

Waiting time phenomena of the Hele-Shaw and the Stefan problem

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Abstract

In this paper we investigate the waiting time phenomena for the one-phase Hele-Shaw and Stefan problems. We identify a general criterion on the growth rate of the initial data for the Hele-Shaw problem which determines the occurrence of a waiting time. For the Stefan problem we show that the waiting time phenomena depends on the balance between the initial data and the geometry of the initial positive phase.

0 Introduction

Let us consider a compact set $K \subset \mathbb{R}^n$ with smooth boundary ∂K . Suppose that a bounded domain Ω contains K and let $\Omega_0 = \Omega - K$ and $\Gamma_0 = \partial\Omega$ (Figure 1). Note that $\partial\Omega_0 = \Gamma_0 \cup \partial K$.

Let u_0 be the harmonic function in Ω_0 with smooth fixed boundary data $u_0 = f > 0$ on K and $u_0 = 0$ on Γ_0 . The classical Hele-Shaw problem, in

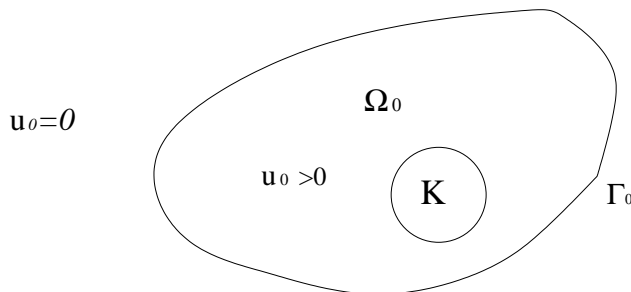


Figure 1.

$n = 2$, models an incompressible viscous fluid which occupies part of the space between two parallel, narrowly placed plates ([ST],[EJ].) In this case u_0 denotes the initial pressure of the fluid and f denotes the rate of injection from K into Ω_0 . Assuming that there is no surface tension, the pressure of the fluid $u(x, t)$ satisfies

$$(HS) \quad \begin{cases} -\Delta u = 0 & \text{in } \{u > 0\} \cap Q, \\ u_t - |Du|^2 = 0 & \text{on } \partial\{u > 0\} \cap Q, \\ u(x, 0) = u_0(x); \quad u(x, t) = f \text{ for } x \in \partial K. \end{cases}$$

where $Q = (\mathbb{R}^n - K) \times (0, \infty)$. Observe that the initial data u_0 in (HS) is determined by the initial domain Ω_0 .

We define

$$\Gamma(u) = \partial\{u > 0\}, \Gamma_t(u) = \partial\{u(\cdot, t) > 0\} - \partial K$$

respectively the *free boundary* of u and the free boundary of u at time t . We also define

$$\Omega(u) = \{u > 0\}, \Omega_t(u) = \{u(\cdot, t) > 0\}$$

respectively the *positive phase* of u and the positive phase of u at time t . When it is obvious from the context we will omit u in the notation of $\Omega_t(u)$ and $\Gamma_t(u)$.

Note that if u is smooth up to the free boundary, then the free boundary moves with normal velocity $V = u_t/|Du|$, and hence the second equation in (HS) implies that $V = |Du|$. Also note that u is determined by Ω_0 and f .

The classical Stefan problem accounts for phase transitions between solid and fluid states, such as the melting of ice in contact with water ([Ru],[Me].) Here we assume that the temperature varies only in fluid and the temperature of the solid is maintained at $0^{\circ}C$. Then the temperature distribution of the fluid $u(x, t)$ with nonnegative initial data u_0 satisfies

$$(ST) \quad \begin{cases} u_t - \Delta u = 0 & \text{in } \{u > 0\}, \\ u_t - |Du|^2 = 0 & \text{on } \partial\{u > 0\} \\ u(x, 0) = u_0(x) \end{cases}$$

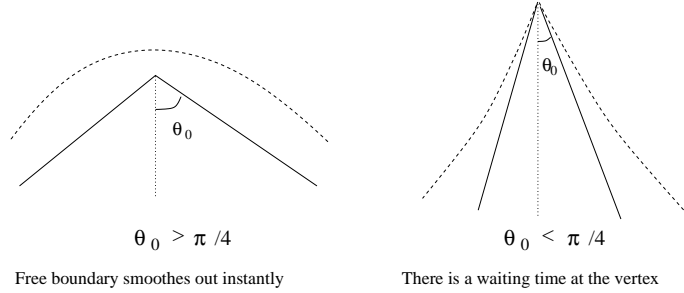


Figure 2.

We use the same notation for the free boundary and the positive phase of u . (Here the free boundary of u is given as $\Gamma_t(u) = \partial\{u(\cdot, t) > 0\}$.) Note that in both problems (HS)-(ST) the free boundary expands in time.

We say that a solution of (HS) or (ST) has a (initial) *waiting time* $t_0 > 0$ at $P \in \mathbb{R}^n$ if $P \in \Gamma_0$ and $P \in \Gamma_t(u)$ for $0 < t < t_0$.

For $n = 2$ [KLV] studied the behavior of self-similar solutions of (HS) with Γ_0 as 'wedges' of the form

$$(0.1) \quad \{(r, \theta) : |\theta| = \theta_0\}.$$

It is proven here that if $\theta_0 \geq \pi/4$ then the free boundary strictly expands and smoothes out, and if $\theta_0 < \pi/4$ then there is a waiting time at the vertex of the wedge (see Figure 2.)

In fact parallel result holds (see [JeKi]) for radially symmetric solutions in \mathbb{R}^n , $n > 2$ with the threshold angle $\theta_0 = \theta_n$, with which the initial data has a quadratic decay rate at the vertex, i.e.,

$$u_0(r, 0) \sim r^2.$$

For solutions of (HS) with Lipschitz initial domain Ω in \mathbb{R}^n , $n \geq 2$ it is recently proved in [CJK] that if the Lipschitz constant of the domain is smaller than a dimensional constant a_n , then the free boundary immediately smoothes out, that is $\Gamma(u)$ becomes analytic in space and time for a small amount of positive time. In particular for $n = 2$ we have $a_2 = 1 = \cot \pi/4$, which corresponds to the threshold angle $\theta_0 = \pi/4$ in the analysis of [KLV]. For $n > 2$ we have a_n smaller than $\cot \theta_n$ due to technical reasons. For the two-dimensional self-similar solutions of (ST), with Γ_0 given as in (0.1) and with $u_0 = 1$ in $\Omega_0(u)$, [K] shows that parallel results holds with a threshold angle $\theta_0 = \pi/6$.

An open and interesting question, positively answered from above results, is whether there is a dichotomy on the free boundary behavior of (HS) and (ST) near $t = 0$, that is whether it is always one of the two cases: either the free boundary immediately smooths out, or there is a waiting time at a point on the initial free boundary. (We mention that the regularity of the free boundary may not last for all time due to the collision of free boundary parts.)

