Viscosity Solutions for the two-phase Stefan Problem

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Abstract

We introduce a notion of viscosity solutions for the two-phase Stefan problem, which incorporates possible existence of a mushy region generated by the initial data. We show that a comparison principle holds between viscosity solutions, and investigate the coincidence of the viscosity solutions and the weak solutions defined via integration by parts. In particular, in the absence of initial mushy region, viscosity solution is the unique weak solution with the same boundary data.

1   Introduction

The classical two-phase Stefan problem models the evolution of temperature in a material with two distinct phases (solid and liquid). The solid-liquid phase transition occurs at a given constant temperature, which we set at zero. The problem is described by an *enthalpy* function h, which represents the internal energy density in each phases (see MEIRMANOV [15] and OLEINIK ET AL. [16]). More precisely, let $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$ with a C^2 boundary. Then $h(x,t) : \Omega \times [0,T) \to \mathbb{R}$ solves the following problem:

$$
\begin{cases}
h_t = \Delta \chi(h) & \text{in } \Omega \times (0,T), \\[2mm]
h(x,0) = h_0(x) & \text{in } \Omega, \\[2mm]
\chi(h) = \theta, & \text{on } \partial\Omega \times (0,T).
\end{cases} \tag{ST}
$$

In this paper we will set $\theta(x,t)$ as a continuous function on $\partial\Omega \times [0,T]$ and h_0 as a bounded measurable function with $\chi(h_0)$ continuous on $\overline{\Omega}$, such that $\chi(h_0) = \theta(\cdot, 0)$ on $\partial\Omega$. Here

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$u(x, t) := \chi(h)(x, t)$ describes the evolving temperature, where χ is defined by

$$\chi(h) = \begin{cases} h, & h > 0, \\ 0, & -1 \le h \le 0, \\ h + 1, & h < -1. \end{cases} \tag{1.1}$$

Note that h is determined by u almost everywhere if $\Gamma(u) := \{u = 0\} = \{-1 \le h \le 0\}$ is of measure zero, but not in general. The interior of zero temperature set $\Gamma(u)$, if exits, is called a *mushy region.* It is well known that mushy region might appear in a Stefan problem with heat sources, see BERTSCH ET AL. [4]. When there is no heat source, if $|\{-1 \le h_0 \le 0\}| = 0$ (here $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$), it is shown in [17] (see also [12] and [1]) that $|\{-1 \le h(\cdot, t) \le 0\}| = 0$ for all $t > 0$ and (ST) can be written in terms of the temperature:

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \{u > 0\} \cup \{u < 0\}, \\[2mm] V_n = |Du^+| - |Du^-| & \text{on} \quad \Gamma(u) = \partial\{u > 0\} = \partial\{u < 0\}, \\[2mm] u(x, 0) = u_0(x) := \chi(h_0(x)) \\[2mm] u = \theta & \text{on} \quad \partial\Omega \times (0, T) \end{cases} \tag{STu}$$

Here u^+ and u^- denotes respectively the limit taken from $\{u > 0\}$ and $\{u < 0\}$, and V_n is the outward normal velocity of the free boundary with respect to $\{u > 0\}$.

For both (ST) and (STu), a smooth initial free boundary may develop singularities in finite time, due to topological changes of the free boundary. Thus it is necessary to introduce a generalized notion of global-in-time solutions.

The first global notion of solutions for (ST) has been derived by a weak formulation via integration by parts: see LADYŽENSKAJA ET. AL. [14] or FRIEDMAN [11]. This approach allows the presence of initial mushy region. It was shown there that a unique weak solution exists for bounded measurable h_0 and θ. Moreover the temperature $u(x, t)$ turns out to be continuous when h_0 and θ are sufficiently smooth – see CAFFARELLI & EVANS [6] and DIBENEDETTO [9].

More recently, the notion of viscosity solutions, first introduced by CRANDALL & LIONS [8], has been proved useful in the qualitative study of free boundary problems, due to its flexibility with nonlinear PDEs and free boundary conditions: see e.g., [2]–[3] and [5] which address a general class of free boundary problems including (STu), and also see [10] where the heat operator is replaced by a class of uniformly parabolic operators. However, the equivalence between viscosity solutions and weak solutions, and in particular the uniqueness of the viscosity solutions of (ST), has not yet been investigated, even in the case of (STu). In this paper we address precisely this point. We introduce a notion of viscosity solutions for (ST), to derive the following theorem:

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Theorem 1.1. *(a) There exists a viscosity solution of* (ST) *in Q. Furthermore, a comparison principle holds between viscosity solutions and thus maximal and minimal viscosity solutions exists. They respectively corresponds to weak solutions of* (ST) *with maximal and minimal initial enthalpy for given temperature u_0.*

(b) Any weak solution of (ST) *in the sense of* [6] *is also a viscosity solution.*

(c) If $|\{-1 \leq h_0 \leq 0\}| = 0$, then there is a unique viscosity solution of (STu) *which coincides with the weak solution of* (ST)*, and also coincides with the notion of viscosity solutions discussed in* [8] *and* [2].

Remark 1.2. *It should be noted that our notion of solutions describes only the temperature evolution (see section 2), and thus requires $|\Gamma_t(u)| = 0$ for all $t > 0$ for the uniqueness result. The comparison principle and the hypothesis in (c) ensure this.*

Previously, one of the authors investigated in [13] viscosity solutions theory for the one-phase version of (STu). While one-phase Stefan problem is a special case of the two-phase Stefan problem, this work does not immediately apply to the two-phase version: we will point out the differences as we proceed in the proof. For example, the free boundary evolution for the two-phase setting is no longer monotone (that is, the positive set of u may shrink or expand over time), which requires a more careful construction of test functions. Moreover, since we are allowing the initial data to have mushy region, $\Gamma(u)$ may have positive measure and must be treated differently.

Below we give an outline of the paper:

In sections 2 and 3 we introduce a definition of viscosity solutions for (ST), and prove that viscosity solutions satisfy a comparison principle for strictly separated initial data. Note that this does not immediately yield uniqueness except under some geometric conditions on the initial data (see Corollary 3.9), however it generates a maximal and minimal viscosity solutions (see Proposition 4.4). These maximal and minimal solutions indeed correspond to the weak solutions with maximal and minimal initial enthalpy.

In section 4, we show that weak solutions are viscosity solutions. Using this result, existence of viscosity solutions is established. Furthermore, using the uniqueness of the weak solutions and barrier arguments near the parabolic boundary of Q (see Lemma 4.5), we establish the uniqueness of viscosity solutions when $|\{u_0 = 0\}| = 0$ and $\theta < 0$.

Remark 1.3. *The proof of the comparison principle (Theorem 3.1) applies to the generalized version of* (STu) *described in* [2]*, where the second equation in* (STu) *is replaced by a general free boundary velocity of the type*

$$V = G(x, |Du^+|, |Du^-|, \eta),$$

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where η is the spatial (outward) unit normal vector of $\Gamma(u)$, $G(x, a, b, \eta)$ is continuous and

$$\frac{\partial G}{\partial a}, -\frac{\partial G}{\partial b} \geq c > 0.$$

As for existence, a slight modification of Perron's method presented in [13] combined with Theorem 3.1 yields the existence of a viscosity solution for the generalized version. This approach is not taken here since we are also interested in the coincidence of weak and viscosity solutions.

2 The Stefan problem

Let $\partial_L Q = \partial\Omega \times [0, T]$ be the lateral boundary of Q and $\partial_P Q = (\Omega \times \{0\}) \cup \partial_L Q$ be the parabolic boundary of Q.

We shall use two notions of generalized solutions for (ST): weak solutions and viscosity solutions.

Definition 2.1. *A bounded measurable function h is called the* weak solution *of* (ST) *with initial data $h_0 \in L^\infty(\Omega)$ and boundary data $\theta \in L^\infty(\partial_L Q)$ if it satisfies (see [15])*

$$\int_Q (h\varphi_t + \chi(h)\Delta\varphi) \; dx \, dt - \int_{\partial_L Q} \theta \frac{\partial\varphi}{\partial\nu} \; ds \, dt + \int_\Omega h_0(x)\varphi(x, 0) \; dx = 0 \qquad (2.1)$$

for all $\varphi \in W_2^{2,1}(Q)$ vanishing on $\partial_L Q$ and at $t = T$.

Due to [11, 14, 15], there exists a unique weak solution h of problem (ST), defined in Definition 2.1, with initial data h_0 and boundary data θ. Furthermore a comparison principle holds between weak solutions ([15]): see the proof of Lemma 4.1 for more precise statement.

As mentioned in the introduction, the temperature $u(x, t)$ is related to the enthalpy h through $u = \chi(h)$. Formal computations yield that (ST) provides the free boundary velocity

$$
\begin{aligned}
V_n &= |Du^+| - |Du^-| &&\text{on } \partial\{u > 0\} \cap \partial\{u < 0\}; \\
V_n &\geq |Du^+| &&\text{on } \partial\{u > 0\} \setminus \partial\{u < 0\}; \qquad\text{(V)}\\
\tilde{V}_n &\leq -|Du^-| &&\text{on } \partial\{u < 0\} \setminus \partial\{u > 0\},
\end{aligned}
$$

where $\tilde{V}_n$ is the outward normal velocity of the free boundary of $\{u \geq 0\}$. The inequalities are due to the (possible) presence of the mushy region and the non-constant distribution of $h(x, t)$. Definitions 2.3 and 2.4 incorporate (V).

