Global Existence and Finite Time Blow-Up for Critical Patlak-Keller-Segel Models with Inhomogeneous Diffusion

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Abstract

The L^1-critical parabolic-elliptic Patlak-Keller-Segel system is a classical model of chemotactic aggregation in micro-organisms well-known to have critical mass phenomena [10, 8]. In this paper we study this critical mass phenomenon in the context of Patlak-Keller-Segel models with spatially varying diffusivity and decay rate of the chemo-attractant. The primary tool for the proof of global existence below the critical mass is the use of pseudo-differential operators to precisely evaluate the leading order quadratic portion of the potential energy (interaction energy). Under the assumption of radial symmetry, blow-up is proved above critical mass using a maximum-principle type argument based on comparing the mass distribution of solutions to a barrier consisting of the unique stationary solutions of the scale-invariant PKS. Although effective where standard Virial methods do not apply, this method seems to be dependent on the assumption of radial symmetry. For technical reasons we work in dimensions three and higher where L^1-critical variants of the PKS have porous media-type nonlinear diffusion on the organism density, resulting in finite speed of propagation and simplified functional inequalities.

1 Introduction

The most widely studied mathematical models of nonlocal aggregation phenomena are the Patlak-Keller-Segel (PKS) models, originally introduced to study the chemotaxis of microorganisms [29, 25, 23, 22]. In this paper we consider critical cases of the form,

$$\begin{cases} u_t + \nabla \cdot (u \nabla c) = \Delta u^{2-2/d} \\ -\nabla \cdot (a(x) \nabla c) + \gamma(x) c = u \\ u(0,x) = u_0(x) \in L^1_+(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d), \quad d \geq 3, \end{cases} \tag{1}$$

where $L^1_+(\mathbb{R}^d;\mu) := \left\{ f \in L^1(\mathbb{R}^d;\mu) : f \geq 0 \right\}$. We define $m := 2 - 2/d$. As weak solutions to (1) conserve mass, we will henceforth refer to $\|u_0\|_1 = \|u(t)\|_1 = M(u)$. For all our work, we assume $a(x)$ is strictly positive which ensures the PDE for c is uniformly elliptic. In this context, critical refers to the approximate balance of the opposing forces of diffusion and aggregation in the limit of L^1 concentration, indicating that there must be a non-zero, but finite, amount of mass concentration at any possible blow-up. This model is a generalization of the classical parabolic-elliptic 2D PKS model which has received considerable attention over the years (see the review [23] and [24, 20, 10, 9, 7]). It is well-known that such models exhibit critical mass phenomena: there exists some $M_c > 0$ such that if $M(u) < M_c$ then the solution exists globally, and if $M(u) > M_c$ then the solution could potentially blow-up in finite time (see for instance [10, 9, 8, 2]). We refer to the special case of

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$a(x) \equiv 1$, $\gamma(x) = 0$ as the *scale-invariant problem*, because solutions are invariant under an L^1 scaling in space and this scaling symmetry plays a fundamental role in the global theory.

In this paper we estimate the critical mass, and under certain restrictions, show that this estimate is sharp for the PKS model (1) with spatially variable coefficients in the chemo-attractant PDE. We could not find a mathematical treatment of the model, however it raises questions which are both mathematically interesting and relevant for biological applications where chemotaxis occurs in a spatially inhomogeneous medium. For homogeneous problems, determining the critical threshold has become a fairly classical procedure; e.g. using sharp functional inequalities to prove global existence below the threshold and Virial methods to prove blow-up above (see more discussion below). However, for inhomogeneous critical problems, these well-established methods generally break down and either must be modified or alternative methods must be found (both shall occur here). This, in turn, sheds some additional light on critical problems in general.

Our results confirm the intuitive fact that any blow-up is essentially a spatially localized phenomenon and does not depend on the global properties of $a(x)$ or $\gamma(x)$. To be precise, we estimate the critical mass to be given by

$$M_c = \left(\frac{2 \min_{x \in \mathbb{R}^d} a(x)}{(m-1)C_\star c_d} \right)^{d/2},$$

where $C_\star$ is the optimal constant in the Hardy-Littlewood-Sobolev inequality discussed in [8] (see (6) below) and c_d is the normalization constant in the Newtonian potential, given explicitly below in (5). This estimate is of course in agreement with the constant coefficient cases discussed in [8] and [2] (see Theorem 2 below for more information). We prove that if $M(u) < M_c$ then the solution is global and uniformly bounded in $L^\infty((0,\infty) \times \mathbb{R}^d)$ (Theorem 5). To complement this result, we prove that if $\gamma = 0$ and $a(x)$ is radially symmetric and monotone non-decreasing in a neighborhood of the origin, then for all $M > M_c$ we may construct a solution with $M(u) = M$ which blows up in finite time (Theorem 6).

We restrict ourselves to the case $\mathbb{R}^d$ for $d \geq 3$ for several technical reasons. First of all, the local existence and uniqueness theory in $\mathbb{R}^2$ does not appear to be worked out anywhere unless $\gamma(x)$ is strictly positive [1]. Second, even when γ is strictly positive, the $\mathbb{R}^2$ case seems to require a refined treatment in the global existence argument (see Remark 5 in §2 for a discussion). Thirdly, there are advantages in the blow-up argument due to the spatial localization provided by the degenerate diffusion in $d \geq 3$. Despite these potential difficulties, we do expect analogous results to hold in $\mathbb{R}^2$.

A key quantity in the study of (1) is the dissipated *free energy*, given by

$$\mathcal{F}(u) = \frac{1}{m-1} \int u^m(x)dx - \frac{1}{2} \int u(x)c(x)dx. \tag{2}$$

The first term is usually referred to as the *entropy* and the latter term is referred to as the *interaction energy* or *potential energy*. Formally, (1) is a gradient flow with respect to the Euclidean Wasserstein distance for (2) (see e.g. [6]), but this will not be relevant for our work (however, see [7] for work on the threshold problem with linear diffusion where this structure is the key tool).

It is well-known that solutions to systems such as (1) exist as long as they remain equi-integrable [24, 13, 8, 2]. The use of sharp functional inequalities to identify when a mixed-sign energy such as (2) is coercive (in the sense that it bounds a controlling norm) is classical, for example [35, 36, 10, 8, 2]. In the context of PKS and similar models, this amounts to using sharp Hardy-Littlewood-Sobolev inequalities to prove that when $M(u) < M_c$ the free energy uniformly controls the entropy (in this case the L^m norm) which in turn rules out any loss of equi-integrability (see [20, 10, 13, 8, 2]). In [2], it was shown for a related class of systems with $c = \mathcal{K} * u$, for interaction kernels $\mathcal{K}$ in a general class, that the critical mass is governed only by the asymptotic expansion of $\mathcal{K}$ at the origin. The idea of the proof there is that the singularity of $\mathcal{K}$ at the origin provides the leading order approximation of the quadratic potential energy $\int uc\,dx$ and the remaining error is 'subcritical' in the sense that it can be controlled by norms weaker than the entropy. Essentially, this is the approach taken here to prove Theorem 5 in §2 below, however for (1), c is not given by a convolution. To recover, we use a pseudo-differential operator to approximate the leading order contribution to the potential energy as a quadratic term of the form $\int u(x)u(y)K(x,y)dxdy$. An asymptotic expansion is derived for $K(x,y)$ along the diagonal $x \sim y$ and the lower-order error term resulting from the approximate inverse is shown to be subcritical. In order to use standard symbol classes we make the assumption that $a(x)$ and $\gamma(x)$ are both smooth with bounded derivatives of all orders. Presumably, one could weaken these hypotheses considerably if necessary. In particular, a full parametrix is not required, only the first approximation of the inverse is required for our work.

Conventionally, the simplest way of proving blow-up for systems such as (1) above the critical threshold is the well-known Virial method, used in for example [32, 8, 2]. Applied to the scale-invariant problem in $d \geq 3$ in [8], this method consists of two steps. First, one uses supercritical mass and a scaling argument to construct initial data with negative free energy. Second, one shows that negative free energy would force the second moment to zero in finite time. The first step can likely be carried out here using the approximate inverse of the chemo-attractant PDE used in the proof of the global existence result (Theorem 5). However, the second step would requires some kind of approximate homogeneity which only leaves errors which can be over-powered by choosing sufficiently concentrated initial data. Although we do not claim that this cannot be done for systems such as (1), we could not carry out this program. Using the approximate inverse introduces nonlocal error terms for which there seems to be no obvious way to control into blow-up. Hence, we instead use a different finite time blow-up proof which is able to treat the local and nonlocal properties of the solution with sufficient precision.

The proof of finite time blow-up is based on a maximum-principle type argument on the mass distribution of solutions. These arguments are primarily influenced by those found in the recent work [27] but are also related to the blow-up argument in the classical work of Jäger and Luckhaus [24]. We compare solutions of (1) to specifically chosen barriers, in fact the extremizers of the sharp Hardy-Littlewood-Sobolev inequality which governs the critical mass, which are also stationary solutions of the scale-invariant PDE (Theorem 4). Note that, in spite of the parabolic nature of the problem, there is, in general, no standard maximum principle between solutions of (1). Although mass comparison principles have been used for (1) before ([17, 18], see also [24, 5]) the use of refined barrier arguments has not been explored, with the exception of [27] and [24]. More details are discussed in Section §3.

The mass comparison argument we adopt is rather delicate and depends on certain regularity properties of solutions as well as the barrier. Among other complications, the degenerate diffusion in (1) implies that classical regularity is not available everywhere and the support of solutions move at a finite speed. In order to deal with the generic presence of the free boundary, we utilize the viscosity solution theory developed for degenerate diffusion equations with drift (see [26]) to prove that strictly positive solutions remain positive until blow-up (see Appendix). Then we need to make a careful approximation argument with smooth, strictly positive solutions, as in [27]. We add that here extra care must be taken due to the finite time blow-up (Lemma 6).

