 < \gamma < |q|$. $\square$

The rest of the proof is parallel to that of Proposition 1 in [K3], using (3.11) and (3.12).

$\square$

Proof of Proposition 3.8(b)

1. Consider the domain

$$\Pi := \{(x,t) : |x| \leq \frac{1}{2} + (n+1)Nt, 0 \leq t \leq \frac{2M\epsilon_0}{|r|\gamma^2}\},$$

where M is the constant given in Proposition 2.9 (a).

Let $u_1 := \breve{u}_{\epsilon;q,r_1}$ as in the proof of (a). Since $r < 0$, u_1 decreases in time. Moreover note that $\bar{u}_{\epsilon;q,r_1} \leq u_1$ since $\Pi \subset Q_1$.

Let q_1 as in (3.5) and choose $\alpha \in [\frac{1}{4}C(n)\gamma, C(n)\gamma]$ so that

$$(1+\alpha)r_1 - (1-\alpha)r = |r|\gamma^2,$$

and define

$$u_2(x,t) := (1-2\alpha)\underline{u}^{\infty}_{\epsilon;q_1,r}(x,(1-\alpha)t - |r|^{-1}(M + C_1\gamma)\epsilon_0)$$

where $C_1 > 0$ is a dimensional constant to be chosen later.

3. Set

$$\tilde{u}_1(x,t) := (1+2\alpha) \sup_{y\in B_{\gamma\epsilon}(x)} u_1(y,(1+\alpha)t); \qquad \tilde{u}_2(x,t) := \inf_{y\in B_{C_1\gamma\epsilon}(x)} u_2(y,t).$$

Note that $\tilde{u}_1$ and $\tilde{u}_2$ are respectively a subsolution and a supersolution of $(P)_\epsilon$. As before, our goal is to prove that

$$\Omega(\tilde{u}_1) \subset \Omega(\tilde{u}_2) \text{ in } \Sigma := \bar{B}_{1/4}(0) \times [0, 2M\epsilon_0/|r|\gamma^2] \tag{3.13}$$

if C_1 and $C(n)$ is sufficiently large. If (3.13) holds, it follows from Lemma 3.4 applied to u_2 that $\Gamma_t(\bar{u}_{2\epsilon;q,r})$ is more than $M\epsilon_0$-away from $l_{q,r}(t)$ for $\frac{M\epsilon_0}{|r|\gamma^2} \leq t \leq \frac{2M\epsilon_0}{|r|\gamma^2}$. For $\frac{2M\epsilon_0}{|r|\gamma^2} \leq t \leq 1$ (b) holds due to Corollary 2.6.

4. Suppose that $\Gamma(\tilde{u}_1)$ contacts $\Gamma(\tilde{u}_2)$ from below at (x_0,t_0) for the first time in Σ. By definition of u_2, $t_0 > M\epsilon_0$. Let μ and $\mathcal{S}$ as before. Arguing as before for the case $r > 0$, and using the fact that u_1 decreases in time leads to

$$\Omega_{t+2\gamma^2\epsilon}(\tilde{u}_1) \subset \Omega_t(\tilde{u}_2) \text{ in } \mathcal{S} \times [0, t_0 + M\epsilon] \tag{3.14}$$

5.

Lemma 3.10.

$$\tilde{u}_1(x,t) \leq \inf_{y\in B_{(C_1\gamma - 4\gamma^2)\epsilon}(x)} u_2(y,t) \tag{3.15}$$

on $\Gamma(\tilde{u}_1) \cap (\mathcal{S} \times [t_0/4, t_0])$ *and*

$$\tilde{u}_1(x,t) \leq \inf_{y\in B_{\gamma\epsilon}(x)} u_2(x,t) \tag{3.16}$$

in $\Omega(\tilde{u}_1) \cap (\tilde{S} \times [t_0/4, t_0])$, *where*

$$\tilde{S} = (\frac{1}{2}(\mathcal{S} + x_0)) \cap \{-|r|t_0 \leq (x-x_0)\cdot\nu \leq \frac{|r|t_0}{2}\} \tag{3.17}$$

Proof of lemma 3.10:

1. The definition of u_2 and the fact that the free boundary speed for u_2 is always greater than -2 yields

$$\inf_{y\in B_{C_1\gamma\epsilon}(x)} u_2(y,t) \leq \inf_{y\in B_{(C_1\gamma-2\gamma^2)\epsilon}(x)} u_2(y,t+\gamma^2\epsilon). \tag{3.18}$$

Now (3.15) follows from (3.18) and (3.14).