In this paper we ask which information on the initial data determines the occurrence of the waiting time at a given point on the initial free boundary for both problems (HS) and (ST).

Our goal in this paper is

- (1) to extend the results of [KLV] on self-similar solutions in \mathbb{R}^2 to solutions with *Non-Tangentially Accessible* initial domains in \mathbb{R}^n and
- (2) to investigate the waiting time phenomena for solutions of (ST) with Lipschitz initial domain in \mathbb{R}^n .

In regards to (1) the main idea for the proof is that the occurrence of waiting time depends on the 'pushing force' at the vertex, which is generated in the neighboring regions and accumulated with respect to the distance to the vertex. (See for example (1.3).)

In regards to (2) we will prove that the waiting time phenomena for (ST) depends on the balance between the initial heat u_0 and the geometry of the initial positive phase Ω_0 of u (see Theorem 1.6- 1.7). Roughly speaking the waiting time occurs when the initial heat u_0 can change into the latent heat (harmonic function) without changing the geometry of the domain too much. It is proven in [K] that with a strong, discontinuous initial heat $u_0 = 1$ in the positive phase and with Γ_0 given as in (0.1), we require the half-angle of the wedge $\theta_0 > \pi/6$ for the free boundary to immediately expand and smooth out. With harmonic initial data, Γ_0 needs to be flatter to smooth out: we require that $\theta_0 > \pi/4$. This is plausible since the initial heat for the first case is much stronger than in the second case, and thus melts the ice more easily.

In Theorem 1.7 we will show that indeed the initial heat needs to be much stronger than the harmonic initial data to make a difference in the waiting time phenomena. More precisely we prove that if the initial data has a degree of regularity depending on the geometry of the initial positive phase, then the occurrence of the waiting time for (ST) will coincide with the case of harmonic initial data. This is because if u_0 is regular enough

then the harmonic measure associated with the evolving positive phase does not change too much while the initial heat changes into 'almost' latent heat. (For further discussion see section 5.) In particular when Γ_0 is given as in (0.1) it follows from Theorem 1.6-1.7 that any Hölder continuous initial data u_0 will generate the same waiting time phenomena with the case of harmonic initial data, that is there is a waiting time if $\theta_0 < \pi/4$ and no waiting time if $\theta_0 > \pi/4$.

In contrast, in the case of the porous medium equation

$$(PME)_m \quad u_t - mu\Delta u - |Du|^2 = 0, u \geq 0,$$

the waiting time occurrence is solely determined by the decay rate of the initial data u_0 (see [A], [AC]), independently of the geometry of the initial positive phase.

For our investigation we use the notion of viscosity solutions, which has been recently introduced in [Ki1]. An important property of viscosity solutions which is frequently used in this paper is the comparison principle (see Theorem 2.7), which enables us to compare our solutions with barriers that we construct in various settings. The main tools we use in the paper include

- (1) comparison principle of solutions for (HS) and (ST),
- (2) estimates on harmonic measures in Lipschitz and NTA domains, and
- (3) properties of caloric functions in Lipschitz domains.

1 Statements of the main results

In this section we summarize the results of this paper. Here we use *viscosity solutions* (see section 2 for definitions and properties) introduced in [Ki1] as our notion of solutions.

1.1 Hele-Shaw problem

For $x \in \mathbb{R}^n$ we denote $B_r(x) := \{y \in \mathbb{R}^n : |y - x| < r\}$. we let Ω be a bounded domain in \mathbb{R}^n such that $0 \in \partial\Omega$ and let $K \subset B_{1/2}(-e_n) \subset \Omega$.

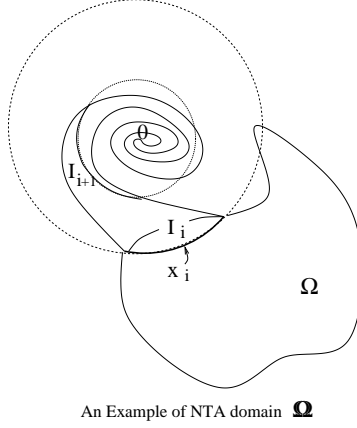


Figure 3.

Definition 1.1. Ω is called non-tangentially accessible (NTA) when there exists a constant $M > 1$ such that:

1. *Corkscrew condition.* For any $\zeta \in \partial\Omega$, $r < 1$ there exists $y = y(r, \zeta) \in \Omega$ such that $r/M < |y - \zeta| < r$ and $B_{r/M}(y) \subset \Omega$.
2. Ω^c satisfies corkscrew condition.
3. *Harnack Chain Condition.* Let $\epsilon > 0$ and let $x_1, x_2 \in \Omega \cap B_{r/4}(\zeta)$ for some $\zeta \in \partial\Omega$ and $r < 1$. If $\text{dist}(x_j, \partial\Omega) > \epsilon$ and $|x_1 - x_2| < 2^k \epsilon$, then there exists a Harnack chain from x_1 to x_2 of length Mk such that the diameter of each ball is bounded below by $M^{-1} \min\{\text{dist}(x_1, \partial\Omega), \text{dist}(x_2, \partial\Omega)\}$.

M is called the NTA constant.

NTA domains include Lipschitz domains and (slowly rotating) spiral domains (see Figure 3.)

Definition 1.2. Let Ω be an NTA domain with NTA constant M and let $0 \in \partial\Omega$, $-e_n \in \Omega$. We denote by I_i the component of $\Omega \cap \partial B_{2^{-i}}(0)$ which separates 0 from $-e_n$, i.e. 0 and $-e_n$ are contained in the different components of $\Omega - I_i$ (see Figure 3). We also denote by x_i a point on I_i such that $B_{2^{-i}/M^2}(x_i) \subset \Omega$.

Note that by (i) and (iii) of the NTA conditions there exists such $x_k \in I_k$.

Throughout Section 1.1 we assume that Ω is NTA with $\{x_i\}_i$ as given in Definition 1.1 and u is the viscosity solution of (HS) with $\Omega_0 = \Omega - K$,

$0 \in \Gamma_0$. In the next two theorems we extend the result of [KLV] and show that if Ω_0 is NTA then the threshold decay rate of u_0 for the waiting time phenomena of (HS) is always quadratic.

