Additional Review problems for Midterm I

• Please explain all of your answers. We will only give partial credit to the answers given without explanation of the reasoning.

1. The population of Whooping Cranes are given as

$$\frac{dy}{dt}(t) = (1-y)(y-2).$$

(a) Sketch the corresponding direction field and phase portrait. Identify the equilibria and specify them as stable, semi-stable or unstable.

(b)The Whooping Crane population crashed in the 1930's. The US is attempting to bring this species back to a viable population by releasing captive-raised birds. For simplicity suppose the release happens at a constant rate. What is the minimal rate of release, $r = r_0$, so that any greater rate of release eventually results in a stable, healthy population , no matter how small the starting population? Please explain your answer.

2. (a) What is the solution of the IVP $dy/dx = 2xy^2, y(0) = 1$?

(b) What is the domain of definition of this solution? (That is, what is the largest a and smallest b so that the solution is defined everywhere between a and b?)

3. Suppose $x(t)$ solves the ODE

$$\ddot{x} + 2b\dot{x} + kx = 0,$$

where b, k are constants.

(a) For which b and k does the solution go to zero as $t \to \infty$?

(b) Suppose $x(0) = 0$. With $b^2 > k \geq 0$, show that unles $x \equiv 0$, $x(t)$ never become zero for $t > 0$.

(c) Now suppose $x(0) < 0$. Can it pass $x = 0$ twice?

4. Suppose $y_1(x)$ solves $\ddot{y} + P(x)y = 0$. Let us assume that $y_2 = y_1 u$ is $\ddot{y} + P(x)y = g(x)$. Find the ODE which $w = \dot{u}$ solves in terms of y_1 and $\dot{y_1}$ (do not solve the ODE).

Answers to the problems

1.(a) The equilibria are $y = 1$ (semistable) and $y = 2$ (stable). Direction fields and phase portraits are omitted.

(b) $r_0 = 1$. With release rate r, the model changes to

$$\frac{dy}{dt} = (1-y)(y-2) + r := g(y)$$

If $r \geq 1$, then the equilibria are $y = a \leq 0$(semistable) and $y = b > 0$ (stable), and thus $y \to b$ as $t \to \infty$, no matter what the initial population is. $\square$

2. (a) $y = \frac{-1}{x^2-1}$. (b) (-1,1).

3.(a) This is true when the exponential parts of the solution has negative exponents (damping case). This happens when $b, k > 0$.

(b) The solution is given as $x(t) = C_1(e^{r_1 t} - e^{r_2 t})$, where $r_1 = -b + \sqrt{b^2 - k}$, $r_2 = -b - \sqrt{b^2 - k}$. If $C_1 > 0$ then $x > 0$ for $t > 0$. If $C_1 < 0$ then $x < 0$ for $t > 0$. If $C_1 = 0$ then $x \equiv 0$.

(c) No, since once it passes $x = 0$ at $t = t_0$, $z(t) := x(t - t_0)$ solves the same equation as x and satisfies $z(0) = 0$, and thus by (b) z can never become zero again.

4.

$$\begin{aligned} \ddot{y_1 u} + P(x)(y_1 u) &= \ddot{y}_1 u + 2\dot{y}_1 \dot{u} + y_1 \ddot{u} + P(x)(y_1 u) \\ &= u(\ddot{y}_1 + P(x) y_1) + 2\dot{y}_1 \dot{u} + y_1 \ddot{u} \\ &= 0 + 2\dot{y}_1 \dot{u} + y_1 \dot{u} = g(x). \end{aligned}$$

Hence $w = \dot{u}$ solves $y_1 \dot{w} + 2\dot{y} w = g(x)$. $\square$