2. Note that by definition of u_1 and u_2,

$$\tilde{u}_1 \leq (1+2\alpha+\gamma\epsilon|q|)P_{q,(1+\alpha)r_1}$$

and

$$\tilde{u}_2 \geq (1-2\alpha-C_1\gamma\epsilon|q|)P_{q,r}.$$

It follows that

$$\tilde{u}_1(x,t) \leq \inf_{y\in B_{\gamma\epsilon}(x)} u_2(x,t)+\gamma\epsilon_0|q| \tag{3.19}$$

in $\{x:(x-x_0)\nu\leq -t_0\}\times[\frac{t_0}{4},t_0]$.

Now (3.17) follows from arguing as in the proof of Proposition 2.9(a) using (3.15) and (3.19), $N>1/r^2\gamma^8$ and $\frac{\epsilon_0}{100}\leq\epsilon\leq\epsilon_0$.

$\square$

6. Let us define

$$w(x,t) := \inf_{y\in B_{\gamma\epsilon\varphi(x)}(x)} u_2(x,t)$$

where φ defined in $\tilde{S}$ satisfies the following properties:

$$\begin{cases} -\Delta(\varphi^{-A_n})=0 & \text{in} \quad \tilde{S}; \\ \varphi=B_n & \text{on} \quad \tilde{S}\cap\{(x-x_0)\cdot\nu=-|r|t_0\}; \\ \varphi=1 & \text{in} \quad \text{on the rest of } \partial\tilde{S}. \end{cases}$$

(See Figure 6.)

Fix $A_n>0$, a sufficiently large dimensional constant. Then due to Lemma 9 in [C1] $w(\cdot,t)$ is superharmonic in $\Omega_t(w)\cap R$ for $0\leq t\leq 2M\epsilon_0/|r|\gamma$. Choose B_n sufficiently large that $\varphi(x_0)>C_1$. Note that $|D\varphi|\epsilon\leq C_2$ where C_2 depends on A_n, M and C_1, where C_1 is given in (3.7).

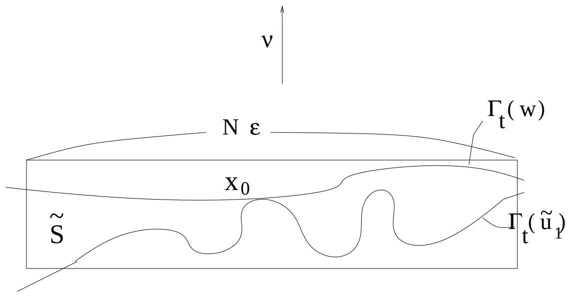

Figure 6: The strip domain for barrier argument

7. Now we compare w and $\tilde{u_1}$ in

$$\diamondsuit := \tilde{S} \times [t_0/4, t_0].$$

where $\tilde{S}$ is as given in (3.17). At $t = t_0/4$, $\tilde{u}_1 \leq w$ in $\tilde{S}$ since $\Gamma_{t_0/4}(w)$ is more than $|r|t_0/2$-away from $\tilde{S}$ due to definition of t_0 and the $M\epsilon$-flatness of $\Gamma(u_2)$. Moreover due to (3.16) and (3.19) $w \leq \tilde{u}_1$ on $\partial\tilde{S} \times [t_0/4, t_0]$.

8. However since $\varphi(x_0) > C_1$, $\tilde{u_1}$ crosses w from below in $\diamondsuit$. This will be a contradiction to Theorem 1.7 if we show that w is a supersolution of $(P)_\epsilon$ in $\diamondsuit$. By arguing as in the proof of Lemma 12 in [K3], one can check that w is a supersolution if $C(n)$ in (3.5) is sufficiently large.

□

Parallel arguments yield the corresponding result for $\underline{u}_{\epsilon;q,r}$:

Proposition 3.11. *Suppose that* q *is a nonzero vector in* $I\!R^n$. *Let* $C(n), \gamma, \epsilon_0$ *given as in Proposition 3.8 and let* $r_1 = (1 + C(n)\gamma)r$, $q_1 = (1 + C(n)\gamma)q$. *Then the following is true:*

(a) Suppose $0 < r \leq \bar{r}(q)$. *Then for* $\epsilon_0/100 < \epsilon < \epsilon_0$ *and*

$$d(\Gamma_t(\underline{u}_{\epsilon;q_1,r_1}), l_{q,r_1}(t) \cap B_{1/4}(0)) > M\epsilon_0$$

for $\frac{M\epsilon_0}{r\gamma^2} \leq t \leq 1$, *where* M *is the constant given in Proposition 2.8.*

(b) Suppose $r \leq \bar{r}(q) < 0$.