Definition 2.2. *A continuous function ϕ is a* classical subsolution *(in a parabolic neighborhood $\Omega_{x_0,t_0} := \{y : |y - x_0| < r\} \times (t_0 - \tau, t_0])$ of* (ST) *if*

(a) *ϕ is $C^{2,1}_{x,t}$ in $\overline{\{\phi > 0\}}$ and in $\overline{\{\phi < 0\}}$,*

(b) *$\Gamma(\phi) = \partial\{\phi > 0\} = \partial\{\phi < 0\}$,*

(c) *$|D\phi^+|$ and $|D\phi^-|$ are nonzero on $\Gamma(\phi)$ (this makes $\Gamma(\phi)$ smooth),*

(d) *$\phi_t - \Delta\phi \leq 0$ in $\{\phi \neq 0\}$,*

(e) *the outward normal velocity V_n of $\Gamma(\phi)$ with respect to $\{\phi > 0\}$ is less or equal to $|D\phi^+| - |D\phi^-|$.*

Similarly one can define a classical supersolution *in a parabolic neighborhood.*

Definition 2.3. *(a) A lower semi-continuous function v is a* viscosity supersolution *of* (ST) *in Q if (i) $v(x,0) \geq \chi(h_0)$, (ii) $v \geq \theta$ on $\partial_L Q$, and if (iii) for any classical subsolution $\phi(x,t)$ in $\Omega_{x_0,t_0} \subset Q$ such that $\phi \leq v$ on the parabolic boundary of Ω_{x_0,t_0}, $\phi \leq v$ in Ω_{x_0,t_0}.*

(b) An upper semi-continuous function u is a viscosity subsolution *of* (ST) *in Q if (i) $u(x,0) \leq \chi(h_0)$, (ii) $u \leq \theta$ on $\partial_L Q$, and if (iii) for any classical supersolution $\phi(x,t)$ in $\Omega_{x_0,t_0} \subset Q$ such that $\phi \geq u$ on the parabolic boundary of Ω_{x_0,t_0}, $\phi \geq u$ in Ω_{x_0,t_0}.*

Observe that if u is a viscosity subsolution of (ST), then $-u$ is a supersolution of (ST).

Definition 2.4. *A continuous function u is a* viscosity solution *of* (ST) *if it is both viscosity subsolution and supersolution.*

Remark 2.5. *Note that above definition of viscosity solutions, unlike that of weak solutions, does not take into account the enthalpy function $h(x,t)$. Such definition ensures that non-uniqueness occurs when the initial data u_0 satisfies $|\{u_0 = 0\}| > 0$. A modified definition, incorporating the presence of $h(x,t)$ and therefore yielding a unique representation of the interface evolution, is under investigation by the authors.*

3 Comparison for strictly separated initial data

A central property for the viscosity solution theory is the *comparison principle*, which is stated in the theorem below. We mention that a corresponding result holds for weak solutions of (ST) (see [11]). As mentioned in the introduction (see Remark 1.3), the proof presented in this section extends to a general class two-phase free boundary problems with nonlinear free boundary velocity.

Theorem 3.1. *Let u and v be respectively a viscosity subsolution and supersolution on Q such that $u < v$ on $\partial_P Q$. Then $u < v$ in Q.*

First we introduce several notations. Let $B_r(x,t)$ denote the closed space-time ball with radius r and center at (x,t),

$$B_r(x,t) = \{(y,s) : |(y,s) - (x,t)| \le r\},$$

and $D_r(x,t)$ the closed space disk with radius r and center at (x,t),

$$D_r(x,t) = \{(y,t) : |y - x| \le r\}.$$

We also speak of the sets

$$\partial D_r(x,t) = \{(y,t) : |y - x| = r\} \quad \text{and} \quad D_r^o(x,t) = \{(y,t) : |y - x| < r\}$$

as the boundary and the interior of $D_r(x,t)$, respectively.

For any $r > 0$, define the sup- and inf-convolutions (see [13])

$$Z(x,t) = \sup_{B_r(x,t)} U(y,s), \qquad U(x,t) = \sup_{D_r(x,t)} u(y,s), \qquad (3.1)$$

$$W(x,t) = \inf_{B_r(x,t)} V(y,s), \qquad V(x,t) = \inf_{D_r(x,t)} v(y,s).$$

Note that Z, U, resp. W, V are also viscosity subsolutions, resp. viscosity supersolutions. For any $\rho > 0$, define

$$\Omega_\rho = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \rho\},$$

The convolutions (3.1) are well defined in $\overline{C_r}$,

$$C_r = \Omega_{2r} \times (r, T - r).$$

Moreover, we fix $r > 0$ small enough so that $Z < W$ on $\partial_P C_r$.

It will be convenient to define $\Phi_r(x,t)$ to be the closed set

$$\Phi_r(x,t) = \bigcup_{(y,s) \in B_r(x,t)} D_r(y,t).$$

Observe that, by definition, the sets $\{Z \geq 0\}$ and $\{W \leq 0\}$ have interior balls of radius r at each point of their boundaries. These balls are centered on the boundaries of $\{U \geq 0\}$ and $\{V \leq 0\}$, respectively. This property of Z and W will be essential in the proofs in this section. It motivates the following definition:

Definition 3.2. *Let E be a closed set, let P be a point on the boundary of E, $P \in \partial E$, and let B be a closed ball containing P. The ball B will be called an interior ball of E at P if $B \subset E$, and it will be called an exterior ball of E at P if $B^o \cap E = \emptyset$.*

To prove that u and v stay ordered, we will argue by contradiction: if $u \geq v$ at some time $\tilde{t}_0$, it must be true for Z and W even earlier. Suppose that there is a time t_0 when the functions Z and W first hit,

$$t_0 = \sup \{ s \geq r : Z(\cdot, s) < W(\cdot, s) \}.$$

Denote $P_0 = (x_0, t_0)$ the point of maximum of $Z - W$ at $t = t_0$. Note that $Z(x_0, t_0) \geq W(x_0, t_0)$. If that were not the case, we would have $Z - W < 0$ at $t = t_0$ and $(Z - W)(x_k, t_k) \geq 0$ for some sequence (x_k, t_k), $x_k \in \Omega$ with $t_k \to t_0^+$. Then there is a converging subsequence $(x_{k_j}, t_{k_j}) \to (x, t_0)$. Set $\varepsilon = -(Z - W)(x, t_0)/2 > 0$. By upper semi continuity there is j_0 such that for all $j \geq j_0$

$$0 \leq (Z - W)(x_{k_j}, t_{k_j}) < (Z - W)(x, t_0) + \varepsilon \leq -\varepsilon,$$

a contradiction.

Note that the positive set of Z (and the negative set of W) may expand discontinuously in time, in the event that the zero set of Z (and W) is of positive measure: this is because our definition only considers test functions which are nondegenerate in both positive and negative phases. Therefore one should put extra care in describing the location of P_0 and the geometry of the free boundaries nearby.

Lemma 3.3. *The set $\{u \geq 0\}$ cannot expand discontinuously, i.e.*

$$\{u \geq 0, \ t = T\} \subset \overline{\{u \geq 0, \ t < T\}} \qquad \forall T > 0.$$

Analogous results hold for $\{v \leq 0\}$, $\{Z \geq 0\}$, $\{U \geq 0\}$, $\{W \leq 0\}$ and $\{V \leq 0\}$.

Proof. Suppose that for some $T > 0$ there is a point $P \in \{u \geq 0, \ t = T\}$ such that $P \notin \overline{\{u \geq 0, \ t < T\}}$. In other words, there exists $\delta > 0$ such that $B_\delta(P) \cap \{u \geq 0, \ t < T\} = \emptyset$. For simplicity we will translate the coordinates so that $P = (0, 0)$. Define $M = \max_Q u$. Note that the ball $B_\delta(P)$ contains a cylinder

$$C = \left\{ (x, t) : |x| \leq h + \eta, \ -\frac{\eta}{2} \min\left(\frac{\eta}{2M}, \frac{h}{n} \right) \leq t \leq 0 \right\}$$
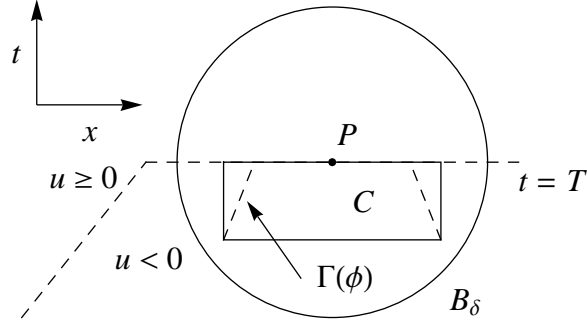
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Figure 1: Situation at point P

with h and η small positive constants. The situation is shown in Figure 1. We will construct a classical supersolution of (ST) in C that crosses u from above to obtain a contradiction.

Define $\omega = 2\max\left(\frac{2M}{\eta}, \frac{n}{h}\right)$. Consider a radially symmetric function in C defined by the formula

$$\phi(x,t) = \begin{cases} a(|x|^2 - (h-\omega t)^2) & \text{for } |x| \le h - \omega t, \\ \\ \frac{2M}{\eta}\left(|x| - h + \omega t\right) & \text{for } |x| > h - \omega t, \end{cases}$$

where $a > 0$ is a small constant specified below.