With some modification, this blow-up proof could also potentially provide estimates from above on how quickly mass concentrates, providing also a lower bound on the blow-up time. Moreover, when applied to the scale-invariant problem, our method can yield blow-up for a class of initial data that the results in [8], obtained via a Virial method, does not cover (see Corollary 1 below). In particular, we exhibit radially symmetric solutions with initially positive free energy that concentrate in finite time. This seems to indicate that the approach we employ could potentially provide different kinds of blow-up results elsewhere, even when straightforward Virial methods can be applied.

The finite time blow-up argument so far only applies to radially symmetric settings. It may be possible to extend this argument to at least locally radial settings. To localize the argument, we would need to carefully estimate the effect of coefficients $a(x)$ from far-away regions on the chemical function $c(x,t)$. Such an estimate, if available in suitable norms, could potentially enable consideration of $c(\cdot,t)$ and $\rho(\cdot,t)$ as "almost" radial profiles, making our blow-up argument feasible.

1.0.1 Notation

In what follows, we denote $\|u\|_p := \|u\|_{L^p(\mathbb{R}^d)}$ where $L^p(\mathbb{R}^d) := L^p$ is the standard Lebesgue space. We will often suppress the dependencies of functions on space and/or time to enhance readability. The standard characteristic function for some $S \subset \mathbb{R}^d$ is denoted $\mathbf{1}_S$ and we denote the ball $B_R(x_0) := \left\{x \in \mathbb{R}^d : |x-x_0| < R\right\}$. In addition, we use $\int f dx := \int_{\mathbb{R}^d} f dx$, and only indicate the domain of integration where it differs from $\mathbb{R}^d$. We also denote the weak L^p space by $L^{p,\infty}$ and the associated quasi-norm

$$\|f\|_{L^{p,\infty}} = \left(\sup_{\alpha>0} \alpha^p \lambda_f(\alpha)\right)^{1/p},$$

where $\lambda_f(\alpha) = |\{f > \alpha\}|$ is the distribution function of f. We use $\mathcal{N}$ to denote the Newtonian potential:

$$\mathcal{N}(x) = \begin{cases} \frac{1}{2\pi}\log|x| & d=2 \\ \frac{\Gamma(d/2+1)}{d(d-2)\pi^{d/2}}|x-y|^{2-d} & d \geq 3. \end{cases}$$

In formulae we use the notation $C(p,k,M,..)$ to denote a generic constant, which may be different from line to line or term to term in the same formula. In general, these constants will depend on more parameters than those listed, for instance those which are fixed by the problem, such as $\mathcal{K}$ and the dimension, but these dependencies are suppressed. We use the notation $f \lesssim_{p,k,...} g$ to denote $f \leq C(p,k,..)g$ where again, dependencies that are not relevant are suppressed.

1.1 Background

The local theory for (1) is studied in [1]. Here we simply discuss the results of that work, which follows closely the work of [10, 31, 3, 4, 2]. We begin with the definition of weak solution, which is stronger than the concept of distribution solutions. The main purpose of this definition is to ensure that weak solutions are unique.

Definition 1 (Weak Solution)**.** A function $u(t,x) : [0,T] \times \mathbb{R}^d \to [0,\infty)$ is a weak solution of (1) if $u \in L^\infty((0,T)\times\mathbb{R}^d) \cap L^\infty(0,T,L^1(\mathbb{R}^d))$, $u^m \in L^2(0,T,\dot{H}^1(\mathbb{R}^d))$, $u\nabla c \in L^2((0,T)\times\mathbb{R}^d)$, $u_t \in L^2(0,T,\dot{H}^{-1}(\mathbb{R}^d))$, and for all test functions $\phi \in \dot{H}^1(\mathbb{R}^d)$ for a.e $t \in [0,T]$,

$$\langle u_t(t), \phi\rangle = \int (-\nabla u^m(t) + u(t)\nabla c(t)) \cdot \nabla\phi \, dx, \tag{3}$$

where $c(t)$ is the strong solution to the PDE $-\nabla \cdot (a(x)\nabla c(t)) + \gamma(x)c(t) = u(t)$ which vanishes at infinity.

We state the following theorem summarizing the local theory of (1), developed in [1] as well as [31, 8, 2].

Theorem 1 (Local Existence and Uniqueness). *Let $d \geq 3$, let $a(x) \in C^1$ be strictly positive such that both $a(x)$ and $\nabla a(x)$ are bounded, let $\gamma(x) \in L^\infty$ be non-negative and $u_0 \in L^1_+(\mathbb{R}^d; (1+|x|^2)dx) \cap L^\infty(\mathbb{R}^d)$. Then there exists a maximal $T_+(u_0) > 0$ and a unique weak solution $u(t)$ to (1) which satisfies $u \in C([0,T]; L^1_+(\mathbb{R}^d; (1+|x|^2)dx)) \cap L^\infty((0,T); L^\infty(\mathbb{R}^d))$ for all $T < T_+(u_0)$ and $u(0) = u_0$. Additionally, $\mathcal{F}(u_0) < \infty$ and we have the energy dissipation inequality,*

$$\mathcal{F}(u(t)) + \int_0^t \int u(s) \left| \nabla \frac{m}{m-1} u^{m-1}(s) - \nabla c(s) \right|^2 dxds \leq \mathcal{F}(u_0). \tag{4}$$

We also have the continuation criterion: if

$$\lim_{k \to \infty} \sup_{t \in [0, T_+(u_0))} \|(u(t) - k)_+\|_1 = 0,$$

then necessarily $T_+(u_0) = \infty$ and $u(t,x) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$.

The results regarding the critical mass in the constant-coefficient case are summarized in the following theorem.

Theorem 2 (Critical Mass [8, 2]). *Suppose $a(x) = a$ and $\gamma(x) = \gamma$ are both constants. Then the sharp critical mass satisfies,*

$$M_c = \left(\frac{2a}{(m-1)C_\star c_d} \right)^{d/2},$$

and if $u(t)$ is a weak solution to (1) with $M(u_0) < M_c$, then $u(t)$ exists globally, e.g. $T_+(u_0) = \infty$, and we have $u(t) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$. Conversely for all $M > M_c$ there exists a solution to (1) which blows up in finite time with $M(u_0) = M$. Here $C_\star$ is the optimal constant in the Hardy-Littlewood-Sobolev inequality (6) below and c_d is the normalization factor in the Newtonian potential:

$$c_d := \frac{\Gamma\left(\frac{d}{2}+1\right)}{d(d-2)\pi^{d/2}}. \tag{5}$$

In [8], Blanchet et. al. exhibit a unique family of stationary solutions to the scale-invariant problem which will be the barriers used in the proof of finite time blow-up here.

Theorem 3 (Stationary Solutions to Scale-Invariant Problem [8]). *There exists a non-negative, radially symmetric, non-increasing function $V(x)$ supported in the ball of radius one with $\|V\|_1 = \left(\frac{2}{(m-1)C_\star c_d}\right)^{d/2}$ which is the unique solution (up to L^1 scaling and translation) of*

$$\Delta V^m = \nabla \cdot (V \nabla \mathcal{N} * V).$$

Remark 1. Note that if $a > 0$ and $\tilde{V} = a^{d/2} V$, then

$$\Delta \tilde{V}^{2-2/d} = \frac{1}{a} \nabla \cdot (\tilde{V} \nabla \mathcal{N} * \tilde{V})$$

and in light of Theorem 2 above, $\tilde{V}$ are the unique (up to L^1 scaling and translation) stationary solutions to the problem

$$\begin{cases} u_t + \nabla \cdot (u \nabla c) = \Delta u^{2-2/d} \\ -a\Delta c = u. \end{cases}$$

In [8], it is also shown that these stationary solutions are the unique extremals of the following Hardy-Littlewood-Sobolev type inequality.

Theorem 4 (Sharp Hardy-Littlewood-Sobolev Inequality [8]). *There exists some optimal $C_\star > 0$ depending only on the dimension such that for all $f \in L^1_+ \cap L^m$,*

$$\int\int f(x)f(y)\,|x-y|^{2-d}\,dydx \leq C_\star \|f\|_1^{2-m}\|f\|_m^m, \tag{6}$$

and equality is achieved if and only if there exists $\alpha_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$, $\lambda_0 \in (0,\infty)$ such that

$$f(x) = \frac{\alpha_0}{\lambda_0^d} V\left(\frac{x-x_0}{\lambda_0}\right).$$

1.2 Summary of Results

We now state the main results.

Theorem 5 (Global Existence). *Let $d \geq 3$ and $a(x) \in C^\infty(\mathbb{R}^d)$ be strictly positive and $\gamma(x) \in C^\infty(\mathbb{R}^d)$ be non-negative such that $D^\alpha a$ and $D^\alpha \gamma$ are bounded for all multi-indices α. Then,*

$$M_c = \left(\frac{2\min_{x\in\mathbb{R}^d} a(x)}{(m-1)C_\star c_d}\right)^{d/2}, \tag{7}$$

and any weak solution $u(t)$ to (1) with $M(u) < M_c$ exists globally and $u(t) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$.

Remark 2. Using the methods of [13, 2], Theorem 5 can be extended to cover more general filtration equation-type nonlinear diffusion on the RHS of the PDE for $u(t)$ (e.g. $\Delta A(u)$, $A \in C^1$ and non-decreasing with $0 < \liminf_{z\to\infty} A'(z)z^{2/d-1} < \infty$, instead of simply $\Delta u^{2-2/d}$).