$$d(\Gamma_t(\underline{u}_{\epsilon;q_1,r_1}), l_{q,r}(t) \cap B_{1/4}(0)) > M\epsilon_0$$

for $\frac{M\epsilon_0}{|r|\gamma^2} \leq t \leq 1$, *where* $M = M(\gamma) > 0$ *is the constant given in Proposition 2.9(a).*

Now we will use Lemma 3.7, Propositions 3.8 and 3.11 to prove that $\bar{r}(q) = \underline{r}(q)$ for any nonzero $q \in I\!R^n$. For given nonzero vector $q \in I\!R^n$, take a sequence of rational vectors q_k which converges to q as $k \to \infty$. Choose q_k such that there exists

$$r^*(q) := \lim_{k\to\infty} r(q_k).$$

Lemma 3.12.

$$\bar{r}(q) = r^*(q) = \underline{r}(q). \tag{3.20}$$

Proof. 1. Let $\nu = \frac{q}{|q|}$. Arguing as in the proof of Lemma 14 in [K3] using Proposition 3.8 and 3.11, One can prove that

$$r_0 := \lim_{a<1, a\to 1} \bar{r}(aq) \leq r^*(q) \leq \lim_{a>1, a\to 1} \underline{r}(aq) := r_1. \tag{3.21}$$

2. Next we prove that

$$\underline{r}(q) = r_1 \text{ when } r^*(q) > 0. \tag{3.22}$$

and

(3.23). $$r_0 = \bar{r}(q) \text{ when } r^*(q) < 0$$

Suppose $0 < r_1 < \underline{r}(q) := r_4$ with $(1+10\gamma)r_1 = r_4$. and consider

$$\tilde{u}(x,t) = (1+\gamma) \inf_{|y-x|<\frac{r_1\gamma\epsilon^2}{10}} \underline{u}_{\epsilon;q,r_1}(y,(1+2\gamma)t)\}. \tag{3.24}$$

By Corollary 3.2, $\tilde{u}$ is a supersolution of $(P)_\epsilon$ with

$$\tilde{u}(x,t) \geq P_1(x,t) := P_{(1+\gamma)q,(1+2\gamma)r_1}(x + r_1\gamma\epsilon^2\nu, t).$$

for any $\epsilon > 0$.

Moreover due to the definition of r_1 and Lemma 2.7, $\Gamma(\tilde{u})$ and $P_1(x,t)$ has a contact point at

$$P_0 = (x_0, t_0) \in B_{1/2}(0) \times [1/2, 1].$$

Let $U_{s,\epsilon}$ be the smallest supersolution of $(P)_\epsilon$ in $(1-2\epsilon)Q_1$ with obstacle

$$P_s(x,t) := P_{q,sr_4}(x+\xi, t), \quad s \in [(1-2\gamma),(1-\gamma)],$$

where $\xi \in \epsilon\mathbb{Z}$, $\xi \cdot \nu \geq 0$ and $|\xi - (\xi\cdot\nu)\nu| \leq 2\epsilon$. Due to the definition of $\underline{r}(q)$ and Lemma 2.7, $U_{s,\epsilon}$ is a solution of $(P)_\epsilon$ away from $\Gamma(P_s)$ and in particular

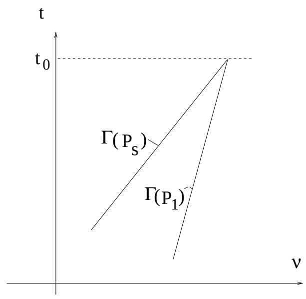

Figure 7: Comparison of minimal supersolutions with different obstacle speed.

in the domain $B_{1/2}(0) \times [1/2, 1]$ for $\epsilon \leq \epsilon_0$, $\epsilon_0 = \epsilon(\gamma) > 0$. For sufficiently small ϵ we can choose s, ξ such that P_s hits P_1 from below at $t = t_0$ (see Figure 7).

Due to Theorem 1.7, $U_s \prec \tilde{u}$ for $t \leq t_0$. Since $U_s \geq P_s$, it follows that

$$P_0 \in \Gamma(U_s) \cap \Gamma(\tilde{u}) \cap P_1 \cap P_s.$$

Note that U_s is a solution in a neighborhood of P_0. Arguing as in the proof of Theorem 1.7 will then yield a contradiction, and we obtain (3.22).

(Here $\tilde{u}$ instead of $\underline{u}_{\epsilon;q,r}$ is used since to proceed as in the proof of Theorem 1.7 since we need interior and exterior ball properties at the contact point P_0 of the two free boundaries. Interior ball property follows from the fact that $P_0 \in \Gamma(P_1)$. Exterior ball property is obtained by definition of $\tilde{u}$ in (3.24).)