Then ϕ satisfies

$$\begin{aligned} (\partial_t - \Delta)\,\phi &= & \frac{2M}{\eta}\left(\omega - \frac{n-1}{|x|}\right) > 0 & \quad \text{for } |x| > h - \omega t \\ & \text{and} & & \\ &= & 2a\left((h-\omega t)\omega - n\right) > 0 & \quad \text{for } |x| < h - \omega t. \end{aligned}$$

Next, note that the free boundary $\Gamma(\phi)$ at time t is the sphere with radius $h - \omega t$, shrinking with normal velocity ω. Therefore we have

$$V_n = \omega > \frac{2M}{\eta} - \left|D\phi^-\right| = \left|D\phi^+\right| - \left|D\phi^-\right|.$$

Hence ϕ is a classical supersolution of (ST) for any $a > 0$.

Lastly, note that the initial boundary $D = C \cap \{t = -\eta/\omega\}$ of the cylinder C is a closed subset of $\{u < 0\}$ and therefore $\max_D u < 0$. We can hence take $a > 0$ small enough to make $\phi > u$ on D.

Thus we have constructed a classical supersolution ϕ on C such that $\phi > u$ on the parabolic boundary of C, while $\phi < 0 \le u$ at $P = (0,0)$. That shows that u crosses ϕ from below inside C, which contradicts the definition of u.

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The conclusion for $\{v \leq 0\} = \{-v \geq 0\}$ can be obtained by repeating the proof for $-v$. Corresponding arguments apply to the sets $\{Z \geq 0\}$, $\{U \geq 0\}$, $\{W \leq 0\}$ and $\{V \leq 0\}$ as well. $\qquad \square$

Lemma 3.4. *The point P_0 lies on the intersection of boundaries of $\{Z \geq 0\}$ and $\{W \leq 0\}$,*

$$P_0 \in \partial\{Z \geq 0\} \cap \partial\{W \leq 0\},$$

and

$$W(P_0) \leq 0 \leq Z(P_0).$$

Proof. We know that $W(P_0) \leq Z(P_0)$ from the definition of P_0. There are three cases:

(a) $0 < W(P_0) \leq Z(P_0)$: in this case $W > 0$ in some neighborhood of P_0 ($\{W > 0\}$ is open as W is a lower-semicontinuous function) and W is a supercaloric function in this neighborhood; this contradicts the fact that Z cannot be crossed from above by any positive caloric function ϕ.

(b) $W(P_0) \leq Z(P_0) < 0$: the same as above for Z subcaloric in a neighborhood of P_0.

Therefore we arrive at

$$W(P_0) \leq 0 \leq Z(P_0). \tag{3.2}$$

Inequality (3.2) immediately yields that $P_0 \in \{Z \geq 0\} \cap \{W \leq 0\}$. Now suppose that P_0 lies in the interior of the set $\{Z \geq 0\}$. That implies $P_0 \notin \overline{\{W \leq 0, \ t < t_0\}}$ since $Z < W$ for $t < t_0$. But that is a contradiction with continuous expansion of $\{W \leq 0\}$, Lemma 3.3. The same happens when P_0 lies in the interior of $\{W \leq 0\}$. Hence the conclusion of the lemma. $\qquad \square$

The lemma states that the situation at point P_0 looks like the one shown in Figure 2. More precisely, there exist $P_1 = (x_1, t_1) \in \partial\{U \geq 0\}$ and $P_2 = (x_2, t_2) \in \partial\{V \leq 0\}$ such that $U(P_1) = Z(P_0)$ and $V(P_2) = W(P_0)$. At P_0, $B_1 = B_r(x_1, t_1) = B_r(P_1)$ is an interior ball of $\{Z \geq 0\}$ and $B_2 = B_r(x_2, t_2) = B_r(P_2)$ is an interior ball of $\{W \leq 0\}$. Moreover, the ball $B_0 = B_r(x_0, t_0) = B_r(P_0)$ is an exterior ball of $\{U \geq 0\}$ at P_1 and $\{V \leq 0\}$ at P_2.

Since $Z < W$ for $t < t_0$, the space projections of B_1 and B_2 are tangent:

$$B_1 \cap B_2 \cap \{t = t_0\} = P_0.$$

Therefore the space projections of P_0, P_1 and P_2 are on a line and we can change the coordinates so that $P_0 = (0, 0)$, $P_1 = (d_1 e_1, t_1 - t_0)$ and $P_2 = (-d_2 e_1, t_2 - t_0)$.
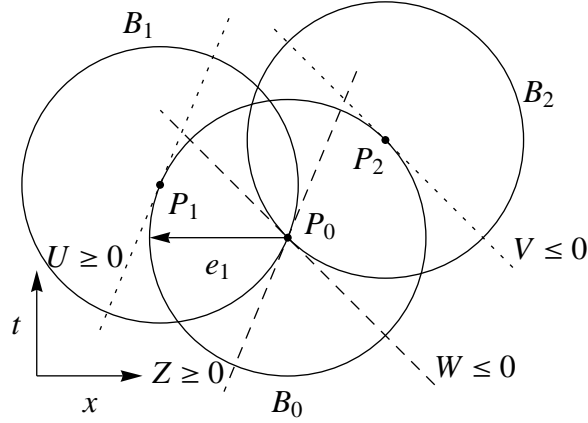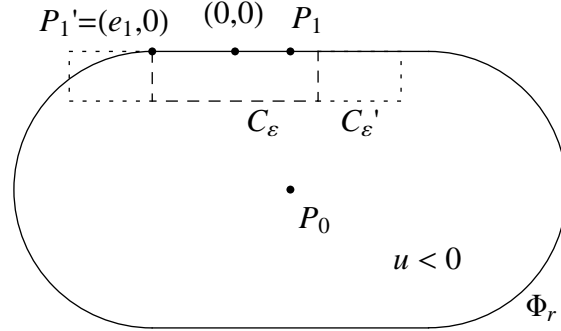
Figure 2: Situation at P_0

Figure 3: Construction of a supersolution for the proof of non-verticality of (P_1, P_0)

Lemma 3.5. *Suppose that $Z(P_0) = 0$. Then the line (P_0, P_1) cannot be vertical (i.e., be parallel to the time-direction) and in the future of P_0. In other words, when $t_1 > t_0$ the space projections of P_1 cannot be equal to the space projection of P_0 or equivalently, $t_1 \neq t_0 + r$. The same holds for (P_0, P_2) when $W(P_0) = 0$.*

Proof. Suppose that (P_1, P_0) is vertical, i.e. $x_1 = x_0$, and $t_1 = t_0 + r$. Thus there is $P_1' \in D_r(P_1)$ such that $u(P_1') = 0$. Lemma 3.4 implies that P_0 lies on the boundary of $\{Z \geq 0\}$. Therefore $u < 0$ in the interior of $\Phi_r(P_0)$. This and Lemma 3.3 yield that $u < 0$ in the interior of $D_r(P_1)$ and P_1' lies on the boundary of $D_r(P_1)$. In particular, for every small $\varepsilon > 0$ we can find a cylinder C_ε with radius ε, top at t_1, height ε^2 and $C_\varepsilon \cap \{u \geq 0\} = P_1'$. Now we use the parabolic scaling $(x', t') \rightarrow (\varepsilon x', \varepsilon^2 t')$, $\varepsilon > 0$, together with a translation and a rotation so that $(0, 0)$ is in the center of the top of C_ε, C_ε is mapped onto $\{|x'| \leq 1, \ -1 \leq t' \leq 0\}$ and $P_1' = (e_1, 0)$. The situation is shown in Figure 3. Let C_ε' be the cylinder with the same axis and height with C_ε, but twice its radius. Since u is a
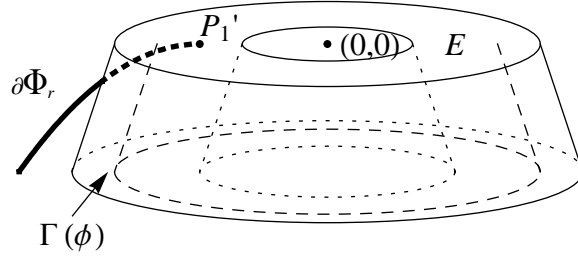
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Figure 4: Local situation at P_1'

upper-semicontinuous function with $u(P_1') = 0$, we can for any $\delta > 0$ find $\varepsilon > 0$ so that $\max_{C_\varepsilon'} u < \delta$.

For simplicity we will drop the primes in (x', t'). Straightforward computation gives a formula for the distance of the lateral boundary of Φ_r from the time-axis, $x = 0$,

$$|x| = \sqrt{-2rt - \varepsilon^2 t^2} + 1, \qquad -1 \le t \le 0.$$

For small $\varepsilon > 0$ and $-1 \le t \le 0$, we have $|x| > \sqrt{-rt} + 1$. Let $\omega = \sqrt{r}/3$ and $h = 1 + \sqrt{r}/3$.

Below we construct a classical supersolution of problem (ST) in a parabolic neighborhood E of P_1', where

$$E := \left\{ (x, t) : -1 \le t \le 0,\ 1 - \sqrt{r}/3 - \omega t \le |x| \le 1 + 2\sqrt{r}/3 - \omega t \right\},$$

(see Figure 4). Note that $E \subset C_\varepsilon'$ for small $r > 0$.