That Theorem 5 is sharp under the hypothesis of radial symmetry is demonstrated by the following result.

Theorem 6 (Finite Time Blow-Up). *Let $\gamma(x) \equiv 0$ and let $a \in C^1(\mathbb{R}^d)$ be radially symmetric, strictly positive and such that both a and ∇a are bounded. Suppose also that $a(0) = \min a(x)$ and that there exists a neighborhood $|x| < \delta_0$ such that $a(x)$ is radially non-decreasing. Then for all $M > M_c$, there exists a solution $u(t)$ with $M(u) = M$ which blows up in finite time, e.g. $T_+(u(0)) < \infty$.*

Remark 3. The requirement that a and ∇a be uniformly bounded are only used to satisfy the hypotheses of Theorem 1, which ensures we have a well-understood local existence, uniqueness and stability theory.

The critical mass (7) only depends on $\min a(x)$, hence one expects the blow-up solution constructed in Theorem 6 should concentrate a sufficient amount of mass near where that minimum is achieved. Exactly how concentrated the initial data is required to be is characterized by (8) in the following proposition, which requires at least part of the initial data be more concentrated than a particular rescaled extremal of the sharp HLS (Theorem 4). Theorem 6 is proved by comparing the true solution against a barrier, and (8) below is the requirement that the solution and the barrier are ordered at time zero. Remark 4 clarifies how (8) requires u_0 to concentrate around where the minimum is achieved at least when $M(u_0) \searrow M_c$.

Proposition 1. *Let $u_0 \in L^1_+(\mathbb{R}^d;(1+|x|^2)dx)\cap C^0(\mathbb{R}^d)$ be radially symmetric such that $M_c < M(u_0)$ and suppose that there is an $R_0 \leq \delta_0$ and an M_0 with $M_c < M_0 < M(u_0)$ such that for $0 \leq r \leq R_0$ we have*
$$\int_{|x|\leq r} \left(\frac{a(R_0)^{1/2}}{R_0}\right)^d V\left(\frac{x}{R_0}\right) dx \leq \left(\frac{M_c}{M_0}\right) \int_{|x|\leq r} u_0(x)dx. \tag{8}$$
Then the weak solution $u(t)$ associated with u_0 blows up in finite time. Moreover, if we define $\mu := M_c/M_0 < 1$, we have the following estimate of the blow-up time:
$$T_+(u_0) \leq \mu^{2/d-1} \frac{\sigma R_0^d}{a(R_0)(d-1)M_c(\mu^{-2/d}-1)} < \infty, \tag{9}$$
where σ is the surface area of the unit sphere in $\mathbb{R}^d$.

Remark 4. In light of Remark 1, (8) implies that
$$M_0\left(\frac{a(R_0)}{a(0)}\right)^{d/2} \leq \int_{|x|\leq R_0} u_0(x)dx.$$
Hence, in order to construct blow-up solutions with $M(u_0) \searrow M_c$, in general we need to choose increasingly concentrated initial data by sending $R_0 \to 0$.

As mentioned above, Theorem 6 and Proposition 1 also provide new results for the homogeneous problem, $a(x) \equiv 1$, $\gamma(x) \equiv 0$. The Virial method used in [8] proves blow-up for all solutions with negative free energy, without the need for radial symmetry. On the other hand, Theorem 6 and Proposition 1 require radial symmetry but do not require any assumptions on the free energy. Indeed we have the following Corollary of Proposition 1, which in particular, shows the existence of solutions with arbitrarily large initial free energy that blow up in finite time.

Corollary 1. *Let $u_0 \in L^1_+(\mathbb{R}^d;(1+|x|^2)dx)\cap C^0(\mathbb{R}^d)$ be radially symmetric, strictly positive in a compact neighborhood of the origin and satisfy*
$$M(u_0) > \left(\frac{2}{(m-1)C_\star c_d}\right)^{d/2}.$$
Then the weak solution $u(t)$ associated with u_0 of the scale-invariant problem ($a(x) \equiv 1$, $\gamma(x) \equiv 0$) blows up in finite time. In particular, for every $F_0 \geq 0$, there exists a solution $u(t)$ with $\mathcal{F}(u(0)) > F_0$ which blows up in finite time.

2 Global Existence

The proof of Theorem 5 hinges primarily on providing a precise decomposition of the potential energy,
$$\int u(x)c(x)dx,$$
into a leading order critical part and another part that is subcritical. This is the purpose of the following section.

2.1 Approximate Inverse of Chemo-attractant PDE

We use the standard symbol classes studied in for example [30], summarized in the following definition.

Definition 2 (Symbol Class S^s, $s \in \mathbb{R}$)**.** Suppose $b(x,\xi) \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ satisfies

$$\left|\partial_x^\beta \partial_\xi^\alpha b(x,\xi)\right| \lesssim_{\beta,\alpha} (1+|\xi|)^{s-|\alpha|},$$

for multi-indices α, β. Then we say both b and the associated pseudo-differential operator (ΨDO) T_b defined by

$$T_b f(x) = \frac{1}{(2\pi)^{d/2}} \int b(x,\xi)\hat{f}(\xi)e^{ix\xi}d\xi$$

are in the symbol class S^s and say the symbol b or the operator T_b are of order s. We also denote $b(x,\xi) = \mathrm{sym}(T_b)$.

Notice that with this definition $S^{s_1} \subset S^{s_2}$ whenever $s_1 < s_2$. Also, since the symbols are required to be smooth, these operators do not carry too much low-frequency information, unlike multiplier or symbol classes that allow singularities at the origin. For the standard relevant facts regarding these symbol classes, such as the symbolic calculus, localization estimates, boundedness on Sobolev spaces and singular integral representations, see Chapter 6 of [30].

Consider the PDE

$$Lc := -a(x)\Delta c - \nabla a(x) \cdot \nabla c + \gamma(x)c = u. \tag{10}$$

By definition, L is a pseudo-differential operator in S^2:

$$Lc = \frac{1}{(2\pi)^{d/2}} \int \Big(\gamma(x) + a(x)\,|\xi|^2 - i\xi \cdot \nabla a(x)\Big)\,\hat{c}(\xi)e^{ix\xi}d\xi.$$

Consider the approximate inverse of L, the S^{-2} class ΨDO

$$A_H u := \frac{1}{(2\pi)^{d/2}} \int \frac{\Phi(\xi)\hat{u}(\xi)e^{ix\xi}}{a(x)\,|\xi|^2 - i\xi \cdot \nabla a(x) + \gamma(x)} d\xi,$$

where $\Phi(\xi) = \prod_{j=1}^d \phi(\xi_j)$ with $\phi(t)$ a smooth function such that $0 \le \phi(t) \le 1$ which is identically one for $|t| \ge 1$ and vanishes in a neighborhood of zero. We remark that if $\gamma(x)$ is strictly positive, we do not need the cut-off $\Phi(\xi)$. By the symbolic calculus [Chapter 6, Theorem 2 [30]],

$$A_H L c = c + T_E c,$$

where the operator $T_E \in S^{-1}$ and the associated symbol E has the following asymptotic expansion

$$E \sim \Phi(\xi) - 1 + \sum_{|\alpha|\ge 1} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \mathrm{sym}(A_H)\partial_x^\alpha \mathrm{sym}(L),$$

in the sense that the error in truncating the series for $N > |\alpha|$ is a symbol of class S^{-1-N}.

Although both A_H and T_E are bounded operators from L^p to itself for $1 < p < \infty$ (Chapter 6 [30]), if $\gamma(x)$ is not strictly positive then L^{-1}, the true inverse, is not. Necessarily, the low-frequency portion of L^{-1} is still present implicitly in $T_E c$. Instead of being bounded on L^p to itself, L^{-1} satisfies the following: for $1 < q < d/2$, $d/(d-2) < p < \infty$ and $\frac{2}{d} + \frac{1}{p} = \frac{1}{q}$,

$$\|c\|_p = \|L^{-1}u\|_p \lesssim \|u\|_q. \tag{11}$$

This can be seen at least formally by multiplying both sides of (10) by $c^{(d-2)p/d-1}$ integrating, and applying the homogeneous Sobolev embedding.

We may formally write down the operator A_H as a singular integral operator by interchanging the integrals:

$$\begin{aligned} A_H u(x) &= \frac{1}{(2\pi)^d} \int \int u(y) \frac{\Phi(\xi) e^{i\xi(x-y)}}{a(x)\,|\xi|^2 - i\xi \cdot \nabla a(x) + \gamma(x)} dy d\xi \\ &= \frac{1}{(2\pi)^d} \int u(y) \left[\int \frac{\Phi(\xi) e^{i\xi(x-y)}}{a(x)\,|\xi|^2 - i\xi \cdot \nabla a(x) + \gamma(x)} d\xi \right] dy \\ &:= \int u(y) K_H(x,y) dy. \end{aligned}$$

The integral for $K_H(x,y)$ is not absolutely convergent so we cannot naively apply Fubini's theorem in the above computation rigorously, but it can be justified by a standard limiting procedure, as in [30]. The key technical lemma for the proof of Theorem 5 is the following characterization of $K_H(x,y)$.