3. Parallel arguments as above proves (3.23),

$$\liminf_{a \to 1} r^*(aq) \geq \underline{r}(q) \text{ for } \underline{r}(q) > 0 \tag{3.25}$$

and

$$\limsup_{a \to 1} r^*(aq) \leq \bar{r}(q) \text{ for } \bar{r}(q) < 0 \tag{3.26}$$

4. Lastly we show that

$$\underline{r}(q) \leq r^*((1+h)q), \quad r^*((1-h)q) \leq \bar{r}(q) \text{ for any } h > 0 \tag{3.27}$$

This follows from parallel arguments as in step 2 using the fact that

$$(1-h) \inf_{|y-x| \leq \frac{h}{10}\epsilon} \underline{u}_{\epsilon;q,r}(y, (1-h)t)$$

and

$$(1+h) \sup_{|y-x|\leq \frac{h}{10}\epsilon} \bar{u}_{\epsilon;q,r}(y,(1+h)t)$$

are respectively sub- and supersolutions of $(P)_\epsilon$.

Due to (3.25)-(3.27) we have

$$\underline{r}(q) \leq \lim_{h\to 0} r^*((1-h)q) \leq \bar{r}(q) \text{ for } r_1 > 0$$

which yields $\underline{r}(q) = \bar{r}(q)$ due to (3.21) and (3.22), and

$$\bar{r}(q) \geq \lim_{h\to 0} r^*((1+h)q) \geq \underline{r}(q) \text{ for } r_0 < 0$$

which yields $\underline{r}(q) = \bar{r}(q)$ by (3.21) and (3.23). Since $r_0 \leq r_1$ by definition, this covers all cases except $r_0 = r_1 = 0$, for which $\bar{r}(q) = \underline{r}(q) = 0$.

□

Let us now define

$$r(q) := \bar{r}(q) = \underline{r}(q).$$

Corollary 3.13.

$$\lim_{a<1,a\to 1} r(aq) = r(q) = \lim_{a>1,a\to 1} r(aq). \tag{3.23}$$

Proof. This follows form (3.21), (3.22) and Lemma 3.12. □

Corollary 3.14. *$r(q)$ is continuous in q in $\mathbb{R}^n - \{0\}$.*

Proof. By Proposition 3.8 and (3.20), arguing as in the proof of Lemma 14 in [K3] yields that for any nonzero $q \in \mathbb{R}^n$ and $\gamma > 0$, if $|\mu - q| \leq r\gamma^{10}$ then

$$r((1-\gamma)q) - \gamma < r(\mu) < r((1+\gamma)q) + \gamma.$$

Now due to (3.27), it follows that $r(\mu) \to r(q)$ as $\mu \to q$. □

For a unit vector $\nu \in \mathbb{R}^n$, we define the *pinning interval* in the direction of ν as below:

$$I(\nu) := \{a > 0 : r(a\nu) = 0\}.$$

Lemma 3.15. *Let $e_1, ..., e_n$ an orthonormal basis in $\mathbb{R}^n$. Let $x_1 = x \cdot e_1$ and suppose $g(x) = g(x_1) \in [1, 2]$ and $g(x_1)$ is periodic with period 1. Then $I(e_1) = [1, 2]$. On the other hand $I(e_i)$ consists of a single point if $i \neq 1$.*

Proof. 1. First let us prove that $1 \in I(e_1)$. Observe that $r(e_1) \leq 0$ since $P_{e_1,0}$ is a supersolution of $(P)_\epsilon$. If $r(e_1) < 0$ then for sufficiently small $\epsilon > 0$ the contact set $\bar{A}_{\epsilon;e_1,r(e_1)/2}$ should be empty due to Proposition 3.8. Comparing

$$u_1(x,t) := \bar{u}_{\epsilon;e_1,r(e_1)/2} \text{ with } u_2(x,t) := P_{e_1,0}(x + (a + N\epsilon)e_1),$$

where $a \in [0, \frac{3}{2}]$ is chosen such that $g(ae_1) = 0$ leads to a contradiction, if we choose the integer N such that u_2 hits u_1 from below at $t_0 \in [1/2, 1]$.

Parallel argument as above yields that $2 \in I(e_1)$. Since $r(q)$ is monotone in increasing in $|q|$, it follows that $[1,2] \subset I(e_1)$. By Lemma 2.7, $I(e_1) = [1,2]$.

