Erratum: Degenerate diffusion with a drift potential: A viscosity solution approach

Inwon C. Kim *

November 28, 2010

Abstract

The earlier paper [KL] contains a Lemma on the lower bound of solution in terms of its L^1 norm, which is incorrect. In this note we explain the mistake and present a correction to it under the restriction that the permeability constant m satisfies $1 < m < 2$. As a consequence, the quantitative estimates on the converge rate (Main Theorem (c) and Theorem 3.6 in [KL]) only hold for $1 < m < 2$. As for $m \geq 2$ a partial convergence rate is obtained.

In the previous paper [KL], the construction of barrier function in step 2. of Lemma 3.4 in [KL] is incorrect: this is due to the fact that the equation $u_t = (m-1)u\Delta u + |Du|^2 - C$ with $C > 0$ is not well-posed when the solution becomes negative. In the case of $1 < m < 2$ we present a corrected and simplified proof of Lemma 3.4, where the aforementioned error is fixed by considering an alternative equation (0.3) in the density form. The validity of Lemma 3.4 in case of $m \geq 2$ remains open. Secondly, we point out that the proof and the statement of Lemma 3.5 has been originally presented in the case of $m = 2$ without clarification. We will correct this by stating the general result as well as the difference in the proof.

Consequently, the result of Main Theorem (c) and Theorem 3.6 in [KL] is only valid for $1 < m < 2$. As for $m \geq 2$, the rate can be only obtained in terms of how far the free boundary of the solution is from the support of the equilibrium (see Theorem 0.4).

Lemma 0.1 (Lemma 3.4 in [KL], corrected version)**.** *Let* $1 < m < 2$ *and* $(x_0, t_0) \in I\!R^n \times (0, \infty)$*. Then there exists small constants* $k, k', a_0 > 0$*, depending on* m*,*n *and the* C^2*-norm of* Φ *in* $B_1(x_0)$*, such that the following is true: Suppose, for* $0 < a < a_0$*,*

$$a^{-n} \int_{B_a(x_0)} \rho(\cdot, t_0) dx \geq a^k.$$

*Department of Mathematics, UCLA. The author is partially supported by NSF 0970072

Then $u(\cdot, t_0 + a) \geq a^{k'}$ *in* $B_a(x_0)$.

Proof. 1. Let us define

$$\tilde{u}(x,t) := u(a(x - x_0), a^2(t - t_0)).$$

Since $u \leq 1$ in $I\!R^n \times [0, \infty)$, $\tilde{u}$ satisfies, in the viscosity sense,

$$\begin{aligned} \tilde{u}_t &\geq (m-1)\tilde{u}\Delta\tilde{u} + |D\tilde{u}|^2 - C_1 a(|D\tilde{u}| + a\tilde{u}) \\ &\geq (m-1)\tilde{u}\Delta\tilde{u} + (1 - C_1 a)|D\tilde{u}|^2 - 2C_1 a, \end{aligned}$$

where the second inequality holds due to Cauchy-Schwartz inequality. Here the constant C_1 depends on the C^2-norm of Φ in $B_a(x_0)$. Hence $\bar{u} := (1 - C_1 a)\tilde{u}$ satisfies

$$\bar{u}_t \geq (\tilde{m} - 1)\bar{u}\Delta\bar{u} + |D\bar{u}|^2 - 2C_1 a, \tag{0.1}$$

where $\tilde{m} = (1 - C_1 a)^{-1}(m - 1) + 1 > m$. Choose a_0 small enough so that $\tilde{m} < 2$.

Therefore the corresponding density function, i.e. $\bar{\rho} = (\frac{\tilde{m}-1}{\tilde{m}}\bar{u})^{\frac{1}{\tilde{m}-1}}$ satisfies

$$\bar{\rho}_t \geq \Delta(\bar{\rho}^{\tilde{m}}) - \frac{2C_1}{\tilde{m} - 1} a\bar{\rho}^{2-\tilde{m}} \geq \Delta(\bar{\rho}^{\tilde{m}}) - C_2 a \chi_{\{\rho \geq 0\}}. \tag{0.2}$$

2. Let $w(x,t)$ denote the weak solution of

$$w_t = \Delta(w|w|^{m-1}) - C_2 a \chi_{|x - x_0| \leq 2} \tag{0.3}$$

with initial data

$$w(x, 0) = \bar{\rho}(x, 0)\chi_{|x - x_0| \leq 1}.$$

The weak solution $w(x,t)$ then exists in $I\!R^n \times [0, \infty)$ by Theorem 5.7 of [V]. Moreover due to [DiBGV], w is uniformly Hölder continuous in $B_2(x_0) \times [1/4, 1/2]$.

Note that any nonnegative solution of the (PME), $\rho_t = \Delta(\rho^{\tilde{m}})$, is a supersolution of (0.3). Therefore using an appropriate Barenblatt profile as a supersolution of (0.3) and using the fact that $\bar{\rho}(\cdot, 0) \leq \chi_{|x - x_0| \leq 1}$, we have

$$\{x : w(x,t) > 0\} \subset \{|x| \leq 2\} \text{ for } 0 \leq t \leq 1/2. \tag{0.4}$$

Therefore it follows that w is a subsolution of (0.2), and thus $w \leq \bar{\rho}$ for $t \in [0, 1/2]$.