We only show the construction for $n \ge 3$. The computation is similar when $n = 2$. Thus consider a radially symmetric function $\phi(x, t)$ on E of the form

$$\phi(x, t) = \begin{cases} a(-|x|^{2-n} + (h - \omega t)^{2-n}) & \text{when } |x| \ge h - \omega t, \\ b(-|x|^{2-n} + (h - \omega t)^{2-n}) & \text{when } |x| < h - \omega t, \end{cases}$$

where $a = \frac{\omega}{2n-4} h^{n-1}$ and $b > 0$ is a small constant to be chosen later.

ϕ then satisfies

$$(\partial_t - \Delta)\phi = \partial_t \phi = c(n-2)\omega(h - \omega t)^{1-n} > 0, \qquad |x| \ne h - \omega t,$$

where $c = a$ or b. The free boundary of ϕ is the shrinking sphere $\Gamma(\phi) = \{|x| = h - \omega t\}$ and therefore the free boundary condition reads as

$$V_n = \omega > a(n-2)(h - \omega t)^{1-n} - \left| D\phi^- \right| = \left| D\phi^+ \right| - \left| D\phi^- \right|.$$

Now define $\delta = \phi(x, t)$ for any point on the exterior lateral boundary of E (where $|x| = h - \omega t + \sqrt{r}/3$). Next choose $\varepsilon > 0$ small enough so that $u < \delta$ on the exterior lateral

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boundary of E, as was noted above. Finally we finish the construction by choosing $b > 0$ small enough so that $\phi(x,t) > u(x,t)$ on the part of parabolic boundary of E where $\phi \leq 0$. This can be done as this is a closed subset of $\{u < 0\}$ and u is upper semi-continuous.

With above choices of b and ε, ϕ is a classical supersolution of (ST) such that $\phi > u$ on the parabolic boundary of E and $\phi(P_1') < 0 = u(P_1')$. Therefore $u - \phi$ has a maximum inside E, which yields a contradiction with u being a viscosity subsolution.

The proof can be repeated for $-W$ in place of Z and other appropriate substitutions. $\qquad \square$

With the previous results, it is now possible to establish the following lemma:

Lemma 3.6.
$$Z(P_0) = W(P_0) = 0.$$

Proof. It is enough to prove $Z(P_0) = 0$ because $W(P_0) = 0$ then follows by repeating the proof for $-W$.

Due to Lemma 3.4, $Z(P_0) \geq 0$. Suppose $Z(P_0) = L > 0$. As in the proof of Lemma 3.5, there is a point $P_1' \in D_r(P_1)$ such that $u(P_1') = L$. First we discuss a couple of extreme cases:

(a) If (P_1, P_0) is vertical with P_1 in the past of P_0, arguing as in the proof of Corollary 3.7 we see that (P_2, P_0) must be vertical with P_2 in the future. So either $W(P_0) = 0$, but that cannot happen by Lemma 3.5, or $W(P_0) < 0$. For $W(P_0) < 0$, one could argue as in (b) with $-W$ in the place of Z (and other appropriate substitutions).

(b) If (P_1, P_0) is vertical with P_1 in the future, the fact that $P_0 \in \partial\{Z \geq 0\}$ implies $u < 0$ in the interior of $\Phi_r(P_0)$, therefore $u < 0$ in the interior of $D_r(P_1)$ by Lemma 3.3 and P_1' must be on the boundary of $D_r(P_1)$. The rest of the argument is parallel to the one shown below.

Now suppose that (P_0, P_1) is not vertical and $t_1 \leq t_0$. This allows us to find a small $\tau > 0$ and points $Q = (\tilde{x}, t_1)$ and $Q_\tau = (\tilde{x}, t_1 - \tau)$ such that $D_h(Q) \cap \{u \geq 0\} = P_1$ and $D_{h_\tau}(Q_\tau) \cap \{u \geq 0\} = \emptyset$, where $h = |P_1 - Q|$, $0 < h - h_\tau \ll 1$ (see Figure 5). For simplicity we translate coordinates so that $Q = (0,0)$.

Denote $M = \max_Q u$. We will now construct a positive classical supersolution ϕ of (ST) in
$$S = \{(x,t) : -\tau < t < 0, \ h + \omega t - \delta \leq |x| \leq h + \omega t + \eta\},$$

where $\eta > 0$ and $\delta > 0$ small are to be chosen and
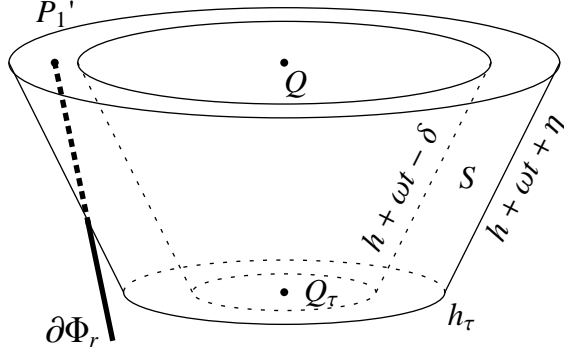$$\omega = \frac{h - h_\tau + \eta}{\tau},$$

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Figure 5: Construction of a supersolution when $Z(P_0) > 0$.

so that ϕ will cross u at P_1, yielding a contradiction.

Consider a spatially radially symmetric function

$$\phi(x,t) = f(|x| - \omega t), \qquad f(s) = -\frac{2c}{3K}\left(-Ks + a\right)^{3/2} + b,$$

where

$$K = 2\left(\frac{h - h_\tau/2}{\tau} + \frac{2n - 2}{h_\tau}\right) > 0$$

and the constants $a, b \in \mathbb{R}$ and $c > 0$ are to be chosen below.

Observe that $g(s) = (f'(s)/c)^2 = -Ks + a$. Choose $0 < \eta < \frac{1}{2}h_\tau$ small enough and $a \in \mathbb{R}$ so that $\frac{1}{2} < g(s) < 1$ for all $h - \eta \le s \le h + \eta$. It is also clear that we can find $c > 0$ and $b \in \mathbb{R}$ such that $0 < f(h) < L$ and $f(h + \eta) > M$. f is continuous and so there is also δ, $0 < \delta < \eta$ such that $f(h - \delta) > 0$.

In particular, with this choice of η, δ, a, b and c, $f(s)$ is positive and strictly increasing in $[h - \delta, h + \eta]$. It follows that ϕ satisfies

$$
\begin{aligned}
(\partial_t - \Delta)\phi(x,t) &= -\left[\left(\omega + \frac{n-1}{|x|}\right)f'(|x| - \omega t) + f''(|x| - \omega t)\right] \\
&= -\frac{1}{f'}\left[\left(\omega + \frac{n-1}{|x|}\right)(f')^2 + f''f'\right] \\
&= -\frac{c^2}{f'}\left[\left(\omega + \frac{n-1}{|x|}\right)g + \frac{1}{2}g'\right] \\
&\ge 0
\end{aligned}
$$

since $\omega + \frac{n-1}{|x|} \le \frac{K}{2}$ in S. Therefore ϕ is positive and strictly supercaloric in S.

To conclude, we note that $\phi > u$ on the parabolic boundary of S. This is clear from the fact that $\phi > 0$ in S and $\phi > M$ on those parts of the parabolic boundary of S that intersect

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$\{u \geq 0\}$. But $(u - \phi)(P_1) > 0$ and therefore u crosses ϕ from below inside S. That is a contradiction with u being a viscosity subsolution of (ST).

If $t_1 > t_0$, including the case when (P_1, P_0) is vertical, the construction of a supersolution is more straightforward but we can also use the above construction with $h_\tau = h$. $\square$

Corollary 3.7. *The vectors $P_0 - P_1$ and $P_0 - P_2$ cannot be vertical (i.e., be parallel to the time-direction). In other words, the space projections of P_1 and P_2 cannot be equal to the space projection of P_0.*

Proof. Observe that if say $t_1 < t_0$ and the line (P_1, P_0) is vertical, then also (P_2, P_0) must be vertical and $t_2 > t_0$. This follows from the strict ordering $Z(\cdot, t) < W(\cdot, t)$ for all $t < t_0$ and B_1 being an interior ball of $\{Z \geq 0\}$ and B_2 being an interior ball of $\{W \leq 0\}$, i.e. $Z \geq 0$ in B_1 and $W \leq 0$ in B_2 which implies $B_1 \cap B_2 \cap \{t < t_0\} = \emptyset$. That is, $B_1 \cap B_2 = P_0$ whenever one of the balls is contained in $\{t \leq t_0\}$. But Lemma 3.5 asserts that (P_2, P_0) cannot be vertical with P_2 in the future of P_0. Similarly when (P_2, P_0) is vertical with $t_2 < t_0$. $\square$

Due to Corollary 3.7, the interior ball B_1 of $\{Z \geq 0\}$ has an interior normal (e_1, m) at $P_0 \in \partial B_1$ with $|m| < \infty$. Note that m denotes the (outward) normal velocity of the space-time ball B_1 at P_0. Therefore, formally speaking, m is smaller than the normal velocity V_n of $\{Z \geq 0\}$ at P_0. Since Z is a subsolution of (ST) and since $P_0 \in \partial\{Z \geq 0\} = \partial\{Z < 0\}$, from the discussion in section 2 (see (V)) formally it must be true that

$$m \leq |DZ^+| - |DZ^-|.$$

Next lemma states that this is indeed true in a weak sense.