2. Let $r(q) = 0$ for $q = ae_i, i \neq 1$. First note that $\underline{u}_{\epsilon;ae_i,0}$ increases in time. In particular $|D\underline{u}_{\epsilon;a_i,0}| \geq 1$ on the free boundary. It follows that for any $a > 0$ and b

$$u_1(x,t) = (1+a)\underline{u}_{\epsilon;q,0}(x - (at - b)e_i, t)$$

is a subsolution of $(P)_\epsilon$ away from $l_{q,a} + be_i$. Moreover $\Gamma(u_1)$ is in $M\epsilon$-neighborhood of $l_{q,a}$ due to Proposition 2.8. It then follows from comparing u_1 and $\underline{u}_{\epsilon;(1+a)q,a}$ with appropriate b and arguing as in the proof of Lemma 3.12 that $\underline{A}_{\epsilon;(1+a)q,a-\tau}$ is empty for small $\epsilon = \epsilon(\tau)$, and thus

$$r((1+a)q) = \underline{r}((1+a)q) \geq a.$$

3. Similarly, note that for any $a > 0$ and b

$$u_2(x,t) = (1-a)\underline{u}_{\epsilon;q,r}(x + (at+b)e_i, t)$$

is a supersolution of $(P)_\epsilon$ and has contact points with $l_{q,-a}$ in $B_{1/2}(0) \times [1/2, 1]$. Hence it follows that $r((1-a)q) = \underline{r}((1-a)q) \leq -a$. $\square$

Lemma 3.16. *For any nonzero unit vector $\nu \in I\!R^n$, $r(a\nu)$ is strictly increasing in a in the set $\{a : r(a\nu) > 0\}$.*

Proof. 1. Suppose not. Then for some $a, b > 0$, $r(a\nu) = r(b\nu), (1 + C(n)\gamma)a = b$ for some $\gamma > 0$, where $C(n)$ is a dimensional constant to be determined later.

2. Suppose $r(a\nu) > 0$. Note that for any r

$$u_1(x,t) = (1+\gamma)\bar{u}_{\epsilon;a\nu,r}(x, (1+\gamma)t)$$

is a subsolution of $(P)_\epsilon$. If we choose $(1+\gamma)^{-1}r(a\nu) < r < r(a\nu)$ then by definition of $r(aq)$ and Lemma 2.7 the contact set $\bar{A}_{\epsilon;a\nu,r}$ is nonempty for any $\epsilon > 0$. It follows from comparison with u_1 that the same is true for $\bar{u}_{\epsilon;b\nu,(1+\gamma)r}$. This contradicts the fact that $r(b\nu) = \bar{r}(b\nu) < (1+\gamma)r$.

$\square$

Remark

It is not clear to the author whether or not $r(a\nu)$ is strictly increasing in a in the set $\{a : r(a\nu) < 0\}$.

4 Convergence to the limiting problem

Recall that the choice of domain $\Omega \in I\!R^n$ containing K determines the initial data u_0 of $(P)_\epsilon$, which is harmonic in $\Omega_0 = \Omega - K$ with boundary data zero on $\Gamma_0 = \partial\Omega$ and $f > 0$ on ∂K. Also recall that $f \in C(I\!R^n \times [0,\infty))$, K satisfies (0.1) and $\overline{Int(\Omega)} = \bar{\Omega}$.

Consider the free boundary problem

$$(P) \qquad \begin{cases} -\Delta u = 0 & \text{in } \{u > 0\} \\ \\ u_t - |Du| r(Du) = 0 & \text{on } \partial\{u > 0\} \end{cases}$$

in $Q = (I\!R^n - K) \times [0,\infty)$, with initial data u_0 and with boundary data f on ∂K. Here $r(q)$ is the continuous function defined in (3.26) for $q \in I\!R^n - \{0\}$. Note that the existence and uniqueness theorems in section 1 applies to both (P) and $(P)_\epsilon$. In particular due to Theorem 1.8 there exists a viscosity solution u_ϵ of $(P)_\epsilon$ with initial data u_0 and fixed boundary data f.

Let us define

$$u_1(x,t) := \lim_{\epsilon_0, r \to 0} \sup\{u^\epsilon(y,s) : \epsilon < \epsilon_0, |(x,t) - (y,s)| \leq r, s \geq 0\}$$

and

$$u_2(x,t) = \lim_{\epsilon_0, r \to 0} \inf\{u^\epsilon(y,s) : \epsilon < \epsilon_0, |(x,t) - (y,s)| \leq r, s \geq 0\}.$$

One can check via a barrier argument using that

$$(4.1) \qquad u_1(x,0) = u_2(x,0) = u_0(x).$$

Our goal in this section is to prove that u_1 and u_2 are respectively sub- and supersolutions of (P).