Lemma 1 (Asymptotic Expansion for $K_H(x,y)$)**.** *Let $K_H(x,y)$ be defined as above by the conditionally convergent integral*

$$K_H(x,y) := \frac{1}{(2\pi)^d} \int \frac{\Phi(\xi) e^{i\xi(x-y)}}{a(x)\,|\xi|^2 - i\xi \cdot \nabla a(x) + \gamma(x)} d\xi.$$

Then we then have the following asymptotic expansion which holds uniformly in $x \in \mathbb{R}^d$,

$$\begin{aligned} K_H(x,y) &= \frac{\Gamma(d/2+1)}{d(d-2)\pi^{d/2} a(x)} |x-y|^{2-d} + o(|x-y|^{2-d}) \quad \text{as } y \to x \\ &= \frac{c_d}{a(x)} |x-y|^{2-d} + o(|x-y|^{2-d}) \quad \text{as } y \to x, \end{aligned} \tag{12}$$

with c_d given above by (5). Moreover, recall that for all $\delta > 0$ and $N > 0$ (see for example pg 235 [30]),

$$|K_H(x,y)| \lesssim_{\delta,N} |x-y|^{-N}, \quad |x-y| > \delta. \tag{13}$$

Proof. The bound (13) is a standard consequence of $A_H \in S^{-2}$. Such localization should not be surprising since the the low frequency contribution of L^{-1} is not included in A_H due to the cut-off Φ. Hence, we focus on (12). Note the trick

$$\frac{1}{D} = \int_0^\infty e^{-tD} dt.$$

Hence,

$$\begin{aligned} K_H(x,y) &= \frac{1}{(2\pi)^d} \int_0^\infty \int \Phi(\xi) e^{i\xi\cdot(x-y+t\nabla a(x)) - ta(x)|\xi|^2 - t\gamma(x)} d\xi dt \\ &= \frac{1}{(2\pi)^d} \prod_{j=1}^d \int_0^\infty e^{-t\gamma(x)} \int_{-\infty}^\infty \phi(\xi_j) e^{i\xi_j(x_j - y_j + t\partial_{x_j} a(x)) - ta(x)\xi_j^2} d\xi_j dt. \end{aligned}$$

Now define the complex change of variable $z_j = (ta(x))^{1/2}\xi_j - i\frac{x_j - y_j + t\partial_{x_j}a(x)}{2(ta(x))^{1/2}}$,

$$\int_{-\infty}^{\infty} \phi(\xi_j) e^{i\xi_j(x_j - y_j + t\partial_{x_j}a(x)) - ta(x)\xi_j^2} d\xi_j dt = \frac{e^{-\frac{\left|x_j - y_j + t\partial_{x_j}a(x)\right|^2}{4ta(x)}}}{(ta)^{1/2}} \int_{\Gamma_j} \phi\left(\frac{\operatorname{Re} z_j}{(ta(x))^{1/2}}\right) e^{-z_j^2} dz_j$$

$$:= \frac{e^{-\frac{\left|x_j - y_j + t\partial_{x_j}a(x)\right|^2}{4ta(x)}}}{(ta(x))^{1/2}} f_j(t),$$

where Γ_j is the contour $\left\{\operatorname{Im} z_j = \frac{x_j - y_j + t\partial_{x_j}a(x)}{2(ta(x))^{1/2}}\right\}$.

Applying the above change of variables to the expression for $K_H(x,y)$ implies

$$K_H(x,y) = \frac{e^{-(x-y)\nabla a(x)/(2a(x))}}{(2\pi)^d (a(x))^{d/2}} \int_0^\infty t^{-d/2} \left[\prod_{j=1}^d f_j(t)\right] e^{-t\gamma(x) - t\frac{|\nabla a(x)|^2}{4a(x)} - \frac{|x-y|^2}{4ta(x)}} dt.$$

We make the following additional change of variables,

$$t = \frac{|x-y|^2}{4a(x)\zeta^2},$$

which then yields

$$\zeta = \frac{|x-y|}{(4ta(x))^{1/2}} \quad \text{and} \quad dt = -\frac{|x-y|^2}{2a(x)\zeta^3} d\zeta.$$

In terms of ζ, $K_H(x,y)$ can be now written as

$$K_H(x,y) = \frac{e^{-(x-y)\nabla a(x)/(2a(x))}}{(\pi)^d (2a(x))} |x-y|^{2-d} \int_0^\infty \zeta^{d-3} \left[\prod_{j=1}^d f_j(t(\zeta))\right] e^{-\frac{|x-y|^2}{4a(x)\zeta^2}\left(\frac{|\nabla a(x)|^2}{4a(x)} + \gamma(x)\right) - \zeta^2} d\zeta.$$

Due to the smoothness of ϕ and a and the strict lower bound on a we have the uniform (in x) convergence of the integral (note that here ζ is fixed and z_j is the complex integration variable)

$$\lim_{y\to x} f_j(t(\zeta)) = \lim_{y\to x} \int_{\Gamma_j} \phi\left(\frac{2|\zeta| \operatorname{Re} z_j}{|x-y|}\right) e^{-z_j^2} dz_j = \int_{\mathbb{R}} e^{-z_j^2} dz_j = \pi^{1/2}.$$

Similarly we also have

$$\lim_{y\to x} \int_0^\infty \zeta^{d-3} \left[\prod_{j=1}^d f_j(t(\zeta))\right] e^{-\frac{|x-y|^2}{4a(x)\zeta^2}\left(\frac{|\nabla a(x)|^2}{4a(x)} + \gamma(x)\right) - \zeta^2} d\zeta = \pi^{d/2} \int_0^\infty \zeta^{d-3} e^{-\zeta^2} d\zeta,$$

uniformly in $x \in \mathbb{R}^d$ due to the uniform continuity of $\nabla a(x), a(x)$, and $\gamma(x)$ as well as the strict positivity of a. Recalling elementary facts about the Gamma function we have,

$$\int_0^\infty \zeta^{d-3} e^{-\zeta^2} d\zeta = \frac{1}{2}\Gamma\left(\frac{d}{2} - 1\right) = \frac{2}{d(d-2)}\Gamma\left(\frac{d}{2} + 1\right).$$

Hence,

$$K_H(x,y) = \frac{\Gamma\left(\frac{d}{2} + 1\right)}{d(d-2)\pi^{d/2} a(x)} |x-y|^{2-d} + o_{y\to x}\left(|x-y|^{2-d}\right),$$

and (12) is proved. $\square$

The error term $T_E c$ in the approximate inverse can be controlled by the following lemma. Here we take advantage of the smoothing nature of $T_E \in S^{-1}$ to show that in the potential energy, this term is subcritical in the sense that the effective power of $\|u\|_m$ associated with the term is strictly less than m. Naturally, one must eventually use (11) in order to prove this lemma.

Lemma 2 (The error term is subcritical)**.** *Let $d \geq 3$, suppose $u \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$ and c is the strong solution of* (10) *which vanishes at infinity. Then,*

$$\int u T_E c dx \lesssim \|u\|_1^{2-2\theta} \|u\|_m^{2\theta}, \tag{14}$$

for some $0 < 2\theta < m = 2 - 2/d$.

Proof. Define

$$p = \frac{4d}{2d-3}.$$

Define the standard Fourier multiplier $\widehat{\langle \nabla \rangle^s f} := \left(1 + |\xi|^2\right)^{s/2} \hat{f}(\xi)$. Since the multiplier $\langle \nabla \rangle^{1/2}$ is self-adjoint we have,

$$\left| \int u T_E c dx \right| \leq \|\langle \nabla \rangle^{-1/2} u\|_{\frac{p}{p-1}} \|\langle \nabla \rangle^{1/2} T_E c\|_p$$

Define

$$\frac{1}{q} = \frac{1}{p} + \frac{2}{d} = \frac{2d+5}{4d} < 1.$$

As mentioned above, Definition 2 implies that since $T_E \in S^{-1}$, we also have $T_E \in S^{-1/2}$. Hence, T_E is a bounded operator L^p to $W^{1/2,p}$ [Chapter 6, Proposition 5 [30]]. Using this and the elliptic L^p estimate (11) we have

$$\|\langle \nabla \rangle^{1/2} T_E c\|_p \lesssim \|c\|_p \lesssim \|u\|_q.$$

One may easily verify that

$$\frac{1}{q} = \frac{p-1}{p} + \frac{1}{2d},$$

and therefore the inhomogeneous Sobolev embedding theorem implies

$$\|\langle \nabla \rangle^{-1/2} u\|_{\frac{p}{p-1}} \lesssim \|u\|_q.$$

Hence, we see the relevance of q as we have in total,

$$\left| \int u T_E c dx \right| \lesssim \|u\|_q^2.$$

In order to interpolate between L^1 and L^m, we obviously need $q < m = 2 - 2/d$, which follows easily from $d \geq 3$. Then, for

$$\theta = \frac{(2d-5)(2d-2)}{4d(d-2)} \in (0,1)$$

we have,

$$\left| \int u T_E c dx \right| \lesssim \|u\|_1^{2-2\theta} \|u\|_m^{2\theta}.$$

To prove subcriticality it remains to confirm that we have $2\theta < m$, which again follows from $d \geq 3$. $\square$

2.2 Proof of Theorem 5

In this section we complete the proof of Theorem 5.

Proof. We prove Theorem 5 by producing a uniform in time bound on the entropy (which is basically just the L^m norm). This in turn proves that the solution is uniformly equi-integrable and Theorem 1 completes the proof.