Theorem 1.3. *If there exists $\epsilon > 0$ such that $|u_0(x_i)| \leq |x_i|^{2+\epsilon}$ for $i = 1, 2, \dots$, then there exists a waiting time $t_0 > 0$ for $u(x, t)$ at 0. In particular, $t_0 \geq C\epsilon^{1/\beta_n}$ where C depends on the NTA constant and β_n is a dimensional constant given in Lemma 3.2.*

Remark There are initial (non-NTA) domains such that $|u_0(x)| \leq |x|^{2+\epsilon}$ but u has no waiting time. For a given angle θ_0 let us define

$$\Omega = B_1(0) \cap \{x_n < 0\} - \cup_{i \geq 1} (\partial B_{2^{-i}}(0) - W(\theta_0, -e_n))$$

where

$$W(\theta_0, -e_n) = \{x \in \mathbb{R}^n : \langle x, -e_n \rangle \geq |x| \cos \theta_0\}.$$

We choose $\theta_0 < \theta_n$, where θ_n is a dimensional constant such that if $\theta_0 = \theta_n$ then the harmonic function associated with Ω has a quadratic decay at $x = 0$. Then for this range of θ_0 we have $|u_0(x)| \leq |x|^{2+\epsilon}$ for some $\epsilon > 0$.

Let $\Omega_0 = \Omega - B_{1/8}(-\frac{3}{4}e_n)$, then u has no waiting time at $0 \in \Gamma_0$ since

$$B_1(0) \cap \{x_n < 0\} \subset \Omega_t \text{ for any } t > 0.$$

Theorem 1.4. *The following statements are true:*

- (a) *If there exists a sequence $n_j \rightarrow \infty$ such that $u(x_{n_j}) \geq |x_{n_j}|^{2-\epsilon_{n_j}}$, then u has no waiting time.*
- (b) *If $u(x_i, 0)$ is comparable to $|x_i|^2$ for $i = 1, 2, \dots$, then u has no waiting time.*

We next consider the borderline case where u_0 has quadratic decay at 0 with logarithmic perturbations. In [KLV] it is proved that if Γ_0 is given as a 'curved wedge'

$\{(r, \theta) : |\theta| \sim \pi/4(1 - \frac{\alpha}{2|\ln(r)|})\}$ near 0 so that

$$(1.1) \quad u_0(z) \sim \operatorname{Re}\left\{\frac{z^2}{(-\ln z)^\alpha}\right\}_+ \text{ as } z \rightarrow 0,$$

then there is a waiting time at the vertex if $\alpha > 1$ and there is no waiting time if $\alpha < 1$. Next theorem extends this result of [KLV].

Theorem 1.5. *Assume that*

(1.2) *There exists $m > 0$ such that for every $\zeta \in \Gamma_0$ there exist balls B_1 and B_2 contained in Ω_0 and $\mathbb{R}^n - \Omega_0$ such that $\text{rad}(B_1) = \text{rad}(B_2) = m|\zeta|$ and $\zeta = \partial B_1 \cap \partial B_2$.*

Then the following statements hold:

- (a) *If $u_0(x_i) \leq |x_i|^2(\frac{1}{i})^\alpha$ for $\alpha > 1$ and $i = 1, 2, \dots$, then u has a waiting time.*
- (b) *If $u_0(x_i) \geq |x_i|^2(\frac{1}{i})^\alpha$ for $\alpha < 1$ and $i = 1, 2, \dots$, then u has no waiting time.*

Remark

1. The case $u_0(x) \sim |x_i|^2(i)^{-\alpha}$ arises, for example, when Ω_0 is given as in (1.1) with $n = 2$. Roughly speaking these are the cases when Ω_0 is a logarithmic perturbation of a domain where the associated harmonic function decays exactly quadratically at zero.

2. (1.2), a C^2 -bound for Γ_0 scaled with respect to the distance to the vertex, is necessary for technical reasons in our proof. It is not clear whether the statement holds without (1.2).

Theorem 1.5 (a) is a special case of the following general statement. Roughly speaking a_k corresponds to the 'vertex-pushing force' generated outside of the 2^{-k} -neighborhood of the vertex.

Proposition 1.6. *Let Ω be NTA with (1.2). Let $a_0 = 1$ and*

$$(1.3) \quad a_k = \prod_{i=0}^{k-1} (1 + a_i) 2^{2k} u_0(x_i).$$

If $\limsup a_k < \infty$, then there exists a waiting time.

1.2 Stefan problem

Throughout section 1.2 we denote u the viscosity solution of (ST) with initial data $u_0(x)$ such that $\{u_0(x) > 0\} = \Omega_0$ is a Lipschitz domain in \mathbb{R}^n . Let K be a compact set in Ω with a smooth boundary and let $h_0(x)$ be the harmonic function in $\Omega_0 - K$ with boundary data 0 on $\Gamma_0 = \partial\Omega_0$ and 1 on ∂K . After a rotation and translation, we may assume that $0 \in \Gamma_0$ and $\Gamma_0 = \{x = (x', x_n) : x_n = f(x')\}$ with a Lipschitz function f in a small neighborhood of 0 with $f(0) = 0$. In this section we denote C as a positive constant depending only on the Lipschitz constant of f and dimension n . Let positive constants $\alpha_1 < \alpha_2$ and C satisfy

$$\frac{1}{C}r^{\alpha_2} \leq h_0(-re_n) \leq Cr^{\alpha_1}$$

for sufficiently small $r > 0$. We call α_1 and α_2 respectively the *minimal* and *maximal* decay rate of h_0 at 0.

Theorem 1.7. *If the maximal decay rate of $h_0(x)$ at 0 is less than 2 then u has no waiting time at 0.*

Theorem 1.8. *Let $u(x, 0) \in C^\delta(\bar{\Omega}_0)$ and let $\alpha_1 \leq \alpha_2$ to be the minimal and maximal decay rate of h_0 at 0. If $\alpha_1 > 2$ and $\alpha_2 - \alpha_1 < \delta/2$ then u has a waiting time at the origin.*

Remark u may not have a waiting time if $\alpha_2 - \alpha_1 \geq \delta/2$, in particular if u_0 is discontinuous. A good example is the one studied in [K], where Ω_0 is given as in (0.1) with $n = 2$ (therefore $\alpha_2 = \alpha_1$) and u_0 is the characteristic function of Ω_0 . In this case there is no waiting time if the wedge angle θ_0 is between $\pi/4$ and $\pi/6$ even though the decay rate of h_0 satisfies $\alpha_1 = \alpha_2 > 2$. In section 5 we construct another example where Ω_0 is an oscillatory domain and $u_0 \in C^\delta$ but $\alpha_2 - \alpha_1 > \delta/2$.

2 Viscosity solutions

In this section we introduce the notion of viscosity solutions for (HS) and (ST) which we will use in this paper. Roughly speaking, viscosity sub and supersolutions are defined by comparison with local (smooth) super and subsolutions. In particular classical solutions of either problems are also viscosity sub and supersolutions.