Lemma 4.1.

$$\bar{\Omega}(u_1) = \limsup_{\epsilon \to 0} \Omega(u^\epsilon).$$

Proof. 1. It is straightforward from definition of u_1 that

$$\bar{\Omega}(u_1) \subseteq \limsup_{\epsilon \to 0} \Omega(u^\epsilon).$$

2. For any $r > 0$, we will show that the $\Omega^1(r)$ is in $4r$-neighborhood of $\limsup_{\epsilon\to 0} \Omega(u^\epsilon, r)$, where $\Omega^1(r)$ and $\Omega(u^\epsilon, r)$ are given by

$$\Omega_t^1(r) := \{y \in \Omega(u_1) : d(y, \Gamma_t(u_1)) > r\}, \quad \Omega_t(u^\epsilon, r) := \{y : d(y, \Omega_t(u^\epsilon)) < r\}.$$

Since r is arbitrary, our conclusion will follow.

3. Fix $T > 0$, $\epsilon > 0$ and for $0 \leq s \leq T$ let $x_\epsilon(s)$ be the furthest point in $\bar{\Omega}(u^\epsilon, r) \cap \{t = s\}$ from $\Omega^1(r) \cap \{t = s\}$ with distance $d_\epsilon^r(s)$. A barrier argument yields that the characteristic functions of $\Omega^1(r)$ and $\Omega(u^\epsilon, r)$ is continuous in time, and thus $d_\epsilon^r(t)$ is continuous in time. Also observe that $d_\epsilon^r(0) \to 0$ as $r \to 0$, by (4.1) and the fact that

$$\overline{Int(\Omega_0)} = \bar{\Omega}_0.$$

If ϵ is sufficiently small with respect to T and r and if

$$d_\epsilon^r(t) \geq m(r) := \max 2r, d_\epsilon^r(0),$$

$(u^\epsilon)^*$ is less than $r/10$ in a space-time neighborhood of $B_{2r}(x_\epsilon(t))$. Moreover by definition of $\Omega(u^\epsilon, r)$ there is the ball $B_r(x_\epsilon(t))$ touching $\Omega_t(u^\epsilon)$ from outside. By a barrier argument with a radially symmetric function, we obtain that $d_\epsilon^r(t)$ decreases in time if $d_\epsilon^r(t) \geq m(r)$. Since $d_\epsilon^r(t)$ is continuous in time, it follows that $d_\epsilon^r(t) \leq m(r)$ for $0 \leq t \leq T$, if $\epsilon \leq \epsilon_0(T, r)$. Since $r > 0$ is arbitrary and $m(r) \to 0$ as $r \to 0$, we can conclude. □

Proposition 4.2. *u_1 and u_2 are respectively a subsolution and a supersolution of (P) with initial data u_0 and fixed boundary data f.*

Proof. Suppose ϕ touches u_1 from above at $P_0 = (x_0, t_0) \in \Gamma(u_1)$ with $|D\phi|(P_0) \neq 0$ and

$$\max(-\Delta\phi, \phi_t - r(q)|D\phi|)(P_0) = C(n)\gamma|D\phi|(P_0) > 0 \text{ for some } \gamma > 0,$$

where $q = -D\phi(x_0, t_0)$. Let

$$r = \frac{\phi_t}{|D\phi|}(x_0, t_0) \geq (1 + C(n)\gamma)r(q), \quad \nu = \frac{q}{|q|}.$$

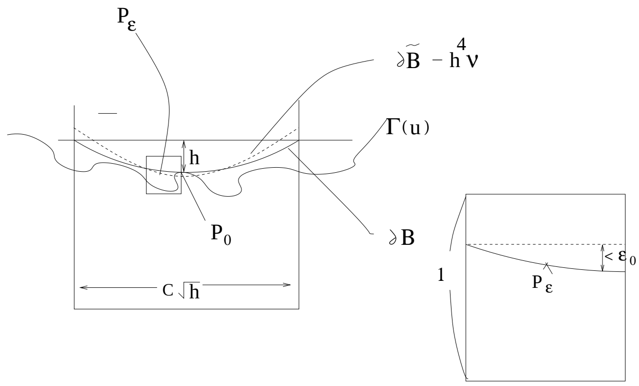

Figure 8: Zoom-up at the contact point of ϵ-solutions

Without loss of generality we may assume that the maximum is zero and strict: otherwise consider, with small $\delta > 0$,

$$\tilde{\phi}(x,t) := \phi(x,t) - \phi(x_0,t_0) + \delta(x-x_0)^4 + \delta(t-t_0)^2.$$

Since ϕ is smooth with $|D\phi|(P_0) \neq 0$, $\Omega(\phi)$ has an exterior ball B at P_0. Without loss of generality we assume the radius of B equals 1. Let us fix $0 < h < \epsilon_0^2$ and consider $\tilde{B}$: a translation of $(1-h)B$ which is inside of B and touches P_0 (see Figure 8.) Since, for small h,

$$S := (\tilde{B}_{\sqrt{h}} - \tilde{B}_{h/2})(P_0) \cap \{t_0 - h \le t \le t_0\} \cup (\tilde{B}_{\sqrt{h}}(P_0) \cap \{t = t_0 - h\}) \subsetneq B,$$

due to Lemma 4.1 $\Omega(u_\epsilon)$ lies strictly away from $\partial\tilde{B} - h^4\nu$ in S for sufficiently small $\epsilon > 0$.