By the energy dissipation inequality (4) and the definition of A_H we have,

$$\frac{1}{m-1}\int u^m dx - \frac{1}{2}\int uA_H u dx - \frac{1}{2}\int uT_E c dx \leq \mathcal{F}(u_0).$$

By (14) (Lemma 2) we then have, for some $0 < 2\theta < m = 2 - 2/d$,

$$\frac{1}{m-1}\int u^m dx - \frac{1}{2}\int uA_H u dx - C\|u\|_1^{2-2\theta}\|u\|_m^{2\theta} \leq \mathcal{F}(u_0).$$

Using Lemma 1, for every $\epsilon > 0$, we may choose a $\delta > 0$ such that

$$\int uA_H u dx \leq \int\int_{|x-y|<\delta}\left(\frac{c_d}{a(x)} + \epsilon\right)\frac{u(x)u(y)}{|x-y|^{d-2}}dxdy + C(\delta)\|u\|_1^2,$$

where c_d, given by (5), is the normalization constant in the Newtonian potential. Hence, by the sharp Hardy-Littlewood-Sobolev inequality (6) we have,

$$\left(\frac{1}{m-1} - \frac{C_\star}{2}\left(\frac{c_d}{\min a(x)} + \epsilon\right)\|u\|_1^{2/d}\right)\|u\|_m^m \leq C\|u\|_1^{2-2\theta}\|u\|_m^{2\theta} + C(\delta)\|u\|_1^2 + \mathcal{F}(u_0).$$

If $M(u) < M_c$ with M_c given by (7) then we may choose ϵ sufficiently small such that the first term is positive. Since $m > 2\theta$ this then implies a global uniform-in-time bound on $\|u(t)\|_m$. This in turn implies a global L^∞ bound on $u(t)$ by the continuation criterion in Theorem 1. $\square$

Remark 5. It appears the proof of Theorem 5 would require some refinement in order to treat the $\mathbb{R}^2$ case. Certainly, the asymptotic expansion of $K_H(x,y)$ along the diagonal $x \sim y$ would need to be refined in order to capture the logarithmic singularity accurately (Lemma 1). Moreover, Lemma 2 would also need to be adjusted in order to yield an error which is subcritical relative to the positive part of the entropy $\int u(\log u)_+ dx$. In the place of the HLS (6), the logarithmic HLS would instead be used [14].

3 Finite Time Blow-Up

As discussed in the introduction, the inability to make obvious use of a Virial method motivates our use of a barrier method based on maximum principle-type arguments. We begin with the following rescaling: let $M_c < M_0 < M(u_0)$ be as in the statement of Proposition 1, define

$$\mu := M_c/M_0 < 1 \tag{15}$$

and let

$$\rho(t,x) := \mu u(\mu^{1-2/d}t, x). \tag{16}$$

Then $M(\rho) = M(u)\mu > M_c$ and ρ solves

$$\begin{cases} \rho_t + \nabla\cdot(\rho\mu^{1-2/d}\nabla c) = \Delta\rho^{2-2/d} \\ -\nabla\cdot(a(x)\nabla c) = \mu^{-1}\rho. \end{cases} \tag{17}$$

As mentioned above, we will use mass comparison arguments involving suitably chosen barriers (sub-solutions) to force the finite time concentration of mass in (17). We define the following comparison function $\bar{u} = \bar{u}(t, x)$,

$$\bar{u}(t,x) = \frac{a(R_0)^{d/2}}{R(t)^d} V\left(\frac{x}{R(t)}\right), \tag{18}$$

where we take $R(t)$ as a solution to the initial value problem

$$\begin{cases} \dot{R}(t) = \frac{M_c(1-M_0^{2/d}M_c^{-2/d})}{a(R_0)\sigma R(t)^{d-1}} \\ R(0) = R_0. \end{cases} \tag{19}$$

Here σ denotes the surface area of the unit sphere. Note that $R(T_\star) = 0$ with

$$T_\star := T_\star(M_0, R_0, d) = \frac{R_0^d \sigma}{(d-1)a(R_0)M_c(\mu^{-2/d} - 1)} < \infty. \tag{20}$$

We define the mass distributions

$$M(t,r) = \int_{|x|\leq r} \rho(t,x)dx, \quad \overline{M}(t,r) = \int_{|x|\leq r} \bar{u}(t,x)dx.$$

Notice that (8) is equivalent to $\overline{M}(r,0) \leq M(r,0)$ for all $r \in [0, R_0]$, which means that the rescaled initial data is initially more concentrated than the barrier $\bar{u}$ on the neighborhood $r \leq R_0$. It is also important to note that the total mass of the barrier $\bar{u}$ is generally more than the critical mass M_c but less than or equal to the total mass of ρ which itself has less mass than the true solution u.

Suppose that $\overline{M}(0,r) \leq M(0,r)$ for all $r \geq 0$. We will show that this ordering is preserved up to the blow-up time of u or $\bar{u}$, e.g.

$$\overline{M}(t,r) \leq M(t,r) \text{ for } 0 \leq t < \min\left(T_\star, T_+(u_0)\right).$$

As alluded to above, in the language of maximum principle-type arguments, $\bar{u}$ plays the role of a *subsolution* in terms of the mass concentration. As $\bar{u}$ concentrates into a delta mass at $t = T_\star$, we must have $T_+(u_0) \leq T_\star$, which will conclude the proof of Theorem 6. The intuition for why the proof ultimately works is based on the fact that the rescaled system (17) pulls mass into the origin faster than the PDE that $a^{d/2}(R_0)V$ solves (see Remark 1). That is, the rescaling (16) transfers the property of having supercritical mass into surplus attractive power when compared against stationary solutions of roughly comparable mass. The surplus attractive power is what gives us the ability to choose $R(t)$ at the rate given in (19) and hence prove that the solution is concentrating fast enough to be squeezed into blow-up by a self-similar barrier $\bar{u}$.

We use mass comparison arguments influenced by those found in [27] to prove that the mass ordering (3) is preserved. By quantifying how much mass is being transported into or away from the origin, such arguments are surprisingly natural and allow easy treatment of advective terms in comparison arguments. It is this additional precision in the treatment of advection that allows the proof to work. The arguments in [27] were also influenced by the mass comparison arguments used in the treatment of the porous media equation (see [34, 33] and the references therein). Our argument is similar also to the subsolution blow-up argument of Jäger and Luckhaus [24], where they treated the homogeneous problem with linear diffusion on bounded domains. However, we feel that our barriers provide a more flexible construction that can be easily applied to different situations. We point out that in our case we compare weak solutions to properly chosen barrier functions but we do not necessarily have comparison between weak solutions.

An important complication arises at the free boundary of the positivity set of $u(t)$. Here, classical regularity breaks down and the mass comparison arguments no longer provide a rigorous argument, or even a convincing formal one. To deal with this technical issue we lift to strictly positive solutions, which are smooth due to uniform parabolicity on compact sets. Comparison with vanishing error is proved against these solutions, for which the formal arguments are rigorous. Passing to the limit requires some standard approximation arguments regarding the stability of (17).

3.1 Preliminaries: Approximation and Regularity

In this section we detail the important approximation and regularity properties of (1). These results are more or less expected, but since they are of independent interest and important for making our arguments rigorous, we include brief sketches of the proofs.

Lemma 3. *Suppose $u_0 \in C^0(\mathbb{R}^d) \cap L^1_+(\mathbb{R}^d; (1+|x|^2)dx)$ and let $u(t)$ be the associated weak solution which satisfies $u(0) = u_0$. Then for all $\epsilon > 0$,*

(a) ∇c is continuous and bounded, and Δc is bounded on $t \in [0, T_+(u_0) - \epsilon)$.

(b) $u(t,x) \in C^0([0, T_+(u_0) - \epsilon) \times \mathbb{R}^d)$.

Proof. (a) By definition of blow-up time T_+, u is bounded up to $t = T_+ - \epsilon$. By a Gagliardo-Nirenberg-Sobolev-type inequality (see e.g. Theorem 7.28 in [21]), for all $1 < p < \infty$,

$$\|\nabla c\|_p \lesssim_{p,d} \|c\|_p + \|D^2 c\|_p.$$

Hence, for any $d < p < \infty$ by Morrey's inequality we have,

$$\|\nabla c\|_\infty \lesssim \|c\|_p + \|D^2 c\|_p.$$

Solutions to the PDE for c satisfy the global elliptic gradient estimate (see for example [1]), for $p > d/(d-2)$,

$$\|D^2 c\|_p \lesssim \|u\|_p + \|u\|_{\frac{dp}{2p+d}}.$$

Hence by the elliptic L^p estimate (11)

$$\|\nabla c\|_\infty \lesssim \|u\|_p + \|u\|_{\frac{dp}{2p+d}}.$$

In fact, Morrey's inequality implies that ∇c is $C^{0,\alpha}$ Hölder continuous for all exponents $\alpha < 1$. From the PDE for c we have,

$$\|a(x)\Delta c\|_\infty \le \|u\|_\infty + \|\nabla a\|_\infty \|\nabla c\|_\infty$$
$$\|\Delta c\|_\infty \le \frac{1}{\min a}\left(\|u\|_\infty + \|\nabla a\|_\infty \|\nabla c\|_\infty\right).$$

(b) Theorem 6.1 of DiBenedetto [19] together with (a) yields (b) (see also Theorem 3.1 of [27]). □

Lemma 4 (Regularity of V)**.** *The extremal V of Theorem 3 is in $C^1(\mathbb{R}^d)$.*

Proof. Recall that V solves the elliptic PDE

$$\Delta V^m = \nabla \cdot (V \nabla \mathcal{N} * V).$$

As shown in [8], it follows from elliptic regularity theory that V is C^2 in its positive set, $\{|x| < 1\}$, and vanishes continuously at $|x| = 1$.