Lemma 3.8.
$$\liminf_{\lambda \to 0+} \frac{Z(P_0 + \lambda e_1, 0)}{\lambda} + \liminf_{\mu \to 0+} \frac{Z(P_0 - \mu e_1, 0)}{\mu} \geq m.$$

Proof. Let us define

$$a := \liminf_{\lambda \to 0+} \frac{Z(P_0 + \lambda e_1, 0)}{\lambda} \quad \text{and} \quad b := \liminf_{\mu \to 0+} \frac{Z(P_0 - \mu e_1, 0)}{\mu}.$$

Observe that $b < 0$. Indeed by Corollary 3.7, we can find a smaller closed ball $B \subset B_2$ such that $B \cap \partial B_2 = P_0$ and $h > 0$ such that $B \cap \{t = t_0 - h\} \neq \emptyset$. Since $Z < W$ for $t < t_0$, there is a smooth function $g \geq 0$ that solves the heat equation inside $B \cap \{t_0 - h \leq t \leq t_0\}$ such that $Z + g < W$ on $B \cap \{t = t_0 - h\}$, $g > 0$ in $B^o \cap \{t = t_0 - h\}$ and $g = 0$ on $\partial B \cap \{t_0 - h \leq t \leq t_0\}$. As Z is subcaloric and W is supercaloric in $B \cap \{t_0 - h \leq t \leq t_0\}$ it follows that $Z + g \leq W$ on $B \cap \{t = t_0\}$ and

$$0 \geq \liminf_{\mu \to 0+} \frac{W(P_0 - \mu e_1, 0)}{\mu} \geq b + |Dg(P_0)| > b.$$
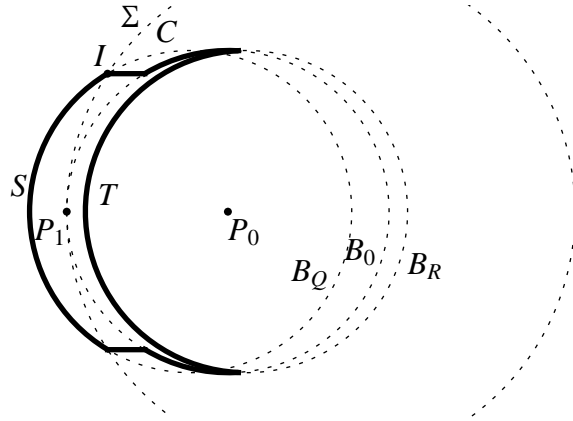
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Figure 6: The slice $\{t = t_1\}$

Suppose $a + b < m$. Since $b < 0$, $\varepsilon > 0$ can be chosen sufficiently small so that

$$a + b < m - 6\varepsilon \quad \text{and} \quad b + 3\varepsilon < 0.$$

We will construct a supersolution of (ST) constructed in a parabolic neighborhood K of P_1, which will be specified later, which crosses U from above, to yield a contradiction. We point out that, due to the two-phase nature of the problem, the construction of such barrier and the domain K requires modifications from the one-phase version constructed in [7] and [13].

From the definitions of a and b, there are sequences $\lambda_k \to 0$, $\mu_k \to 0$ with $\lambda_k > 0$ and $\mu_k > 0$ such that

$$Z(x_0 + \lambda_k e_1, t_0) \leq (a + \varepsilon)\lambda_k,$$
$$Z(x_0 - \mu_k e_1, t_0) \leq (b + \varepsilon)\mu_k.$$

In what follows, $\lambda \in \{\lambda_k\}_{k=1}^{\infty}$ and $\mu \in \{\mu_k\}_{k=1}^{\infty}$. Now denote $Q = P_0 + (\lambda e_1, 0)$, $R = P_0 + (-\mu e_1, 0)$ and B_Q, resp. B_R the balls centered at Q, resp. R, with radius r. The situation is illustrated in Figure 6.

From the definition of Z, it follows that $U(x, t) \leq Z(y, s)$ for any $(y, s) \in B_r(x, t)$ and thus

$$U \leq \begin{cases} Z(Q) \leq (a + \varepsilon)\lambda & \text{in } B_Q, \\ Z(P_0) = 0 & \text{in } B_0, \\ Z(R) \leq (b + \varepsilon)\mu & \text{in } B_R. \end{cases}$$

In fact, $U \leq 0$ on a slightly larger set. Because $Z \leq W$ at $t = t_0$ and $W \leq V(P_2) = 0$ in B_2, we can conclude that $Z \leq 0$ on $D = B_2 \cap \{t = t_0\}$. It means that $U \leq 0$ on Σ, the union of
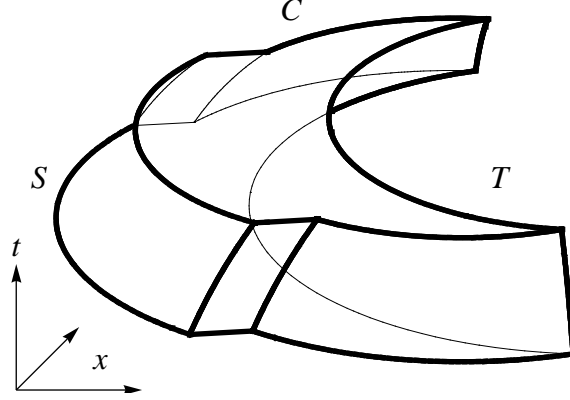
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Figure 7: Domain K for $n = 2$

balls of radius r with a center in D,

$$\Sigma = \bigcup_{(x,t)\in D} B_r(x,t).$$

Finally, we have an improved bound,

$$U \leq \begin{cases} (a+\varepsilon)\lambda & \text{in } B_Q, \\ 0 & \text{in } \Sigma, \\ (b+\varepsilon)\mu & \text{in } B_R. \end{cases} \tag{3.3}$$

Now we will compare U with a supersolution in a domain $K := K_\tau^{\lambda,\mu}$. Fix $\tau > 0$ small. $K_\tau^{\lambda,\mu}$ is the subset of $\{t_1 - \tau \leq t \leq t_1\}$ with the shape depicted in Figure 7 for $n = 2$. The shape for each t is depicted as the thick curve in Figure 6. The boundary of the slice is given by the arc $S = \partial B_Q \setminus \Sigma$, the arc $T = \partial B_R \cap B_0$, and finally of the curve (surface) C that connects S and T. C at any given time is composed of $C_1 = \partial L \cap \Sigma \setminus B_R$, where L is a cylinder with axis e_1 connecting $I = \partial B_Q \cap \partial \Sigma$ with ∂B_0, and $C_2 = \partial B_0 \setminus (B_R \cup L)$.

Let $\phi(x,t) = \phi(\rho,t)$, where $\rho = |x - x_0|$. We need $U \leq \phi$ on the parabolic boundary of $K_\tau^{\lambda,\mu}$. By Taylor expansion in λ and $(t - t_1)$, it follows that the spatial distance of the point of intersection I from x_0 can be found explicitly for each time t as

$$R_I(t) = d_1 + \lambda - m(t - t_1) + o(\lambda) + o(t - t_1) \leq d_1 + \lambda - m(t - t_1).$$

and $R_T(t) \leq |x - x_0| \leq R_S(t)$ in K, where

$$R_S(t) = d_1 + \lambda - m(t - t_1) + o(t - t_1) \text{ and } R_T(t) = d_1 - \mu - m(t - t_1) + o(t - t_1).$$

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Lastly, the radius of the space-ball $B_0 \cap \{t = t\}$ is given by

$$R_1(t) = d_1 - m(t - t_1) + o(t - t_1).$$

Now we are ready for the construction of the supersolution. Take $\alpha = a + 3\varepsilon$, $\beta = b + 3\varepsilon$, $\tilde{R} = d_1 = R_1(t_1)$ and $\tilde{m} = \alpha + \beta < m$. Let $\phi(x, t)$ be the function $\phi^{\alpha,\beta}_{\tilde{R},\tilde{m}}(x - x_0, t - t_0)$ from Appendix. Let $\Sigma_{\tau,\delta}$ be the set from Appendix where ϕ exists and has the required properties.

Let $\tilde{R}(t) = \tilde{R} - \tilde{m}(t - t_1)$ be the radius of the free boundary of ϕ. We claim that for sufficiently small λ, μ and τ we have $U \leq \phi$ on the parabolic boundary of $K^{\lambda,\mu}_\tau$.

To prove the claim, first note that for sufficiently small $\lambda, \mu > 0$

$$K^{\lambda,\mu}_\tau \subset \Sigma_{\tau,\delta} = \left\{ (x, t) : \tilde{R}(t) - \delta \leq |x - x_0| \leq \tilde{R}(t) + \delta, \ t_1 - \tau \leq t \leq t_1 \right\}.$$

We will compare ϕ with U on two parts of the parabolic boundary.