Definition 2.1. *We say that a pair of functions $u_0, v_0 : \bar{D} \rightarrow [0, \infty)$ are (strictly) separated (denoted by $u_0 \prec v_0$) in $D \subset \mathbb{R}^n$ if*

- (i) *the support of u_0 , $\text{supp}(u_0) = \overline{\{u_0 > 0\}}$ restricted in \bar{D} is compact and*
- (ii) *in $\text{supp}(u_0) \cap \bar{D}$ the functions are strictly ordered:*

$$u_0(x) < v_0(x).$$

For a nonnegative real valued function $u(x, t)$ defined in a cylindrical domain $D \times (a, b)$, we define

$$\Omega(u) = \{(x, t) : u(x, t) > 0\}, \quad \Omega_t(u) = \{x : u(x, t) > 0\},$$

$$\Gamma(u) = \partial\{(x, t) : u(x, t) = 0\}, \quad \Gamma_t(u) = \partial\{x : u(x, t) = 0\}.$$

Let $Q = (\mathbb{R}^n - K) \times (0, \infty)$ and let Σ be a cylindrical domain $D \times (a, b) \subset \mathbb{R}^n \times \mathbb{R}$, where D is an open subset of \mathbb{R}^n . The following definitions are introduced in [Kil].

Definition 2.2. *A nonnegative upper semicontinuous function u defined in Σ is a viscosity subsolution of (HS) if*

- (a) *for each $a < T < b$ the set $\overline{\Omega(u)} \cap \{t \leq T\}$ is bounded; and*
- (b) *for every $\phi \in C^{2,1}(\Sigma)$ such that $u - \phi$ has a local maximum in $\overline{\Omega(u)} \cap \{t \leq t_0\} \cap \Sigma$ at (x_0, t_0) ,*
 - (i) $-\Delta\phi(x_0, t_0) \leq 0$ *if $u(x_0, t_0) > 0$.*
 - (ii) $(\phi_t - |D\phi|^2)(x_0, t_0) \leq 0$ *if $(x_0, t_0) \in \Gamma(u)$ if $-\Delta\phi(x_0, t_0) > 0$.*

Note that because u is only upper lowercontinuous there may be points of $\Gamma(u)$ at which u is positive.

Definition 2.3. *A nonnegative lower semicontinuous function v defined in Σ is a viscosity supersolution of (HS) if for every $\phi \in C^{2,1}(\Sigma)$ such that $v - \phi$ has a local minimum in $\Sigma \cap \{t \leq t_0\}$ at (x_0, t_0) ,*

- (i) $-\Delta\phi(x_0, t_0) \geq 0$ *if $v(x_0, t_0) > 0$,*
- (ii) *If $(x_0, t_0) \in \Gamma(v)$, $|D\phi|(x_0, t_0) \neq 0$ and $-\Delta\phi(x_0, t_0) < 0$,*

then

$$(\phi_t - |D\phi|^2)(x_0, t_0) \geq 0.$$

Definition 2.4. *u is a viscosity subsolution of (HS) with initial data u_0 and fixed boundary data $f > 0$ if*

- (a) *u is a viscosity subsolution in \bar{Q} ,*
- (b) *$u = u_0$ at $t = 0$; $u \leq f$ on ∂K ;*

$$(c) \overline{\Omega(u)} \cap \{t = 0\} = \overline{\Omega(u_0)};$$

Definition 2.5. u is a viscosity supersolution of (HS) with initial data u_0 and fixed boundary data f if u is a viscosity supersolution in \bar{Q} with $u = u_0$ at $t = 0$ and $u \geq f$ on ∂K .

For a nonnegative real valued function $u(x, t)$ defined in a cylindrical domain $D \times (a, b)$,

$$u^*(x, t) = \limsup_{(\xi, s) \in D \times (a, b) \rightarrow (x, t)} u(\xi, s).$$

Definition 2.6. u is a viscosity solution of (HS) (with boundary data u_0 and f) if u is a viscosity supersolution and u^* is a viscosity subsolution of (HS) (with boundary data u_0 and f .)

Viscosity solutions of (ST) are similarly defined (see [Ki1] for definition). The following properties of viscosity solutions are frequently used in our paper.

Theorem 2.7. (comparison principle, [Ki2]) Let u, v be respectively viscosity sub- and supersolutions (of (HS) or (ST)) in $D \times (0, T) \subset Q$ with initial data $u_0 \prec v_0$ in D . If $u \leq v$ on ∂D and $u < v$ on $\partial D \cap \bar{\Omega}(u)$ for $0 \leq t < T$, then $u(\cdot, t) \prec v(\cdot, t)$ in D for $t \in [0, T)$.

Theorem 2.8. (a) For a given domain Ω_0 in \mathbb{R}^n , there are the maximal and minimal viscosity solutions of (HS) in Q with boundary data 1 and initial data u_0 . If the minimal viscosity solution does not have an initial waiting time, then it is the unique viscosity solution of (HS) with given boundary data.

(b) u is harmonic in $\Omega(u)$. Indeed $u(x, t) = h_t(x)$, where

$$h_t(x) = \inf\{v \in \mathcal{P} \text{ with } v = 1 \text{ on } \partial K \text{ and } v \geq 0 \text{ on } \Gamma_t\}.$$

where \mathcal{P} is the set of superharmonic functions in Ω_t which are lower semicontinuous in $\bar{\Omega}_t$.

(c) For a given initial data $v_0 \geq 0$ with its positive set $\Omega_0(v)$ bounded in \mathbb{R}^n there are the maximal and minimal viscosity solutions of (ST) in $\mathbb{R}^n \times [0, \infty)$. If the minimal viscosity solution does not have an initial waiting time, then it is the unique viscosity solution of (ST) with given boundary data.

Remark

It is an open question that whether or not there is a unique viscosity solution of (HS) or (ST) when there is a waiting time.

3 Proofs of the main result: Hele-Shaw problem

In this section we prove the main results for Hele-Shaw problem. Let $0 \in \Gamma_0$ and $K \subset B_{1/2}(-e_n) \subset \Omega \subset B_{10}(0)$. Throughout the paper we denote $A_k = B_{2^{-k}}(0) - B_{2^{-k-1}}(0)$ ($k \geq 1$).

The key observation that will be used throughout this section is that there is a waiting time at $0 \in \Gamma_0$ if and only if there exists $t_0 > 0$ such that $\Gamma_{t_0} \cap A_k \neq \emptyset$ for every $k \geq 1$. In order to decide whether there exists such $t_0 > 0$ or not, we will use induction and estimate the change of the harmonic measure for each annulus A_k in time. First we prove Theorem 1.3.

Definition 3.1. $\omega(x, \cdot, \Omega)$ is the unique probability measure on $\partial\Omega$ such that $\omega(x, E, \Omega) = w(x)$ where w is the harmonic function in Ω , which has boundary value 1 on E and 0 elsewhere on its boundary.

First we state properties of NTA domains we use in the proof.