On the other hand by definition of u_1 and by Lemma 4.1, for sufficiently small $0 < \epsilon < h^{1/2}\epsilon_0$ $\Gamma_t(u_\epsilon)$ contacts $(\partial\tilde{B} - h^4\nu)$ for the first time at $P_\epsilon = (x_\epsilon, t_\epsilon)$ in

$$\Sigma := B_{\sqrt{h}}(P_0) \times (t_0 - h, t_0].$$

Note that $u_\epsilon \le f$ in $\Sigma \cap \{t \le t_\epsilon\}$, where $f(\cdot, s)$ is the harmonic function in $(\Sigma - (\tilde{B} - h^4\nu)) \cap \{t = s\}$ with boundary data zero on $\partial(\tilde{B} - h^4\nu)$ and ϕ on the lateral boundary of Σ. Observe that, due to the regularity of ϕ,

$$q_\epsilon := -Df(P_\epsilon) = q + O(h^{1/2}), \quad r_\epsilon := \frac{f_t}{|Df|}(P_\epsilon) = r + O(h^{1/2}).$$

Now let

$$v_{\epsilon_0}(x,t) := \alpha^{-1} u_\epsilon(\alpha x + y_\epsilon, \alpha t + t_\epsilon), \quad \alpha = \epsilon/\epsilon_0,$$

where $|y_\epsilon - x_\epsilon| < 2\epsilon$, $y_\epsilon \in \epsilon\mathbb{Z}^n$. Then v_{ϵ_0} is a solution of $(P)_{\epsilon_0}$ in $B_1(0) \times [-1,1]$ with

$$\alpha^{-1}(x_\epsilon - y_\epsilon) \in \Gamma_0(v_{\epsilon_0}) \cap B_{2\epsilon_0}(0).$$

Moreover, since $h \leq \epsilon_0^2$, the tangent plane to $\tilde{B}$ at P_ϵ has its normal direction $\nu + O(\epsilon_0)$, and thus

$$v_{\epsilon_0}(x,t) \leq (1 + O(h^{1/2}))P_{q,r+O(h^{1/2})}(x - 2\epsilon_0\nu, t) \text{ in } B_1(0) \times [-1,0],$$

which contradicts Proposition 3.8 if ϵ_0 is sufficiently small. □

Corollary 4.3. *(a) If any subsequence of $\{u_\epsilon\}$ locally uniformly converges to u as $\epsilon \to 0$, then u is a viscosity solution of (P).*

(b) If there is a unique viscosity solution u of (P) for given initial positive domain Ω_0 and boundary data $f > 0$, then the whole sequence $\{u^\epsilon\}_\epsilon$ locally uniformly converges to u in space-time as $\epsilon \to 0$.

(c) In general u_1 and u_2 lies between maximal and minimal viscosity solutions of (P).

(d) For given sequence of smooth domains $\Omega_1 \subset\subset \Omega_2 ... \subset\subset \Omega_0$, there exists a sequence $\epsilon_k \to 0$ such that the viscosity solution u_{ϵ_k} of $(P)_{\epsilon_k}$ with initial domain Ω_k uniformly converges to the minimal solution of (P) with initial domain Ω_0.

(e) Corresponding statement holds for maximal solution of (P).

Proof. 1. (a) follows from Proposition 4.2.

2. Let us consider a sequence of smooth domains $\{\Omega_k\}$ such that

$$\Omega_1 \subset\subset \Omega_2 ... \subset\subset \Omega_0,$$

and a sequence of smooth domains $\{\tilde{\Omega}_k\}$ such that

$$\Omega_0 \subset\subset \tilde{\Omega}_1 \subset\subset \tilde{\Omega}_2 ...$$

To prove (b), let us define w_1 and w_2 of (P) by

$$w_1 = \lim_{k\to\infty} v_k, \quad w_2 = \lim_{k\to\infty} \tilde{v}_k,$$

where v_k and $\tilde{v}_k$ solve (P) with initial domain Ω_k and $\tilde{\Omega}_k$. Note that v_k and $\tilde{v}_k$ respectively increases and decreases in k. Arguing as in the proof of Proposition 4.2 it follows that $(w_1)_*$ and $(w_2)_*$ are respectively minimal and maximal viscosity solutions of (P). By Proposition 4.2 and Theorem 1.7

$$w_2 \leq u_2 \leq u_1 \leq w_1,$$

and thus if $w_1 = w_2$, then $u_1 = u_2$.