Part 1: On $\partial(K^{\lambda,\mu}_\tau \cap \{t_1 - \tau \leq t \leq t_1\})$:

From (3.3) we need

$$\phi(\rho, t) \geq \begin{cases} (a + \varepsilon)\lambda & \text{for } R_I(t) \leq \rho \leq R_S(t), \\ 0 & \text{for } R_1(t) \leq \rho \leq R_I(t), \\ (b + \varepsilon)\mu & \text{for } R_T(t) \leq \rho \leq R_1(t). \end{cases} \tag{3.4}$$

The second condition is satisfied as long as $\tilde{R}(t) \leq R_1(t)$ and that is true for all $t \in [t_1 - \tau, t_1]$ if τ is small enough. Third condition is trivial for $\tilde{R}(t) \leq \rho \leq R_1(t)$. For $R_T(t) \leq \rho \leq \tilde{R}(t)$ we use the fact that

$$\phi(\rho, t) \geq (\beta - \varepsilon) \left(\tilde{R}(t) - \rho \right)$$
$$\geq (\beta - \varepsilon) \left(\tilde{R}(t) - R_T(t) \right)$$
$$\geq (\beta - \varepsilon)\mu > (b + \varepsilon)\mu.$$

Lastly the first condition in (3.4) holds since

$$\phi(\rho, t) \geq (\alpha - \varepsilon) \left(\rho - \tilde{R}(t) \right) \geq (\alpha - \varepsilon) \left(R_I(t) - \tilde{R}(t) \right)$$
$$\geq (\alpha - \varepsilon)(d_1 - m(t - t_1) + \lambda - \eta - d_1 + \tilde{m}(t - t_1))$$
$$= (\alpha - \varepsilon) \left((\tilde{m} - m)(t - t_1) + \lambda - \eta \right)$$
$$\geq (\alpha - \varepsilon)(\lambda - \eta) > (a + \varepsilon)\lambda$$

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if τ and λ are small compared to ε since $\eta = o(\lambda) + o(t - t_1)$.

Part 2: On $K_\tau^{\lambda,\mu} \cap \{t = t_1 - \tau\}$:

From the bounds on U in (3.3) we need

$$\phi(\rho, t) \geq \begin{cases} (a + \varepsilon)\lambda & \text{for } R_1(t_1 - \tau) \leq \rho \leq R_S(t_1 - \tau), \\ 0 & \text{for } R_T(t_1 - \tau) \leq \rho \leq R_1(t_1 - \tau). \end{cases}$$

The second inequality can be satisfied by taking τ and μ small so that $\tilde{R}(t_1 - \tau) \leq R_T(t_1 - \tau)$. The first inequality is then satisfied by taking λ small.

We demonstrated that $U \leq \phi$ on the parabolic boundary of $K_\tau^{\lambda,\mu}$ and $U \geq \phi$ at P_1 if λ, μ and $\tau > 0$ are small enough. Since ϕ is in fact a strict classical supersolution due to Appendix, we can perturb ϕ into $\tilde{\phi}$ near P_1 such that $\tilde{\phi}$ is still a classical supersolution with $U \leq \tilde{\phi}$ on the parabolic boundary of $K_\tau^{\lambda,\mu}$ and $U > \tilde{\phi}$ at P_1. This is a contradiction with the definition of U.

$\square$

We recall that $Z \leq W$ at $t = t_0$. Therefore Lemma 3.8 as well as considering g in the beginning of its proof yields

$$\limsup_{\lambda \to 0+} \frac{W(P_0 + \lambda e_1, 0)}{\lambda} + \limsup_{\mu \to 0+} \frac{W(P_0 - \mu e_1, 0)}{\mu} \tag{3.5}$$
$$\geq \liminf_{\lambda \to 0+} \frac{Z(P_0 + \lambda e_1, 0)}{\lambda} + \liminf_{\mu \to 0+} \frac{Z(P_0 - \mu e_1, 0)}{\mu} + |Dg(P_0)|$$
$$> m.$$

Recall that strict ordering $Z < W$ for $t < t_0$ implies that B_2 has an exterior normal $(e_1, \hat{m})$ with $\hat{m} \leq m$. With this result in mind, we can repeat the argument in Lemma 3.8 to get a contradiction by proving that the quantity (3.5) must be less or equal than m.

This establishes the desired contradiction and we conclude that Z and W stay strictly separated for all times.

Note that Theorem 3.1 yields ordering only between strictly separated initial data. Therefore it does not directly yield the uniqueness result for viscosity solutions, even with the additional assumption $|\{u_0 = 0\}| = 0$. However under some restrictions on initial data, uniqueness readily follows from Theorem 3.1.

Corollary 3.9. *Let $u(x, t)$ be a viscosity solution of* (ST) *with initial data u_0 and boundary data $\theta(x, t) = \theta(x) < 0$, where u_0 and θ satisfies one of the following conditions:*

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(a) u_0 *is strictly concave;*

(b) $\theta(x,t) \equiv \theta < 0$ *and both* Ω *and* u_0 *is star-shaped: i.e. for sufficiently small* $\varepsilon > 0$,

$(1+\varepsilon)^{-1}\Omega \subset \Omega$ *and* $u_0(x) \leq u_0((1+\varepsilon)^{-1}x)$, *where equality only holds at* $x = 0$.

Then $u(x,t)$ *is unique.*

Proof. 1. Let us begin by mentioning that for any a, b and $c > 0$, if u is a viscosity sub (super)-solution of (ST), then so is

$$au(bx, ab^2t + c).$$

2. Now let u and v be two viscosity solutions of (ST) with initial data u_0. In the case of (a), one can verify via barrier arguments that $v(x,t) \leq u_0(x)$ if $0 \leq s < t$ with equality holding only at $\partial_L Q$. In particular, for sufficiently small $\varepsilon > 0$, we have $(1+\varepsilon)v(x,\varepsilon) < u_0(x)$ (recall that θ is negative). Therefore observation made in step 1 and Theorem 3.1 yields

$$(1+\varepsilon)v(x,(1+\varepsilon)t+\varepsilon) < u(x,t) \quad \text{in } Q$$

for any $\varepsilon > 0$. Now we can send $\varepsilon \to 0$ to obtain $v \leq u$. Similarly we can argue with v and u switched to obtain $u \leq v$, and therefore $u = v$.

In the second case, note that for any $\varepsilon > 0$ and $\delta > 0$,

$$(1+\delta)^{-1}u_0(x) < u_0((1+\varepsilon)^{-1}x) \text{ in } \Omega.$$

Also, since u_0 is star-shaped, the minimum of u_0 (and u and v) equals θ_0. Since u and v are negative caloric functions near $\partial\Omega$, Hopf's lemma yields that the inward normal derivative of $u(\cdot,t)$ and $v(\cdot,t)$ on $\partial\Omega$ is strictly positive. In particular for any $T > 0$ there exists $\delta = O(\varepsilon)$ such that

$$(1+\delta)^{-1}\theta_0 < u((1+\varepsilon)^{-1}x,t) \text{ for } (x,t) \in \partial_L Q.$$

Hence Theorem 3.1 again yields that, for sufficiently small $\varepsilon < 0$,

$$(1+\delta)^{-1}v(x,(1+\delta)^{-1}t) < u((1+\varepsilon)^{-1}x,(1+\varepsilon)^{-2}t) \quad \text{in } Q.$$

Therefore, by sending $\varepsilon \to 0$, we can conclude that $v \leq u$. As before, by switching v and u in above argument one can conclude $v = u$.

$\square$

For a general uniqueness result using the comparison principle we refer to section 4 in [K] where rather technical barrier arguments take place. We have not attempted to follow this approach since, as mentioned before, we are interested in the coincidence of the two generalized solutions for (ST).

4 Correspondence between weak and viscosity solutions

Recall that, due to [11,14,15], there exists a unique weak solution h of problem (ST), defined in Definition 2.1, with initial data h_0 and boundary data θ. Let us define $u = \chi(h)$ to be the *weak temperature* solution of (ST) associated with initial enthalpy function h_0, see [11].

Lemma 4.1. *Any weak temperature solution u of* (ST) *with initial data $u_0 = \chi(h_0)$ is a viscosity solution of* (ST).

Proof. The analysis in [6] yields that weak temperature solutions are continuous for continuous $\chi(h_0)$. Moreover, a comparison principle holds for weak solutions. Indeed, if h_1 and h_2 are two weak solutions with $h_1 \le h_2$ on $\partial_P Q$, then $h_1 \le h_2$ in Q. See [15], for instance.

The comparison principle can be extended to accommodate weak sub and supersolutions. A function h_1 is a *weak subsolution* if it satisfies (2.1) in Definition 2.1 with $\ge$ instead of equality for all $\varphi \in W_2^{2,1}(Q)$, $\varphi \ge 0$, $\varphi = 0$ on $\partial_L Q \cup \{t = T\}$. A function h_2 is a *weak supersolution* if it satisfies (2.1) with $\le$ instead of equality for all φ as above. A comparison principle still holds. If $h_1 \le h_2$ on the parabolic boundary $\partial_P Q$ then $h_1 \le h_2$ in Q.

A standard integration by parts also shows that a classical subsolution ϕ in Definition 2.2 corresponds to a weak subsolution $\chi^{-1}(\phi)$, and also that a classical supersolution φ is a weak supersolution $\chi^{-1}(\varphi)$: note that, by definition, ϕ and φ have no mushy region and thus $\chi^{-1}\phi$ and $\chi^{-1}\varphi$ are well defined almost everywhere. We deduce from this that we can compare weak temperature solutions with classical sub and supersolutions. Hence we conclude that weak temperature solutions are viscosity solutions of (ST) in the sense of Definition 2.4. $\qquad\square$

Corollary 4.2. *There exists a viscosity solution of* (ST) *for continuous $\chi(h_0)$ and θ.*

Unfortunately it is not clear to the authors how to find a corresponding weak solution for a given viscosity solution when the initial data has a fat zero set. However it is possible to identify the maximal and minimal viscosity solution with maximal and minimal weak temperature solutions (see Proposition 4.4). To prove this we first need a uniform control on the expansion of the sets $\{u \le 0\}$ and $\{u \ge 0\}$ at $t = 0$ in terms of the L^∞-norm of the initial data.