Using (0.3) and the definition of weak solution (or formally integration by parts) yields that

$$\int w(x,t)dx = \int w(x,0)dx - c_n C_2 a t \geq \frac{a^k}{2} - C_3 a t,$$

where c_n equals the volume of the n-dimensional ball with radius 2. Since $k < 1$, for small a we have $\int w(x, 1/2)dx \geq a^k/4$. Let x^* be the point where $w(\cdot, 1/2)$ assumes its maximum, then from (0.4) it follows that $|x - x^*| \leq 2$ and $w(x^*, 1/2) \geq C_4 a^k$ for some dimensional constant C_4. Due to the Hölder regularity of $w(\cdot, 1/2)$, there exists $0 < \gamma < 1$ depending only on m and n such that

$$\bar{\rho}(\cdot, 1/2) \geq w(\cdot, 1/2) \geq \frac{C_4}{2} a^k \text{ in } B_{a^{k_2}}(x^*), k_2 = \frac{k}{\gamma}. \tag{0.5}$$

3. Let now $U(x,t) := B(x,t;1,C) = \frac{(C(t+1)^{2\lambda} - \frac{\lambda}{2}|x|^2)_+}{(t+1)}$ be the Barenblatt profile given in Lemma 2.18 of [KL], with $0 < \lambda = ((m-1)d+2)^{-1} < 1/2$. Let us fix $C = a^{\lambda/2}$ such that

$$C = a^{\lambda/2} \text{ (initial height) and } \sqrt{\frac{2C}{\lambda}} = \sqrt{\frac{2}{\lambda}} a^{\lambda/4} \text{ (initial support size).}$$

If k is sufficiently small, then $U(x - x^*, 0) \leq \tilde{u}(\cdot, 1/2)$ due to (0.5). Moreover, a straightforward computation yields that $aU(\cdot, t), |DU|(\cdot, t) \leq c(t) := \sqrt{C}(t+1)^{\lambda-1}$ for $0 \leq t \leq a^{-1}$. Now let

$$\tilde{U}(x,t) := (U(x - x^*, t) - 2C_1 a \int_0^t c(s)ds)_+$$

Then, since $U(\cdot, t)$ is concave, we obtain

$$\tilde{U}_t \leq (m-1)\tilde{U}\Delta\tilde{U} + |D\tilde{U}|^2 - C_1 a(|D\tilde{U}| + a\tilde{U}) \text{ in } \overline{\{\tilde{U} > 0\}}.$$

Hence, by the comparison principle, $\tilde{u}(x, t+1/2) \geq \tilde{U}(x,t)$ in $I\!R^n \times [0, \infty)$. In particular

$$\tilde{u}(\cdot, a^{-1}) \geq \tilde{U}(\cdot, a^{-1} - 1/2) \geq a^{1-\lambda} \text{ in } B_1(x_0) \subset B_3(x^*).$$

We now conclude by scaling back to the original variable. $\square$

Next, we make corrections to the proof and statement of Lemma 3.5.

Lemma 0.2 (Lemma 3.5 in [KL], corrected version)**.** *Let $\mathcal{K}$ be a compact subset of $I\!R^n$ with $u = 0$ outside of $\mathcal{K}$ for all time. Then there exists a constant $C > 0$ depending on $m > 1$, $\sup \rho$ and $\max_{x \in \mathcal{K}} \Delta\Phi(x)$ such that the following holds: suppose*

$$\int_{B_C(0)} \rho(\cdot, t)dx \leq c_0 \textit{ for } t_1 \leq t \leq t_2 := t_1 + \log(1/c_0).$$

then $\rho(\cdot, t_2) \leq C c_0^k$ in $B_1(0)$ with $k = \frac{2}{m(n+1)}$.

Remark 0.3. *1. In step 2. of the original proof, where we let $\tilde{\rho} = \tilde{\rho}_1 + \tilde{\rho}_2$, the initial data should be divided as follows: $\tilde{\rho}_1(\cdot, 0) = \frac{\rho_0}{a}$ and $\tilde{\rho}_2(\cdot, 0) = 1/10$. The rest of the proof is the same.*

2. The proof is written in the case of $m = 2$ without clarification: as for $m \neq 2$ one has to replace the scaling for $\tilde{\rho}$ in the proof of step 2. by $\tilde{\rho}(x, t) := a^{-1}\rho(a^{m/2}x, at)$. Proceeding as before with this scaling yields the above statement. We note that in the original statement $k = 1/(n + 1)$.

Using Lemma 0.1 - 0.2 and proceeding as in the proof of Theorem 3.6 in [KL], the following holds.

Theorem 0.4 (Theorem 3.6 in [KL], corrected)**.** *Let Φ and u_∞ be as in Theorem 3.2. Then there exists K and $\alpha > 0$ depending on $m, \sup u_0, k_0, M_1$, $A := min_{\Phi(x)>C_0}|D\Phi|$ and n such that the following is true:*

(a) $\Gamma_t(u) = \partial\{u(\cdot, t) > 0\}$ is in the $Ke^{-\alpha t}$-neighborhood of the positive set $\{u_\infty > 0\}$.

(b) If $1 < m < 2$, then $\Gamma_t(u)$ is in the $Ke^{-\alpha t}$-neighborhood of $\Gamma(u_\infty)$.

References

[DiBGV] E. DiBenedetto, U. Gianazza and V. Vespri, *Harnack estimates for quasi–linear degenerate parabolic differential equations,* Acta Math., **200** (2008), 181–209.

[KL] Inwon C. Kim and Helen K. Lei, *Degenerate diffusion with a drift potential: a viscosity solution approach,* DCDS-A **27** (2010), no.2, pp. 767-786.

[V] J. L. Vazquez, "The Porous Medium Equation: Mathematical Theory," Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.