Lemma 3.2. ([JK], Lemma 4.1) Let Ω be NTA contained in $B_{10}(0)$. There exists a dimensional constant $\beta_n > 0$ such that for any $\zeta \in \partial\Omega$, $0 < 2r < 1$ and positive harmonic function u in $\Omega \cap B_{2r}(\zeta)$, if u vanishes continuously on $B_{2r}(\zeta) \cap \partial\Omega$, then for $x \in \Omega \cap B_r(\zeta)$,

$$u(x) \leq C \left(\frac{|x - \zeta|}{r} \right)^{\beta_n} \sup\{u(y) : y \in \partial B_{2r}(\zeta) \cap \Omega\}$$

where C depends only on the NTA constant.

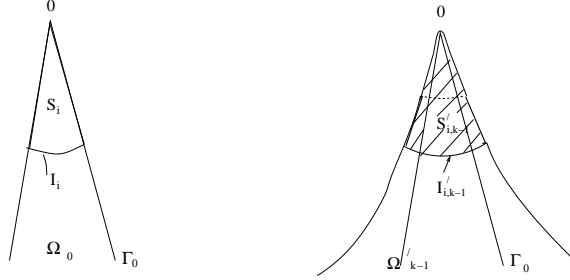
Lemma 3.3. ([JK], Lemma 4.11) Let Ω be NTA contained in $B_{10}(0)$. For $\zeta_0, \zeta \in \partial\Omega$ and $r < 1$, let $B_s(\zeta) \subset B_{r/2}(\zeta_0)$. If $x \in \Omega - B_{2r}(\zeta_0)$ and x_0 is a point in the middle of $B_r(\zeta_0) \cap \Omega$, then

$$\omega(x_0, B_s(\zeta), \Omega) \approx \frac{\omega(x, B_s(\zeta), \Omega)}{\omega(x, B_r(\zeta_0), \Omega)}$$

where $C_1 \approx C_2$ means that the ratio between C_1 and C_2 is bounded above and below by a constant that depends on the NTA constant.

Proof of Theorem 1.3.

Recall that I_i ($i \geq 1$) is the component of $\Omega_0 \cap \partial B_{2^{-i}}(0)$ which separates 0



Construction of the initial positive phase of the barrier function in the proof of Theorem 1.3

Figure 4.

from $-e_n$. Denote by M the NTA constant of Ω . Let S_i be the component of $\Omega_0 - I_i$, which contains 0 on its boundary (see Figure 4).

Let $a = c_0 \epsilon^{1/\beta_n}$ where $c_0 = c_0(M)$ will be determined later and β_n is the dimensional constant given in Lemma 3.2. We will show that there exists $t_0 > 0$ such that for every $i \geq 1$,

$$(3.1) \quad \Gamma_{t_0} \subset (\cup_{j=1}^i \mathcal{N}_j) \cup \tilde{\mathcal{N}}_{i+1}$$

where \mathcal{N}_j is the $a2^{-j}$ -neighborhood of $\Gamma_0 \cap \partial(S_j - S_{j+1})$ and $\tilde{\mathcal{N}}_{i+1}$ is the $a2^{-i}$ -neighborhood of $\Gamma_0 \cap \partial(S_{i+1})$.

Let us choose t_0 such that

$$t_0 = c_1 a$$

where $c_1 = c(M, c_0)$ will be also determined later.

By a barrier argument, we can see that (3.1) holds when $i = 1$ for t_0 if c_1 is chosen small enough. We will use an induction for the proof of (3.1). Assume that (3.1) holds for $i = k - 1$. We will show that (3.1) also holds with the same t_0 for $i = k$.

Since Ω_0 is an NTA domain, if we let $a \ll c(M)$ and if (3.1) holds for $i = k - 1$, then we can see that Ω_{t_0} is contained in a NTA domain Ω'_{k-1} such that (3.1) is satisfied with Γ_{t_0} replaced by $\partial\Omega'_{k-1}$ and for $i = k - 1$.

Let v be a harmonic function in $\Omega'_{k-1} - K$ with boundary value 1 on ∂K and 0 on $\partial\Omega'_{k-1}$, then $u(x, t_0) \leq v(x)$. We will show that $v(x_k) \leq |x_k|^{2+\epsilon'}$ for some $\epsilon' > 0$. (Recall that x_k is a point on I_k such that $B_{2^{-i}/M^2}(x_k) \subset \Omega_0$.)

Define $I'_{i,k-1}$ be the component of $\Omega'_{k-1} \cap \partial B_{2^{-i}}(0)$, which contains I_i and let $S'_{i,k-1}$ be the component of $\Omega'_{k-1} - I'_{i,k-1}$, which contains S_i (see Figure 4.) For simplicity we will omit the k -dependence on $I'_{i,k-1}$ and $S'_{i,k-1}$ from

now on. Let

$$\Omega_i = \Omega_0 \cup S'_i$$

and let v_i be the harmonic function in Ω_i with boundary value 1 on ∂K and 0 on $\partial\Omega_i - \partial K$. Observe that $v_{i-1} - v_i$ is a positive harmonic function in Ω_i with boundary value v_{i-1} on $\partial\Omega_i \cap \Omega_{i-1}$ and 0 elsewhere on its boundary.

Claim 1. *Assume that (3.1) holds for $i = k - 1$. Then for $2 \leq i \leq k$*

$$\frac{v_{i-1}(x_k)}{v_i(x_k)} \leq 1 + Ca^{\beta_n}$$

where C depends only on the NTA constant M .

Proof of Claim 1. Let $2 \leq i \leq k$. Since $S'_{i-1} - S'_i$ is contained in the $2a2^{-(i-1)}$ -neighborhood of $\partial\Omega_{i-1}$, Lemma 3.2 implies that for $y \in \partial\Omega_i \cap \Omega_{i-1}$

$$(3.2) \quad \begin{cases} v_{i-1}(y) \leq C \left(\frac{\text{dist}(y, \partial\Omega_{i-1})}{2^{-(i-1)}} \right)^{\beta_n} \sup_{x \in S'_{i-1} - S'_i} v_{i-1}(x) \\ \leq Ca^{\beta_n} \sup_{x \in S'_{i-1} - S'_i} v_{i-1}(x) \end{cases}$$

and

$$(3.3) \quad \sup\{v_{i-1}(x) : S'_{i-1} - S'_i\} \leq Cv_{i-1}(x_{i-1}).$$

Also we obtain

$$(3.4) \quad v_{i-1}(x_{i-1}) \approx v_i(x_{i-1}).$$

since Lemma 3.2 and Lemma 3.3 imply that for $1 < j < i - 1$ and $y \in \partial B_{2^{-j}}(0) \cap \Omega$

$$\frac{\omega(y, \partial B_{2^{-j+1}}(0), \Omega^{i-1})}{\omega(y, \partial B_{2^{-j+1}}(0), \Omega^i)} \leq 1 + C2^{(j-i)\beta_n}.$$