Lemma 4.3. *Let u be a viscosity solution of* (ST) *with initial data u_0. Further assume that $|\{u_0 > 0\}| = 0$. Then for any $r > 0$ the set $\{u \le 0\}$ does not expand by more than r before the time $t_r = c(n, M)r^2$, $M = \max_{\partial_P Q} |u_0|$. In other words,*

$$\{x : u(x,t) \le 0\} \subset \{x : \operatorname{dist}(x, \{u_0 \le 0\}) \le r\}$$

for all $t \in [0, t_r]$.

The same is true for $\{u \ge 0\}$.

Proof. We only prove the lemma for $\{u \leq 0\}$. Parallel argument holds for $\{u \geq 0\}$.

Set

$$A_r = \{x : \text{dist}(x, \{u_0 \leq 0\}) > r\}.$$

We need to show that $u(x, t) > 0$ in $A_r \times [0, t_r]$, for t_r specified below.

Thus choose $x_0 \in A_r$. Define the function

$$f(\rho) = \begin{cases} \frac{1}{\rho^{n-2}} - 1, & n \geq 3, \\ -\log \rho, & n = 2. \end{cases}$$

Note that f is decreasing and $f(1) = 0$. For simplicity we show the following computation for $n \geq 3$ only. The case $n = 2$ is similar.

Set $K = \frac{M}{|f(2)|}$ and $v = \frac{8K(n-2)}{r}$. We will construct a classical subsolution of (ST) in the set

$$C = \{(x, t) : |x - x_0| < r - vt, \ 0 \leq t \leq t_r\}$$

that is below u on the parabolic boundary of C and is positive at x_0 for $0 \leq t \leq t_r$.

Define the radially symmetric function $\phi = \phi(x, t)$ in C as

$$\phi(x, t) = \begin{cases} Kf\left(\frac{2|x-x_0|}{r-vt}\right), & 2|x - x_0| > r - vt, \\ \tilde{\phi}(x, t), & 2|x - x_0| < r - vt, \end{cases}$$

where $\tilde{\phi}(x, t)$ is a solution of the heat equation in the domain

$$C_{\frac{1}{2}} = \left\{(x, t) : |x - x_0| < \frac{1}{2}(r - vt), \ t \in (0, t_r)\right\}$$

with smooth positive initial data $\tilde{\phi}(x, 0) < u(x, 0)$ in the interior of $D = \{x : |x - x_0| < r/2\}$ and boundary data 0.

The modulus of the gradient $|D\phi^-|$ on the free boundary $\Gamma(\phi)$ is

$$\left|D\phi^-\right| = K\left|f'(1)\right| \frac{2}{r - vt} = \frac{2K(n-2)}{r - vt}.$$

For $t < t_r = \frac{r}{2v} = c(n, M)r^2$,

$$\left|D\phi^-\right| < \frac{4K(n-2)}{r} = \frac{v}{2} = -V_n.$$

Therefore ϕ is a classical subsolution of (ST) in the set C. Also, $\phi < u$ on the parabolic boundary $\partial_P C$. Theorem 3.1 implies $\phi < u$ in the set C and therefore $u > 0$ at x_0 for $0 \leq t \leq t_r$. $\qquad \square$

Proposition 4.4. *There exist maximal and minimal viscosity solutions* $U_1(x,t)$ *and* $U_2(x,t)$ *of* (ST) *satisfying the following:*

$$U_2 \leq u \leq U_1 \quad \text{for any viscosity solution } u \text{ of (ST).}$$

Furthermore, U_1 *is the unique weak temperature solution of* (ST) *with the intial enthalpy*

$$h_0^{max} := \begin{cases} u_0 & \text{if } u_0 \geq 0, \\ u_0 - 1 & \text{if } u_0 < 0, \end{cases}$$

and U_2 *is the unique weak temperature solution of* (ST) *with the initial enthalpy*

$$h_0^{min} := \begin{cases} u_0 & \text{if } u_0 > 0, \\ u_0 - 1 & \text{if } u_0 \leq 0. \end{cases}$$

Proof. 1. Take an approximation $w_0^\varepsilon < u_0 < v_0^\varepsilon$ such that the zero set of the approximation has zero measure. This can be done, for example, with simply $w_0^\varepsilon = u_0 - \varepsilon$ and $v_0^\varepsilon = v_0 + \varepsilon$ for along a sequence of $\varepsilon_0 \to 0$. Indeed the function $F(\varepsilon) := |\{u_0 > 0\}| - |\{u_0 > \varepsilon\}|$ is a monotone, bounded function. therefore it can only have countable number of jumps. Except at these jumps, $|\{u_0 = \varepsilon\}| = 0$.

Let us choose a viscosity solution v^ε (and w^ε) of (ST) respectively with initial data v_0^ε (resp. w_0^ε) and lateral boundary data $\theta + \varepsilon$ (resp. $\theta - \varepsilon$). In particular (excluding a countable number of ε) the functions v_0^ε and w_0^ε have no mushy region, and therefore $h_0^{1,\varepsilon} := \chi^{-1}(w_0^\varepsilon)$ and $h_0^{2,\varepsilon} := \chi^{-1}(v_0^\varepsilon)$ are well-defined almost everywhere, hence the corresponding weak solutions $h^{1,\varepsilon}$ and $h^{2,\varepsilon}$ exist. Now, by Lemma 4.1, the weak temperature solution $v^\varepsilon := \chi(h^{1,\varepsilon})$ (resp. $w^\varepsilon := \chi(h^{2,\varepsilon})$) is a viscosity solution of (ST) with initial data v_0^ε (resp. w_0^ε). Let u be a viscosity solution of (ST) with initial data u_0. Let $\hat{u}$ be the unique weak temperature solution of (ST) with boundary data $h_0 := \chi^{-1}(u_0)$. For convenience, let us denote the initial and lateral boundary data $g := (u_0, \theta)$.

2. The proof of the following lemma will be presented in the Appendix:

Lemma 4.5. v^ε *(or* w^ε*), along a subsequence, uniformly converge to a continuous function* u *with boundary data* g .

Due to the stability of viscosity solutions, it is straightforward to verify that u is a viscosity solution of (ST) with boundary data g.

On the other hand, due to stability result of [15], for each time $t > 0$ the weak soution $h^{1,\varepsilon}(\cdot, t)$ converges to $h(\cdot, t)$ in L^1-norm, where $h(\cdot, 0) = h_0^{max}$.

Using the uniform convergence of $v^\varepsilon := \chi(h^{1,\varepsilon})$, it is easy to check that $u = \chi(h) = U_1$ a.e.

3. Lastly, Theorem 3.1 yields that any viscosity solution u of (ST) with initial data u_0 satisfies
$w^\varepsilon < u < v^\varepsilon$, and thus taking $\varepsilon \to 0$ we obtain $U_2 \leq u \leq U_1$.

□

The following Theorem is obtained as a corollary of Propostion 4.4

Theorem 4.6. *The viscosity solution u of* (ST) *with* $|\{u_0 = 0\}| = 0$ *coincides with the weak temperature solution and it is unique.*

Appendix

First we construct "two-phase" test functions for the barrier arguments in the proof of the comparison principle, Theorem 3.1.

A. Test functions. Here we provide a construction of a class of radially symmetric classical sub- and supersolutions for that purpose. We will be working in spherical coordinates and thus we shall denote $\rho = |x|$.

Let $\alpha \neq 0$, $\beta \neq 0$, $\varepsilon > 0$, $R > 0$ and m be given parameters satisfying

$$\operatorname{sign} \alpha \neq \operatorname{sign} \beta \qquad \text{and} \qquad \alpha + \beta \neq m \operatorname{sign} \beta. \tag{4.1}$$

We show below that one can find constants $\tau > 0$, $\delta > 0$ and q such that a test function $\phi(x,t)$ of the form

$$\phi(x,t) = \phi_0(|x| - R - mt).$$

with

$$\phi_0(s) = \begin{cases} \alpha s + q\frac{s^2}{2} & s \geq 0, \\ -\beta s + q\frac{s^2}{2} & s < 0, \end{cases}$$

defined in the domain

$$\Sigma_{\tau,\delta} = \{(x,t) : R + mt - \delta \leq |x| \leq R + mt + \delta, \ -\tau \leq t \leq 0\},$$

is a classical subsolution (or supersolution) of (ST) on $\Sigma_{\tau,\delta}$, see Figure 8. Moreover, $\phi(x,t)$ is "ε-close" to a linear function of ρ on both sides of the free boundary $\Gamma(\phi)$, see (4.5).