Now let us prove that the gradient of $V(x)$ uniformly vanishes to zero as x approaches $|x| = 1$. Integration by part in the ball $\{|x| \leq r\}$ as well as the radial symmetry of V yields the following for $|x| < 1$:

$$-\frac{x}{|x|} \cdot \nabla V^m(x) = (V|\nabla V^{m-1}| + V^{m-1}|\nabla V|) = -\frac{x}{|x|} \cdot \nabla(\mathcal{N} * V)V$$

Dividing by V, we obtain

$$|\nabla V^{m-1}| + V^{m-2}|\nabla V| \leq |\nabla(\mathcal{N} * V)| \text{ for } |x| < 1.$$

Note that $|\nabla(\mathcal{N} * V)|$ is bounded since V is bounded and compactly supported. It follows that $V^{m-2}|\nabla V|$ is bounded as $|x| \to 1$. Since $m < 2$, V^{m-2} diverges to infinity as $|x| \to 1$: it follows that $|\nabla V|$ must vanish as $|x| \to 1$, and we can conclude. $\square$

Lemma 5 (Regularity of Strictly Positive Solutions)**.** *Let $u_0 \in L^1_+(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ be strictly positive a.e.. Then for all $\epsilon > 0$, the associated weak solution $u(t)$ with $u(0) = u_0$ is smooth and strictly positive on $(0, T_+(u_0) - \epsilon) \times \mathbb{R}^d$.*

Proof. Due to Lemma 8 in the appendix, u stays strictly positive for all $0 \leq t < T_+(u_0)$. Consequently u is a solution of a uniformly parabolic quasilinear PDE of divergence form, and the regularity of u follows from classical regularity theory [28]. $\square$

Lemma 6 (Stability of the blow-up time)**.** *Let $\{u_0^n\} \subset L^1_+(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d)$ and $u_n(t)$ denote the associated weak solutions of* (1) *with $u_n(0) = u_0^n$ defined on the intervals $[0, T_+(u_0^n))$. Suppose further that*

(a) $\sup_n (\|u_0^n\|_\infty + \|u_0^n\|_1) < \infty$ *and*

(b) $u_0^n \to u_0$ *strongly in* L^1 *for some* $u_0 \in L^1_+(\mathbb{R}^d; (1 + |x|^2)dx) \cap L^\infty(\mathbb{R}^d)$.

Let $u(t)$ be the solution to (1) *with $u(0) = u_0$, and define $T_0 > 0$ such that,*

$$T_0 := \sup \left\{ T \in (0, \infty) : \liminf_{n\to\infty} \sup_{t\in[0,T]} \|u_n(t)\|_\infty < \infty \right\}.$$

Then for all $T < T_0$, there exists a subsequence such that $u_{n_k} \to u$ in $C([0, T], L^p(\mathbb{R}^d))$ for all $1 \leq p < \infty$. If additionally $u_0^n \in C^0(\mathbb{R}^d)$ and $u_0^n \to u_0$ locally uniformly, then we also have $u_{n_k} \to u$ locally uniformly on $[0, T] \times \mathbb{R}^d$. Moreover, $T_+(u_0) = T_0 \leq \liminf_{n\to\infty} T_+(u_0^n)$.

Proof. From the local existence theory and the fact $\sup_n (\|u_0^n\|_\infty + \|u_0^n\|_1) < \infty$, it is assured that $T_0 > 0$ (see [2] or [8]). Let $0 < T < T_0$. By the precompactness arguments of the local existence theory (see [2, 1]) we may extract a subsequence $\{u_{n_k}\}$ which converges in $C([0, T]; L^p(\mathbb{R}^d))$ for all $1 \leq p < \infty$ to a weak solution, and by uniqueness of weak solutions, the limit must be $u(t)$. In particular, we may extend $u(t)$ to include any time interval such that

$$\liminf_{n\to\infty} \sup_{t\in[0,T]} \|u_n(t)\|_\infty < \infty.$$

By the proof of Lemma 3 and Theorem 6.1 of [19], if $u_0^n \to u_0$ locally uniformly, it moreover follows that $u_{n_k}(t) \to u(t)$ locally uniformly up to extraction of an additional subsequence. By the continuation criterion in Theorem 1, it is necessary that $T_0 \leq \liminf_{n\to\infty} T_+(u_0^n)$ but there is no a priori reason for them to be comparable (for instance, T_0 could be finite, but $T_+(u_0^n) \equiv \infty$). However, blow-up times are at least semi-continuous. From the first part of the lemma, $u(t)$ exists on all compact time intervals of $[0, T_0)$. Since $u_{n_k}(t) \to u(t)$ locally uniformly on this time interval, if $T_0 < \infty$, then we necessarily have $\liminf_{t\nearrow T_0} \|u(t)\|_\infty = \infty$ and therefore $T_+(u_0) = T_0$. This proves the lemma. $\square$

3.2 Mass Comparison

In this section we develop the comparison arguments and prove Theorem 6.

First, note that $\bar{u}(t)$ is a classical solution to the transport equation

$$\partial_t \bar{u} + \nabla \cdot (\bar{u}\frac{\dot{R}}{R}x) = 0. \tag{21}$$

Of course, for all t, $\bar{u}(t,x)$ is also a weak solution to the scale-invariant problem (Remark 1),

$$\frac{1}{a(R_0)}\nabla \cdot (\bar{u}\nabla\mathcal{N} * \bar{u}) = \Delta \bar{u}^m. \tag{22}$$

We have the following lemma which describes the PDE satisfied by the mass function corresponding to $\rho(t)$.

Lemma 7 (Evolution of Mass Function)**.** *Let $\rho(t,x)$ be a smooth radially symmetric solution to* (17)*. Then, $M(t,r) := \int_{|x|\leq r} \rho(t,x)dx$ satisfies*

$$\partial_t M(t,r) = \sigma r^{d-1}\partial_r \left(\frac{\partial_r M(r)}{\sigma r^{d-1}}\right)^m + \frac{\mu^{-2/d}}{a(r)}\frac{M(r)}{\sigma r^{d-1}}\partial_r M(r), \tag{23}$$

where σ is the surface area of the unit sphere in $\mathbb{R}^d$.

Proof. By radial symmetry

$$\rho(t,r) = \frac{1}{\sigma r^{d-1}}\partial_r M(t,r)$$

and by the divergence theorem and the radial symmetry of a,

$$a(r)\int_{|x|=r} \partial_r c(t,x) dS = -\frac{\mu^{-1}}{\sigma r^{d-1}} M(t,r). \tag{24}$$

Now, using the divergence theorem and radial symmetry,

$$\begin{aligned}
\partial_t M(t,r) &= \int_{|x|=r} \frac{x}{|x|}\cdot\left(\nabla\rho^m - \mu^{1-2/d}\rho\nabla c\right) dS \\
&= \sigma r^{d-1}\partial_r\left(\frac{\partial_r M(t,r)}{\sigma r^{d-1}}\right)^m - \frac{\mu^{1-2/d}}{\sigma r^{d-1}}\partial_r M(t,r)\int_{|x|=r}\partial_r c(t,r)dS \\
&= \sigma r^{d-1}\partial_r\left(\frac{\partial_r M(t,r)}{\sigma r^{d-1}}\right)^m + \frac{\mu^{-2/d}}{a(r)}\frac{\partial_r M(t,r)}{\sigma r^{d-1}} M(t,r).
\end{aligned}$$

$\square$

Suppose u_0 satisfies (8) and let $\rho_\epsilon(t)$ solve (17) with initial data

$$\rho_\epsilon(0) = \rho_0 + \epsilon \frac{e^{-|x|^2/4}}{(4\pi)^{d/2}}.$$

By Lemma 5, $\rho_\epsilon(t,r)$ remain smooth and positive on their time interval of existence $(0, T_+(\rho_\epsilon(0)))$.

We are now ready to state and prove the mass comparison result which will complete the proof of Theorem 6.

Proposition 2. *Suppose that $T < \min(T_\star, T_+(\rho_\epsilon(0)))$ and let $M_\epsilon(t,r) := \int_{|x|\le r} \rho_\epsilon(t,x)dx$. Further suppose $\overline{M}(0,r) \le M_\epsilon(0,r)$ for all $r \ge 0$. Then we have*

$$\overline{M}(t,r) - M_\epsilon(t,r) \le 0 \text{ in } [0,T] \times \mathbb{R}^d.$$

Proof. Similar to the proof of Lemma 7, by (22) we have,

$$\sigma r^{d-1} \partial_r \left(\frac{\partial_r \overline{M}(r)}{\sigma r^{d-1}} \right)^m + \frac{1}{a(R_0)} \frac{\overline{M}(r)}{\sigma r^{d-1}} \partial_r \overline{M}(r) = 0, \tag{25}$$

and by (21),

$$\begin{aligned} \partial_t \overline{M}(r) &= -\int_{|x|=r} \bar{u}(r) \frac{r}{R} \dot{R} dS \\ &= -\frac{r}{R} \dot{R} \partial_r \overline{M}(r). \end{aligned} \tag{26}$$

By Lemma 4, these equations both hold in the strong sense everywhere, but for this proof we will only need them in the positivity set. For notational simplicity, define $M(t,r) := M_\epsilon(t,r)$.

Consider the space-time region $(t,r) \in Q_T$, where

$$Q_T = \{(t,r) : t \in [0,T], \ \ r \in [0, R(t)]\}.$$

As $M(\bar{u}) \equiv \overline{M}(t,r)$ in $|x| \ge R(t)$, we need only prove the comparison result in Q_T, from which the result on $(0,T) \times \mathbb{R}^d$ follows.

For a given constant $\lambda > 0$ (to be chosen later), let us consider the function

$$f(t,r) := (\overline{M}(t,r) - M(t,r))e^{-\lambda t} \text{ in } [0,T] \times \mathbb{R}^+.$$

Note that $f(0,r) \le 0$. If $f(t,r) \le 0$ in Q_T there is nothing to prove, so suppose that it is positive somewhere. Then $f(t,r)$ has a strictly positive maximum in Q_T, which is achieved at some point $(t_\star, r_\star)$. Necessarily, $r_\star > 0$. If $r_\star = R(t_\star)$ then by $r_\star$ being the location of the maximum, we must have

$$0 = \partial_r \overline{M}(t_\star, r_\star) \ge \partial_r M(t_\star, r_\star) = \sigma r_*^{d-1} u^\epsilon(t_\star, r_\star).$$

Since u_ϵ is strictly positive, it follows that $r_\star < R(t_\star)$. This implies that due to maximization,

(A) $\partial_t(\overline{M}(t_\star, r_\star) - M(t_\star, r_\star)) \ge \lambda(\overline{M}(t_\star, r_\star) - M(t_\star, r_\star))$.