Since $v_{i-1} - v_i$ is a positive harmonic function in Ω_i with boundary value v_{i-1} on $\partial\Omega_i \cap \Omega_{i-1}$ and 0 elsewhere on its boundary, (3.2), (3.3) and (3.4) imply that

$$\begin{aligned} v_{i-1}(x_k) - v_i(x_k) &\leq \omega(x_k, \partial B_{2^{-i}}(0), B_{2^{-i}}(0) \cap \Omega_i) \sup_{\partial\Omega_i \cap \Omega_{i-1}} v_{i-1}(y) \\ &\leq Ca^{\beta_n} \omega(x_k, \partial B_{2^{-i}}(0), B_{2^{-i}}(0) \cap \Omega_i) v_{i-1}(x_{i-1}) \\ &\approx Ca^{\beta_n} \omega(x_k, \partial B_{2^{-i}}(0), B_{2^{-i}}(0) \cap \Omega_i) v_i(x_{i-1}) \\ &\leq Ca^{\beta_n} v_i(x_k). \end{aligned}$$

for some $C > 0$ depending on the NTA constant M . □

Claim 1 combined with $\frac{v(x_k)}{v_1(x_k)} \leq C$ and $\frac{v_k(x_k)}{u_0(x_k)} \leq C$ yields that

$$\frac{v(x_k)}{u_0(x_k)} \leq C_1(1 + C_2 a^{\beta_n})^k$$

where C_1 and C_2 depend only on M .

Hence if we choose c_0 in the definition of a small enough such that $C_2 a^{\beta_n} < (1/10)\epsilon$, then

$$(3.5) \quad v(x_k) \leq C u_0(x_k) (1 + (1/10)\epsilon)^k \leq C |x_k|^{2+\epsilon} (1 + (1/10)\epsilon)^k \leq C |x_k|^{2+\epsilon'}$$

for $\epsilon' = \epsilon - \log_2(1 + (1/10)\epsilon) > 0$.

Now using (3.5) and the relation $u(x, t_0) \leq v(x)$, we prove (3.1) for $i = k$. Let B be a ball in Ω_0^c such that

$$(1/M)(a/2)2^{-k} \leq \text{rad}(B) \leq \text{dist}(B, S_k) \leq (a/2)2^{-k}.$$

Let $(c(M)/a)B$ be the concentric ball containing B with the radius of $(c(M)/a)\text{rad}(B)$. Let ϕ be a radially symmetric solution of (HS) in $\Sigma := (c(M)/a)B \times [0, t_0)$ with fixed boundary data $C(2^{-k})^{2+\epsilon'}$ on $\partial((c(M)/a)B)$ and with initial positive set $(c(M)/a)B - B$. Then (3.5) and Lemma 3.2 imply that $u \leq \phi$ on the parabolic boundary of Σ . Hence by Theorem 2.7 $u \leq \phi$ in Σ . Since

$$t_0 := c_1 a \leq c_1 a 2^{\epsilon' k},$$

it follows from the comparison with u and ϕ that if we choose c_1 small enough then (3.1) holds for $i = k$. Now we can conclude. □

Proof of Theorem 1.4

We will prove that for both (a) and (b), for any $t > 0$ there is $k_0 = k_0(t)$ such that $\Gamma_t(u) \cap B_{k_0}(0) = \emptyset$. First let us assume (a). Due to the *Corkscrew condition* on Ω , for sufficiently big $n > 0$, there exists y_n such that $|y_n| \simeq |x_n|$ and $B_{|x_n|/M}(y_n) \subset \Omega_0$. Moreover due to Lemma 3.2 and the Harnack inequality,

$$u_0(x) \geq C |x_n|^{2-\epsilon_n} \text{ in } B_{|x_n|/2M}(y_n).$$

Now if one considers a radially symmetric solution ϕ of (HS) with $K = B_{|x_n|/2M}(y_n)$ and $\Omega = B_{|x_n|/M}(y_n)$, $\phi \leq u$ by Theorem 2.7. Since $|D\phi| \geq C |x_n|^{1-\epsilon}$ on $\Gamma_t(\phi)$ up to $t = t_0$ when $\Gamma_{t_0}(\phi) = B_{2|x_n|}(y_n)$, one obtains

$$t_0 \leq C \frac{2|x_n|}{|x_n|^{1-\epsilon}} = C|x_n|^\epsilon.$$

Due to Theorem 2.7, $\Omega_t(\phi) \subset \Omega_t(u)$ for each $t > 0$ and it follows that $\Gamma_t(u) \cap B_{|x_n|}(0) = \emptyset$ if $t > t_0$. Since $t_0 \rightarrow 0$ as $n \rightarrow \infty$ we can conclude.

Now for the case (b), for any sufficiently small $r > 0$ due to Definition 1.1 condition 2, there is $1/M < m < 2$ and $y \in \Omega$ such that $B_{mr}(y) \subset \Omega$ and $\bar{B}_{mr}(y) \cap \partial\Omega$ is nonempty. Let ϕ be a radially symmetric solution of (HS) with $\Omega = B_{mr}(y)$, $K = B_{mr/2}(y)$ and $f = r^2$. By comparing u to ϕ for each $r > 0$, it follows that for any $t_0 > 0$ there is $k_0 > 0$ such that for $k > k_0$ a fixed portion of $\Gamma_0(u)$ in A_k expands in the direction of $e_n = (0, \dots, 1) \in \mathbb{R}^n$ by distance $C2^{-k}$. In particular if $0 \in \Gamma_{t_0}(u)$, then for any $t_0 > 0$ there exists $\epsilon > 0$ such that

$$\sup_{x \in I_i} u(x, t_0) \geq |2^{-i}|^{2-\epsilon} \text{ for sufficiently large } i.$$

Hence by (a) $u(0, t_0 + s) > 0$ for any $s > 0$. Since t_0 is arbitrary, we can conclude. \square

Proof of Proposition 1.6.

Let $t_0 > 0$ be sufficiently small. Suppose that there exist $a_i < \frac{m}{10}$ ($0 \leq i \leq k-1$) such that $\Gamma_{t_0} \cap A_i$ is contained in the $a_i 2^{-i}$ -neighborhood of $\Gamma_0 \cap A_i$ for $0 \leq i \leq k-1$. By a similar argument as in Claim 1

$$u(x_k, t_0) \leq C \prod_{i=0}^{k-1} (1 + a_i) u(x_k, 0)$$

since we can take the dimensional constant β_n in Claim 1 as 1 due to our extra assumption on Ω_0 .