We shall denote $R_t = R + mt$ since the set $\Gamma(\phi) = \{(x,t) : r = R + mt\}$ is the free boundary of ϕ. Observe that on $\Gamma(\phi)$, we have

$$\left|D\phi^+\right| = \max\left(\alpha, \beta\right), \qquad \left|D\phi^-\right| = -\min\left(\alpha, \beta\right).$$

and thus

$$\left|D\phi^+\right| - \left|D\phi^-\right| = \alpha + \beta. \tag{4.2}$$
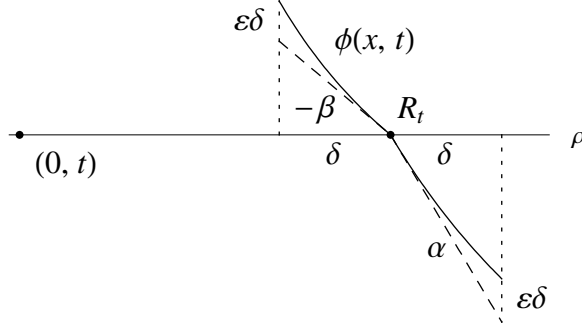
Figure 8: Test function $\phi(r,t)$ at time t

We can also deduce the direction of the outer normal of $\{\phi > 0\}$, and hence the sign of the normal velocity V_n, from the signs of α and β due to (4.1). As $\Gamma(\phi)$ is the space ball expanding with speed m, we recognize two situations:

$$V_n = \begin{cases} -m & \text{when } \beta < 0, \\ m & \text{when } \beta > 0. \end{cases} \tag{4.3}$$

Now we can express the free boundary condition in the light of (4.2) and (4.3) as

(a) $m \operatorname{sign} \beta \geq \alpha + \beta$ for a classical supersolution,

(b) $m \operatorname{sign} \beta \leq \alpha + \beta$ for a classical subsolution.

Suppose that the condition in (a) is satisfied and we are thus constructing a classical supersolution. In this case, the function ϕ has to satisfy also $(\partial_t - \Delta)\phi \geq 0$ in $\{\phi \neq 0\}$.

Straightforward computation yields

$$(\partial_t - \Delta)\, \phi(x,t)\big|_{\rho=R_t^+} = -\alpha \left(m + \frac{n-1}{R_t} \right) - q,$$

$$(\partial_t - \Delta)\, \phi(x,t)\big|_{\rho=R_t^-} = \beta \left(m + \frac{n-1}{R_t} \right) - q.$$

Continuity allows us to choose small $\tau > 0$ and $\delta > 0$ and a suitable constant q such that

$$(\partial_t - \Delta)\, \phi(x,t) \geq 0 \qquad \text{in } \Sigma_{\tau,\delta} \setminus \Gamma(\phi) \tag{4.4}$$

and

$$\begin{aligned} \left| \frac{\phi(x,t)}{\rho - R_t} - \alpha \right| &< \varepsilon, & \text{when } \rho > R_t, \\ \left| \frac{\phi(x,t)}{R_t - \rho} - \beta \right| &< \varepsilon, & \text{when } \rho < R_t, \end{aligned} \tag{4.5}$$

in $\Sigma_{\tau,\delta}$.

Thus let $\phi_{R,m}^{\alpha,\beta}$ denote the classical supersolution constructed above with the parameters R, m, α and β. Also let $\delta\left(\phi_{R,m}^{\alpha,\beta}, \varepsilon\right)$ and $\tau\left(\phi_{R,m}^{\alpha,\beta}, \varepsilon\right)$ denote the constants from above.

The case when the condition in (b) is satisfied and we are constructing a subsolution is analogous. The constants $\tau > 0$, $\delta > 0$ and q are chosen in such a way that the function ϕ satisfies $(\partial_t - \Delta)\, \phi(x,t) \leq 0$ in (4.4). And correspondingly, let $\phi_{R,m}^{\alpha,\beta}$ be the constructed classical subsolution. $\qquad\square$

We next present the proof of the lemma used in the proof of Proposition 4.4.

B. Proof of Lemma 4.5. We will only prove the Lemma for v^ε. The argument for w^ε is parallel. Let $u^k := v^{1/k}$ and let g^k the corresponding initial and lateral boundary data of $v^{1/k}$.

[6] provide us with a uniform bound on the $W_2^{1,0}(Q)$ norm and with a uniform modulus of continuity ω_d for v^ε in the domains away from $\partial_P Q$:

$$Q_d = \{(x,t) \in Q : \mathrm{dist}((x,t), \partial_P Q) > d\}, \qquad d > 0.$$

Therefore we need a uniform modulus of continuity on $\partial_P Q$ to be able to extract a subsequence of u^k converging uniformly. We will obtain this by comparing u^k to solutions of the heat equation near the parabolic boundary of Q.

First find k_0 for which $g^{k_0} < 0$ on $\partial_L Q$. Theorem 3.1 implies u^k decreases in k. In particular

$$\left\{u^{k+1} \geq 0\right\} \subset \left\{u^k \geq 0\right\} \subset \left\{u^{k_0} \geq 0\right\}.$$

Since $\left\{u^{k_0} \geq 0\right\}$ is a closed set and $g^k < 0$ for $k \geq k_0$, there exists $\eta > 0$ such that

$$\left\{u^k \geq 0\right\} \subset Q^\eta := \{(x,t) \in Q : \mathrm{dist}(x, \partial\Omega) > \eta\} \qquad \text{for all } k \geq k_0.$$

Note that $Q^\eta := \Omega^\eta \times (0, T)$ where

$$\Omega^\eta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \eta\}.$$

Define ψ_+^k, resp. ψ_-^k, to be the solution of the heat equation in Q with boundary data g^{k+}, resp. $-g^{k-}$. Here g^{k+} and g^{k-} are the positive part and the negative part of g^k, respectively.

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The maximum principle for the heat equation yields

$$\psi_-^k \le u^k \le \psi_+^k \qquad \text{in } Q. \tag{4.6}$$

A uniform modulus of continuity for u^k on the lateral boundary $\partial_L Q$ then follows from comparing u^k with φ^k that solves the heat equation

$$\begin{cases} \varphi_t - \Delta\varphi = 0, & \text{in } Q \setminus Q^\eta, \\ \varphi = g^k, & \text{on } \partial_L Q, \\ \varphi(x,0) = g^k(x,0)\zeta(x), & \text{on } \Omega \setminus \Omega^\eta. \end{cases}$$

Here $\zeta(x)$ is a smooth function on $\Omega \setminus \Omega^\eta$ such that $\zeta \in [0,1]$, $\zeta = 1$ on $\Omega \setminus \Omega^{\eta/2}$ and $\zeta = 0$ on $\partial\Omega^\eta$.

Observe that (4.6) and comparison for the heat equation implies that

$$\psi_-^k \le u^k \le \varphi^k \qquad \text{on } Q \setminus Q^\eta.$$

Since $\psi_-^k = u^k = \varphi^k = g^k$ on $\partial_L Q$, the uniform continuity follows.

The proof for $t = 0$ is more involved. We split $u^k = u^{k+} - u^{k-}$ and consider each of $u^{k\pm} \ge 0$ separately.

Choose $r > 0$ small and fix k. Let $A^k = \{x \in \Omega : g^k > 0\}$. Define

$$A_r^k = \{x \in A : \operatorname{dist}(x, \partial A \cap \Omega) > r\}.$$

By Lemma 4.3, there is $t_r > 0$ such that $A_r^k \subset \{x : u^k(x,t) > 0\}$ for all $t < t_r$. Moreover, t_r can be chosen independent of k. Construct a solution φ^k to the heat equation in $A_r^k \times (0, t_r)$ with boundary data 0 on $(\partial A_r^k \cap \Omega) \times (0, t_r)$ and $g^k(x,t)\zeta^k(x)$ on the rest of the parabolic boundary of $A_r^k \times (0, t_r)$. ζ^k is chosen to be a smooth cutoff function with values in $[0,1]$ on A_r^k such that $\zeta^k = 1$ on A_{2r}^k and $\zeta^k = 0$ on $\partial A_r^k \cap \Omega$. It is also chosen in such a way that $\left\|\zeta^k\right\|_{C^{0,1}}$ is uniformly bounded in k. Define $\varphi_r^k = 0$ in the complement of $A_r^k \times (0, t_r)$. The maximum principle for heat equation then yields

$$\varphi_r^k \le u^{k+} \le \psi_+^k \qquad \text{in } \Omega \times [0, t_r].$$

Thus the standard parabolic estimates for the heat equation ([14]) imply

$$\left| u^{k+}(x_1, t_1) - u^{k+}(x_0, 0) \right| = \left| u^{k+}(x_1, t_1) - g^{k+}(x_0) \right|$$
$$\le \omega_g(2r) + \omega_H \left(|x_1 - x_0|^2 + t_1 \right),$$

for all $t_1 < t_r$, $x_0, x_1 \in \Omega$, modulus of continuity ω_H independent of k and r, where ω_g is a modulus of continuity of g. This implies uniform modulus of continuity for u^{k+} at $t = 0$.

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A similar consideration yields a uniform modulus for u^{k-} at $t = 0$. Those together give us a uniform modulus of continuity for u^k.

Now as in [6], we can find a subsequence $\left\{u^{k_j}\right\}_{j=1}^{\infty}$ such that

$$
\begin{aligned}
\beta\left(u^{k_j}\right) \to \beta\left(\tilde{u}\right) && \text{weakly in } L^2(Q), \\
Du^{k_j} \to D\tilde{u} && \text{weakly in } L^2(Q), \\
u^{k_j} \to \tilde{u} && \text{uniformly on } \overline{Q}.
\end{aligned}
$$

$\square$

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