3. Note that for given k, one can choose ϵ_k such that

$$\tilde{v}_k - 1/k \leq u_{\epsilon_k} \leq v_k + 1/k$$

due to Proposition 4.2. Now we obtain (c) by sending $k \to \infty$. $\square$

Due to Theorem 1.8 (c) and (d), the following holds:

Corollary 4.4. *Let K be star-shaped with respect to the origin and let the fixed boundary data $f = 1$. Then the whole sequence $\{u^\epsilon\}$ locally uniformly converges to a unique viscosity solution u of (P) with initial data u_0 and if (a) Ω is star-shaped with respect to the origin or if (b) $|Du_0| > 2$ or $|Du_0| < 1$ on Γ_0. In the case of (a) $\Omega_t(u)$ is star-shaped for all times.*

References

[C1] L. Caffarelli, A Harnack inequality approach to the regularity of free boundaries, Part I: Lipschitz free boundaries are $C^{1,\alpha}$, *Rev. Mat. Iberoamericana***3** (1987), no. 2, 139-162

[C] L. A. Caffarelli, *A note on Nonlinear Homogenization,* Comm. pure. Appl. Math. (1999), 829-838.

[CIL] M. Crandall, H. Ishii and P. Lions, *User's guide to viscosity solutions of second order differential equations,* Bull. Amer. Math. Soc. **27** (1992), 1-67.

[CL] L. A. Caffarelli and K. Lee, *Homogenization of the oscillating free boundaries: the Elliptic Case,* submitted.

[CLM1] L. A. Caffarelli, K. Lee and A. Mellet, *Singular limit and homogenization for flame propagation in periodic excitable media,* Arch. Ration. Mech. Anal. **172** (2004), 153-190.

[CLM2] L. A. Caffarelli, K. Lee and A. Mellet, *Homogenization and flame propagation in periodic excitable media: the asymptotic speed of propagation,* Comm. Pure. Appl. Math **59**(2006), 501-525.

[CM1] L. A. Caffarelli and A. Mellet, *Capillary drops on an inhomogeneous surface,* M3AS, to appear.

[CM2] L. A. Caffarelli and A. Mellet, *Capillary drops on an inhomogeneous surface: Contact angle hysteresis,* Cal. Var. PDE, to apeear.

[CSW] L. A. Caffarelli, P. E. Souganidis and L. Wang, *Homogenization of fully nonlinear, uniformly elliptic and parabolic partial differential equations in stationary ergodic media,* Comm. Pure Appl. Math.**58** (2005), 319-361.

[DM] G. Dal Maso and L. Modica, *Nonlinear stochastic homogenization and ergodic theory,* J. Reine Angew. Math. **368** (1986), 28-42.

[E] L. C. Evans, *Periodic homogenisation of certain fully nonlinear partial differential equation,* Proc. Roy. Soc. Edinburgh Sect. A. **120** (1992), 245-265.

[G] K. S. Glasner, *Homogenization of contact line dynamics,* Int. Free boundaries, to appear.

[K1] I. C. Kim, *Uniqueness and Existence result of Hele-Shaw and Stefan problem,* Arch. Rat. Mech.Anal.**168** (2003), pp. 299-328.

[K2] I. C. Kim, *Regularity of the free boundary for the one phase Hele-Shaw problem,* J. Diff. Equations **223** (2006), 161-184.

[K3] I. C. Kim, *Homogenization of the free boundary velocity,* Arch. Rat. Mech. Anal, to appear.

[Ko] S. M. Kozlov, *The method of averaging and walk in inhomogeneous environments*, Russian Math. Surveys **40**(1985), 73-145.

[PV] Papanicolaou and Varadhan, *Boundary value problems with rapidly oscillating random coefficients,* Proceed. Colloq. on Random Fields, Rigours results in statistical mechanics and quantum field theory, J. Fritz, J.L. Lebaritz, D. Szasz (editors), Colloquia Mathematica Societ. Janos Boyai **10** (1979) 835-873.

[S] S. Spagnolo, *Sul limite delle soluzioni di problemi di Cauchy relativi all'equazione del calore(On the limit of solutions of Cauchy problems for the heat equation),* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **21** (1967), 657-699.