A barrier argument as in the proof of Theorem 1.3 yields that $\Gamma_{t_0} \cap A_k$ is contained in the $a_k 2^{-k}$ -neighborhood of $\Gamma_0 \cap A_k$ where

$$a_k := C \frac{\prod_{i=0}^{k-1} (1 + a_i) u(x_k, 0)}{2^{-2k}} \cdot t_0.$$

Observe that if we can show $a_k < \frac{m}{10}$, then by induction there exists a waiting time at 0. Since we can take $t_0 > 0$ sufficiently small, we can conclude that if $\limsup \tilde{a}_k < \infty$ for the sequence $\tilde{a}_k = \prod_{i=0}^{k-1} (1 + \tilde{a}_i) 2^{2k} u(x_k, 0)$, $\tilde{a}_0 = 1$, then there exists a waiting time. \square

(a) of Theorem 1.5 follows from Proposition 1.6. Next, we prove the second part of Theorem 1.5.

Definition 3.4. For $\zeta \in \Gamma_0$, let

$$H(\zeta, t) := \text{dist}(\zeta, \Gamma_t).$$

Proof of (b) of Theorem 1.5.

1. Let $t_0 > 0$ and partition $[0, t_0]$ into $0 = s_0 < s_1 < \dots < s_P = t_0$. By a simple barrier argument with a radially symmetric test function we obtain that

$$H(\zeta, s_1) \geq C \frac{u(x_i, 0)}{2^{-i}} \cdot s_1$$

for every $\zeta \in \Gamma_0 \cap A_i$. Hence by a similar argument as in Claim 1

$$u(x_k, s_1) \geq \prod_{i=0}^{k-1} \left(1 + C \frac{u(x_i, 0)}{2^{-2i}} \cdot s_1\right) \cdot u(x_k, 0).$$

2. Repeating our argument, we also obtain that for $1 \leq m \leq P$

$$(3.6) \quad u(x_k, s_m) \geq \prod_{i=0}^{k-1} \prod_{j=0}^{m-1} \left(1 + C \frac{u(x_i, s_j)(s_{j+1} - s_j)}{2^{-2i}}\right) \cdot u(x_k, 0)$$

and for any $\zeta \in \Gamma_0 \cap A_k$

$$(3.7) \quad H(\zeta, s_P) \geq C \sum_{m=0}^{n-1} \frac{u(x_k, s_m)(s_{m+1} - s_m)}{2^{-k}}.$$

3. In terms of integration, (3.6) yields

$$(3.8) \quad \begin{aligned} u(x_k, t) &\geq u_0(x_k) \prod_{i=1}^{k-1} \left(1 + C 2^{2i} \int_0^t u(x_i, s) ds\right) \\ &\geq 2^{-2k} k^{-\alpha t} \sum_{i=1}^{k-1} 2^{2i} u_0(x_i) \\ &\geq 2^{-2k} k^{-\alpha t} \sum_{i=1}^{k-1} i^{-\alpha} \\ &\geq 2^{-2k} k^{1-2\alpha t} \end{aligned}$$

where the second and third inequality holds due to the assumption on the decay rate of u . Observe that $2\alpha - 1 < \alpha$ since $\alpha < 1$. Hence if we define a sequence b_1, b_2, \dots such that $b_1 = \alpha$ and $b_{n+1} = 2b_n - 1$, then there exists $l = l(\alpha)$ such that $b_l \leq 0$.

4. Now we are ready to show that there is no waiting time for u at the origin. Let us fix $t_0 > 0$ and define

$$b_j = j \frac{t_0}{2l}, j = 0, \dots, l.$$

Then by inductively applying the estimate (3.8) replacing $u(x_k, t)$ by $u(x_k, b_j)$ and $u(x_i, 0)$ by $u(x_i, b_{j-1})$ for $j = 1, 2, \dots, l$ we have

$$u(x_k, \frac{t_0}{2}) = u(x_k, b_l) \geq 2^{-2k} [\prod_{i=1}^l (b_i - b_{i-1})^{2^{l-i}}] k^{-b_l} \geq 2^{-2k} (\frac{t_0}{2l})^{2^l}$$

In other words u decays slower than quadratically at $t = t_0/2$. Hence due to Theorem 1.3, $u(0, t_0) > 0$. Since t_0 is arbitrary, we can conclude. \square

4 Proofs of the main result: Stefan problem

In this section we assume Γ_0 to be locally Lipschitz along the direction $e_n = (0, \dots, 0, 1)$ in a neighborhood of the vertex 0. More precisely in a small neighborhood of the origin we assume that Ω_0 is given as $\{(x', x_n) : x_n > f(x', 0)\}$ where f is Lipschitz with Lipschitz constant L .

The following lemmas are important in our analysis in this section. In particular, Lemma 4.3 shows that the initial heat with fixed Lipschitz positive phase changes to be (almost) harmonic over the amount of time $t \sim d^2$, where d is the distance between the given point in the positive phase and the free boundary. In our free boundary problem (ST), this change will affect the expansion of the initial positive phase over time and vice versa. Lemma 4.1 gives us a control over the change of the initial heat before it changes to be (almost) harmonic, that is for the time interval $0 \leq t \leq d^2$.

Lemma 4.1. [Di]

Suppose $u_0(x) \in C_0(\mathbb{R}^n) \cap C^\delta(\mathbb{R}^n)$ for some $\delta > 0$ and let $u(x, t)$ solve the heat equation in $\mathbb{R}^n \times [0, \infty)$ with initial data u_0 . Then there exists a constant A depending only on n, δ and the Hölder constant of u_0 in $B_1(0)$ such that

$$|u_t|(x, t) + |u_{x_i x_j}|(x, t) \leq A t^{\delta/2-1}$$

in $B_1(0) \times (0, 1)$ for any $i, j = 1, 2, \dots, n$.

Let Q_a be the a -cube $B_a(0) \times (-a, a)$ in \mathbb{R}^{n+1} .

Lemma 4.2. [ACS] *Let u be a caloric function in $Q_1 \cap D$, where $D \cap Q_1 = \{x_n > f(x', t)\}$ where f is Lipschitz and $(0, 0) \in \partial D$. Then there exists a constant $\delta > 0$ depending on n , the Lipschitz constant L of D , and the ratio r between the supremum of u in Q_1 and $u(-e_n, 0)$ such that $\nabla u \cdot e_n \geq 0$ in $D \cap Q_\delta$.*

Lemma 4.3. [ACS] *Let u, D and r, L be as in Lemma 4.3. Then there exist $\epsilon, \delta > 0$ depending on n, r, L such that*

$$u + u^{1+\epsilon}, \quad u - u^{1+\epsilon}$$

are subharmonic and superharmonic, respectively, at each time in $Q_\delta \cap D$.