(B) $\partial_r \overline{M}(t_\star, r_\star) = \partial_r M(t_\star, r_\star)$.

(C) $\partial_{rr} M(t_\star, r_\star) \ge \partial_{rr} \overline{M}(t_\star, r_\star)$.

Using the mass equations satisfied by each function we have, by (23), (26) and (25),

$$\begin{aligned}\partial_t(\overline{M}-M)(t_\star,r_\star) &= \sigma r_\star^{d-1}\partial_r\left(\frac{\partial_r\overline{M}}{\sigma r_\star^{d-1}}\right)^m + \frac{1}{a(R_0)}\frac{\overline{M}}{\sigma r_\star^{d-1}}\partial_r\overline{M} \\ &\quad - \sigma r_\star^{d-1}\partial_r\left(\frac{\partial_r M}{\sigma r_\star^{d-1}}\right)^m - \frac{\mu^{-2/d}}{a(r_\star)}\frac{M}{\sigma r_\star^{d-1}}\partial_r M - \partial_r\overline{M}\frac{r_\star}{R(t_\star)}\dot{R}(t_\star).\end{aligned}$$

where all terms are evaluated at $(t_\star, r_\star)$. By (B) and (C), we can order the higher order nonlinearity coming from the diffusion as well as relate the advection terms,

$$\partial_t(\overline{M}-M)(t_\star,r_\star) \leq \frac{\partial_r\overline{M}}{\sigma r_\star^{d-1}}\left(\frac{1}{a(R_0)}\overline{M} - \frac{\mu^{-2/d}}{a(r_\star)}M\right) - \partial_r\overline{M}\frac{r_\star}{R(t_\star)}\dot{R}(t_\star).$$

Using that $r_\star \leq R_0$ and by assumption $a(r)$ is non-decreasing on $r \in [0, R_0]$ we have,

$$\begin{aligned}\partial_t(\overline{M}-M)(t_\star,r_\star) &\leq \frac{1}{a(r_\star)}\frac{\partial_r\overline{M}}{\sigma r_\star^{d-1}}(\overline{M}-M)(t_\star,r_\star) \\ &\quad + \partial_r\overline{M}\left[\frac{(1-\mu^{-2/d})}{a(r_\star)}\frac{M}{\sigma r_\star^{d-1}} - \frac{r_\star}{R(t_\star)}\dot{R}(t_\star)\right](t_\star,r_\star).\end{aligned}$$

Since V is radial and non-increasing, we have

$$\partial_r(r^{1-d}\partial_r\overline{M}(t,r)) = \sigma^{-1}\partial_r(\bar{u}(t,r)) \leq 0.$$

In addition we have $\overline{M}(0,t) = 0$ and $\overline{M}(R(t),t) = M_c$. Note that $h(r) := M_c(\frac{r}{R(t)})^d$ solves $\partial_r(r^{1-d}\partial_r h(r)) = 0$ in $(0, R(t))$ with boundary data $h(0) = 0$ and $h(R(t)) = M_c$. Therefore it follows from a maximum principle argument for the elliptic equation $\partial_r(r^{1-d}\partial_r h) = 0$ that

$$\overline{M}(r,t) \geq M_c(\frac{r}{R(t)})^d. \tag{27}$$

Using the above observation we have

$$\begin{aligned}\partial_t(\overline{M}-M)(t_\star,r_\star) &\leq \frac{1}{a(r_\star)}\frac{\partial_r\overline{M}}{\sigma r_\star^{d-1}}(\overline{M}-M)(t_\star,r_\star) \\ &\quad + \frac{\partial_r\overline{M}}{r_\star^{d-1}}\left[\frac{(1-\mu^{-2/d})}{a(r_\star)}\frac{M}{\sigma} - \frac{\overline{M}}{M_c}R(t_\star)^{d-1}\dot{R}(t_\star)\right](t_\star,r_\star).\end{aligned}$$

Due to (19), and using the fact that $a(r)$ is non-decreasing, we have

$$\begin{aligned}\partial_t(\overline{M}-M)(t_\star,r_\star) &\leq \frac{1}{a(r_\star)}\frac{\partial_r\overline{M}}{\sigma r_\star^{d-1}}(\overline{M}-M)(t_\star,r_\star) \\ &\quad + \partial_r\overline{M}\left[\frac{(\mu^{-2/d}-1)}{a(r_\star)\sigma r_\star^{d-1}}(\overline{M}-M)\right](t_\star,r_\star).\end{aligned}$$

Since $\lambda(\overline{M}-M) \leq \partial_t(\overline{M}-M)$, the result follows by choosing

$$\lambda > \sup_{t\in[0,T]}\|\bar{u}(t)\|_\infty\frac{1}{a(0)\mu^{2/d}}.$$

$\square$

We may now prove Theorem 6.

Proof. (Theorem 6) Let u_0 satisfy the hypotheses of Proposition 1 and let $\{\rho_\epsilon\}$ and $T_\star$ be given as above. By the hypotheses of Proposition 1,

$$\overline{M}(0,r) \leq M(0,r) \leq M_\epsilon(0,r) \text{ for } r \in [0,\infty) \text{ for any } \epsilon > 0.$$

Let T_0 be defined as in Lemma 6, which satisfies $T_+(\rho_0) = T_0 \leq \liminf_{\epsilon\to 0} T_+(\rho_\epsilon(0))$. Therefore, it suffices to show $T_0 \leq T_\star$.

To this end, suppose $T_0 > T_\star$, which implies that $\{\rho_\epsilon(t)\}$ and $\rho(t)$ exist on $[0,T_\star]$ for sufficiently small ϵ. Moreover, by Lemma 6, there exists a sequence $\rho_{\epsilon_k} \to \rho$ in $C([0,T_\star];L^1(\mathbb{R}^d))$ and locally uniformly. Combined with Proposition 2, this implies

$$\overline{M}(t,r) \leq M(t,r), \tag{28}$$

for all $0 \leq t < T_\star$. Since $\bar{u}$ concentrates at time $T_\star$, (28) implies that $\rho(t)$ must also concentrate at $T_\star$, contradicting the assumption $T_+(u_0) = T_0 > T_\star$. □

We briefly sketch a proof of Corollary 1.

Proof. (Corollary 1) Let u_0 be as in the statement of Corollary 1. We only need to verify (8) in order to apply Proposition 1. Let R_1 be such that $M_0 = \int_{|x|\leq R_1} u_0 dx > M_c$. Then obviously (8) holds for any $r > R_1$. By assumption, $M_c M_0^{-1} u_0$ is strictly positive on some compact ball $\{|x| \leq r_1\}$. Hence we may choose R_0 sufficiently large such that $R_0^{-d} V(R_0^{-1}x) < M_c M_0^{-1} u_0(x)$ for $|x| \leq r_1$ and clearly (8) holds up to at least $r = r_1$. As for $r_1 \leq r \leq R_1$, $M_c M_0^{-1} \int_{|x|\leq r} u_0(x) dx$ is non-decreasing and hence bounded below on the compact annulus by the value at $r = r_1$. Hence, we may choose R_0 even larger to ensure that (8) holds also for $r_1 \leq r \leq R_1$ and therefore everywhere. Hence, we may apply Proposition 1 and the result follows.

We now prove that we may construct blow up solutions with arbitrarily large initial free energy. We follow a similar procedure as Lemma 3.7 in [8]. Let $\mathcal{V}_M \subset L^1_+ \cap L^m$ be the set of non-negative, radially symmetric non-increasing functions in $L^1 \cap L^m$ with mass M. By the above reasoning, if $u_0 \in \mathcal{V}_M$ is continuous with finite second moment then the associated solution $u(t)$ to the scale-invariant problem with $u(0) = u_0$ blows up in finite time. Now we prove that

$$\sup_{h\in\mathcal{V}_M} \mathcal{F}(h) = +\infty.$$

Suppose for contradiction that

$$A := \sup_{f\in\mathcal{V}_M} \mathcal{F}(h) < \infty. \tag{29}$$

Following the same scaling argument as in Lemma 3.7 of [8], we may use the HLS (6) to show (29) implies a reverse Hölder-type inequality for any $h \in L^1_+ \cap L^m$ which is radially symmetric non-increasing:

$$\|h\|_m^m \|h\|_1^{2/d} \lesssim_M \|h\|_{2d/(d+2)}^2.$$

However, this inequality is clearly false, as $m > 2d/(d+2)$ implies we may easily construct a sequence of functions with uniformly bounded $L^{2d/(d+2)}$ norm and unbounded L^m norm. Indeed, consider $f_\delta = (\delta + |x|)^{-\alpha} \mathbf{1}_{B(0,1)}(x)$ with $\alpha \in (d/m, (d+2)/2)$ with $\delta \in [0,1]$. Then $\|f_\delta\|_1 \geq \|f_1\|_1 > 0$ but $\|f_\delta\|_{2d/(d+2)}$ is uniformly bounded and $\lim_{\delta\to 0} \|f_\delta\|_m = \infty$. Hence it follows by contradiction that $A = +\infty$. By density, we may restrict to continuous functions with finite second moment in $\mathcal{V}_M$ and show that there are solutions to the scale-invariant problem with initial free energy arbitrarily large which blow up in finite time. □

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Appendix: Pressure Form Comparison

By Lemma 3, for any $\epsilon > 0$ we have

$$L := \sup_{\epsilon \leq t \leq T^* - \epsilon} |\Delta c| < \infty. \tag{30}$$

Let us define the pressure form of u:

$$v = \frac{m}{m-1}(u)^{m-1}. \tag{31}$$

Then formally v solves the following equation:

$$v_t = (m-1)v\Delta v + |Dv|^2 + \nabla v \cdot \nabla c + (m-1)v\Delta c.$$

We proceed to prove the"viscosity solution" property of the pressure v. The notion of viscosity solutions are first introduced for Hamilton-Jacobi equations by Crandall-Lions ([16]) and later for fully nonlinear elliptic-parabolic equations ([15]) as well as free boundary type problems (see [11], [12] and [26] for porous medium-type problems.) The advantage of the approach lies in pointwise control of solutions and, in our setting, their free boundaries. More specifically we will show that the initially positive solutions cannot touch down to zero at later times, i.e. that contact lines cannot be nucleated.

Since c is not C^2 up to the zero set of u, Δc is not well-defined on the free boundary $\partial\{u > 0\} = \partial\{v > 0\}$. This causes a technical problem for directly applying a standard notion of viscosity solutions to v. Hence we will directly prove the necessary properties to be used in our analysis in the next section.

Definition 3. For nonnegative functions u and v defined in a small neighborhood Σ of (x_0, t_0), we say

(a) *u crosses v from below* at (x_0, t_0) if

$$u \leq v \text{ in } \Sigma \cap \{t \leq t_0\} \text{ and } u(x_0, t_0) = v(x_0, t_0).$$

(b) *u crosses v from above* at (x_0, t_0) if

$$u \geq v \text{ in } \Sigma \cap \{t \leq t_0\} \text{ and } u(x_0, t_0) = v(x_0, t_0).$$

Proposition 3. *For any given domain* $\Sigma \subset \mathbb{R}^d \times [0, T^* - \epsilon)$*, let* ϕ *be a nonnegative continuous function in* Σ *which is* $C^{2,1}$ *in* $\overline{\{\phi > 0\}}$*, with* $|D\phi| > 0$ *on* $\partial\{\phi > 0\}$*. Let* v *be the pressure form of* u *as defined in* (31)*. Then the following holds:*

(a) Suppose v *crosses* ϕ *from below at* (x_0, t_0) *in* $\overline{\{u > 0\}} \cap \{t \leq t_0\}$ *in* Σ*. Then we have*

$$\phi_t - (m-1)\phi\Delta\phi - |D\phi|^2 - \nabla\phi \cdot \nabla c - (m-1)L\phi \leq 0$$

(b) Suppose v *crosses* ϕ *from above at* (x_0, t_0) *in* $\overline{\{u > 0\}} \cap \{t \leq t_0\}$ *in* Σ*. Then we have*

$$\phi_t - (m-1)\phi\Delta\phi - |D\phi|^2 - \nabla\phi \cdot \nabla c + (m-1)L\phi \geq 0.$$

Here the constant L *is as given in* (30)*.*

Proof. 1. Note that u and v are smooth in their positive set, and there the result follows easily. Hence, the only difficult case is when $(x_0, t_0) \in \partial\{v > 0\}$.

2. Let us take $c_\epsilon := c * \eta_\epsilon$, where η_ϵ is the standard mollifier. Let u_ϵ be the weak solution of

$$(u_\epsilon)_t = \Delta(u_\epsilon)^m + \nabla \cdot (u_\epsilon \nabla c_\epsilon),$$

with initial data $u_\epsilon(x, 0) = u(x, 0)$. Since c_ϵ is C^2, it follows from [26] that u_ϵ is a viscosity solution in the sense defined in therein. In particular, it is shown in [26] that the statements in the proposition hold for v_ϵ: the pressure form of u_ϵ. Below we will approximate u by u_ϵ to prove the proposition. Since c_ϵ is uniformly bounded in $C^{1,1}$ norm, u_ϵ is equi-continuous due to Theorem 6.1 of [19]. Using this fact, parallel arguments leading to Proposition 3.3 of [27] yields that u_ϵ uniformly converges to u, and thus v_ϵ to v.

3. Let us now show (a) when v crosses a nonnegative function $\phi \in C^{2,1}(\overline{\{\phi > 0\}})$ from below at $(x_0, t_0) \in \partial\{v > 0\}$. Let us perturb ϕ so that v really crosses ϕ, not just touching. This can be done by replacing ϕ with

$$\tilde{\phi}(x, t) = (\phi(x, t) - a(t - t_0 - b))_+,$$

where a and b are small positive constants.

4. If (a) fails then, since $\phi(x_0, t_0) = 0$, then ϕ satisfies

$$(\phi_t - (m-1)\phi\Delta\phi - |D\phi|^2 - \nabla\phi \cdot \nabla c - (m-1)L\phi)(x_0, t_0) > 0.$$

Now let us pick a small $\delta > 0$ and take $\phi_\delta(\cdot, t) := (\phi_+)(\cdot, t) * \eta_\delta + m(\delta)$ where $\eta(x)$ is a standard mollifier which is smooth and has exponential decay at infinity, and $m(\delta)$ is a constant. Choose $m(\delta)$ accordingly so that v is strictly below ϕ_δ at $t = t_0$ but crosses ϕ_δ from below at $t = t_0 + O(\delta)$. Note that this is possible because η does not have a compact support.

Then due to continuity of the derivatives of ϕ in its support and the corresponding convergence of ϕ_δ to ϕ, ϕ_δ satisfies

$$(\phi_\delta)_t - (m-1)\phi_\delta\Delta\phi_\delta - |D\phi_\delta|^2 - \nabla\phi_\delta \cdot \nabla c_\epsilon - (m-1)L\phi_\delta > 0. \tag{32}$$

in $O(\delta_0)$-neighborhood of (x_0, t_0) if $\epsilon, \delta << \delta_0$.

Since v_ϵ converges uniformly to v as $\epsilon \to 0$, v_ϵ crosses ϕ_δ from below at (x_δ, t_δ), which lies in $O(\delta_0)$-neighborhood of (x_0, t_0) if ϵ and δ are chosen small enough.

Note that at (x_0, t_0) we have

$$(v_\epsilon)_t \geq (D\phi_\delta)_t, \quad |Dv_\epsilon| = |D\phi_\delta| \text{ and } \Delta v_\epsilon \leq \Delta\phi_\delta.$$

this contradicts (32) and the fact that v_ϵ satisfies

$$(v_\epsilon)_t - (m-1)v_\epsilon \Delta v_\epsilon - |Dv_\epsilon|^2 - \nabla v_\epsilon \cdot \nabla c_\epsilon - (m-1)Lv_\epsilon \leq 0$$

□

Corollary 2 (Local comparison in pressure variable)**.** *In any given parabolic, cylindrical neighborhood Σ, let ϕ be a $C^{2,1}$ function in $\overline{\{\phi > 0\}}$ with $|D\Phi| > 0$ on $\partial\{\phi > 0\}$.*

(a) Suppose that ϕ satisfies

$$\phi_t - (m-1)\phi\Delta\phi - |D\phi|^2 - \nabla\phi \cdot \nabla c - (m-1)L\phi > 0 \text{ in } \Sigma.$$

Then v cannot cross ϕ from below in Σ.

(b) Suppose ϕ satisfies

$$\phi_t - (m-1)\phi\Delta\phi - |D\phi|^2 - \nabla\phi \cdot \nabla c + (m-1)L\phi < 0 \text{ in } \Sigma.$$

Then v cannot cross ϕ from above in Σ.

An immediate consequence of the above proposition is the preservation of positivity for u.

Lemma 8. *Let $u(x,t)$ be the weak solution associated with the strictly positive continuous initial data $u_0(x) > 0$. Then $u(x,t)$ is strictly positive everywhere up to the blow-up time $T_+(u_0)$.*

Proof. Let v be the corresponding pressure form of u. Let us recall that the Barenblatt profile is given as

$$B(x,t) := t^{-\lambda}\left(C - k\frac{|x|^2}{t^{2\mu}}\right)_+$$

where $C > 0$ is a positive constant and

$$\lambda = \frac{d(m-1)}{d(m-1)+2}, \mu = \frac{\lambda}{d}, k = \frac{\lambda}{2d}.$$

$B(x,t)$ then solves the porous medium equation in its pressure form in the viscosity sense (see [26]):

$$B_t - (m-1)B\Delta B - |DB|^2 = 0.$$

Let us now define

$$\tilde{B}(x,t) := e^{-Mt} \sup_{y \in B_{M-Mt}(x)} B(x,t) \text{ for } 0 \leq t \leq 1.$$

Then due to Proposition 2.13 in [26] $\tilde{B}$ satisfies

$$\tilde{B}_t - (m-1)\tilde{B}\Delta\tilde{B} - |D\tilde{B}|^2 + M|D\tilde{B}| + M\tilde{B} \leq 0$$

for $0 \leq t \leq 1$. Let us choose

$$M = (m-1)\max\left(\|\nabla c\|_{L^\infty}, L\right).$$

(Note that the first term in above upper bound is bounded before the blow-up time).

Since $B(x,t)$ vanishes uniformly to zero as $C \to 0$, so does $\tilde{B}$. Hence for any $\tau > 0$ one can choose $C = C(\tau)$ sufficiently small so that $\tilde{B}(x,\tau) \leq u_0$. Then Corollary 2 yields that

$$\tilde{B}(x, t+\tau) \leq u(x,t) \text{ for } 0 \leq t \leq 1.$$

Since τ can be arbitrarily large and $\tilde{B}$ has its support expanding to the whole domain as τ grows to infinity, we conclude that u is strictly positive for $0 \leq t \leq 1$.

We can iterate above argument up to the blow-up time to conclude (since the solution and L remain bounded until blow-up).

$\square$

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