Lemma 4.4. (Dahlberg, see [D]) *Let u_1, u_2 be two nonnegative harmonic functions in a domain D of \mathbb{R}^n of the form*

$$D = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < 2, |x_n| < 2M, x_n > f(x')\}$$

with f a Lipschitz function with constant less than M and $f(0) = 0$. Assume further that $u_1 = u_2 = 0$ along the graph of f . Then for

$$D_{1/2} = \{|x'| < 1, |x_n| < M, x_n > f(x')\}$$

We have

$$0 < C_1 \leq \frac{u_1(x', x_n)}{u_2(x', x_n)} \cdot \frac{u_2(0, M)}{u_1(0, M)} \leq C_2$$

with C_1, C_2 depending only on M .

Lemma 4.5. (Caffarelli, see [C1]) *Let u be as in Lemma 4.4. Then there exists $c > 0$ depending only on M such that for $0 < d < c$ $u_n(0, d) := \frac{\partial u}{\partial x_n}(0, d) \geq 0$ and*

$$C_1 \frac{u(0, d)}{d} \leq u_n(0, d) \leq C_2 \frac{u(0, d)}{d}$$

where $C_i = C_i(M)$.

Let u and h_0 to be as in section 1.2.

Lemma 4.6. *If $h_0(x)$ has less than quadratic decay at 0, then there is no waiting time for u .*

Proof

1. First we choose a small $t_0 > 0$. Below we will show that, $u(0, t) > 0$ for $t > t_0$. Since t_0 is arbitrary we can then conclude. Also without loss of generality, we may assume that Γ_0 is Lipschitz in B_1 along the direction e_n .

2. Let w solve the heat equation in the domain

$$\Sigma := \Omega_0 \times [0, t_0]$$

with initial data $u(x, 0)$. Since Σ is Lipschitz in space and time, In a small neighborhood $B_h(0) \times [t_0, t_0 + h^2]$, $h = h(t_0)$ there is ϵ depending only on the Lipschitz constant of Γ_0 in B_1 such that $w_1 := w - w^{1+\epsilon}$ is superharmonic in Σ .

3. Let us choose $C = C(t_0)$ large enough that $Cw_1 \geq w$ at $(-he_n, t_0)$. Then by continuity of w in time and by Lemma 4.4 applied at each t ,

$$w \geq w_1 \geq Ch_0 \text{ in } B_h(0) \times [t_0, t_0 + \epsilon]$$

for small $\epsilon > 0$.

In particular it follows that, for some $k > 0$,

$$w(-re_n, t) \geq Ch_0(-re_n, t) \geq Ch_0(-re_n, 0) > Cr^{2-k} \text{ for } t \in [t_0, t_0 + \epsilon].$$

4. Lastly, observe that $w \leq u$ by the maximal principle of the heat equation. Hence for sufficiently small $r > 0$

$$u(-re_n, t) \geq Cr^{2-k} \text{ for } t \in [t_0, t_0 + \epsilon].$$

Since Ω_0 is Lipschitz, $B_{Cr}(-re_n) \subset \Omega_0$.

Next we consider a radially symmetric subsolution $\phi(x, t)$ of (ST) in $\Sigma := (\mathbb{R}^n - B_{Cr/2}(-re_n)) \times [t_0, t_0 + \epsilon]$ whose initial and fixed boundary data are given as

$$-\Delta\phi(x, t_0) = 0 \text{ in } B_{Cr}(-re_n) - B_{Cr/2}(-re_n),$$

$$\phi(\cdot, t_0) = 0 \text{ on } \partial B_{Cr}(-re_n), \quad \phi(\cdot, t) = Cr^{2-k} \text{ on } \partial B_{Cr/2}(-re_n).$$

By comparing u and ϕ in Σ , it follows that

$$u(0, t_0 + r^{k/2}) > 0 \text{ for sufficiently small } r > 0.$$

Hence we conclude. □

Proving the occurrence of the waiting time is more involved since we need to observe the behavior of the solution during a time period $0 < t < t_0$.

Proof of Theorem 1.8

1. Recall that we assume $u_0(x) \in C^\delta(\bar{\Omega}_0)$ and the minimal and maximal decay rate a and b of h_0 satisfies $2 < \alpha_1 \leq \alpha_2$ and $\alpha_2 - \alpha_1 < \delta/2$, where h_0 is the harmonic function in $\Omega_0 - K$, K a compact subset of Ω with boundary value 0 on $\partial\Omega_0$ and 1 on ∂K . Throughout the proof we will denote C as a positive constant which only depends on δ, L and n .

2. The plan is to construct a supersolution \tilde{v} of (ST) in a neighborhood of the origin which has a waiting time at the origin and to compare \tilde{v} with u to conclude our theorem. For this purpose first we will construct a supersolution \tilde{h} of (HS) such that $u \leq \tilde{v} \leq \tilde{h}$ in a small neighborhood of 0 and for a small time.

3. Let $r > 0$ be sufficiently small. First we construct a supersolution $h(x, t)$ of (HS) in $B_r(0) \times [0, 1)$ such that

(i) $\Omega_0(u) \cap B_r(0) \subset \Omega_0(h) \cap B_r(0)$ and $\bar{\Omega}_0(u) \cap \partial B_r(0) \subset \Omega_0(h)$;

(ii) $\Gamma(h)$ has a waiting time at $(0, 0)$;

(iii) $\beta_2 - \beta_1 < \delta/2$, where β_1 and β_2 are the maximal and minimal rate of $h(x, 0)$;

(iv) $\Gamma(h)$ is Lipschitz in space-time.

Note that we cannot use a solution of (HS) with initial free boundary Γ_0 as a barrier for u most importantly since we do not know if it satisfies property (iv) and hence we cannot apply Lemma 4.2 and 4.3.

4. To construct such h satisfying (i)-(iv), first we choose a sufficiently small $a > 0$ and define Ω' to be the support of the following function:

$$U(x) := \sup_{y \in B_{2a|x|}(x)} u(y, 0).$$

Since $h_0(x)$ has its minimal decay rate more than 2 near the origin, the harmonic function in $\Omega' - K$ with boundary value 0 on $\partial\Omega'$ and 1 on ∂K has a minimal decay rate $2 + \epsilon'$ for some $\epsilon' > 0$, if $a > 0$ is sufficiently small. (Its proof is the same as that of Theorem 1.2.) Next for $0 \leq t \leq 1$ we define

$$(4.1) \quad U^t(x) := \inf_{y \in B_{a(t)|x|}(x)} U(y)$$

where $a(t) := 2a - \frac{a}{2}t|x|^{\epsilon'/2}$. Next we let

$$\Omega'_t = \Omega(U^t)$$

and let $h^t(x)$ be the harmonic function in $\Omega'_t - K$ with boundary data 0 on $\partial\Omega'_t$ and 1 on K .

If we define $h(x, t) := h^t(x)$, then (i),(ii),(iv) follow from its construction and (iii) is also satisfied if a is sufficiently small, since $2 + \epsilon' \rightarrow \alpha_1$ as $a \rightarrow 0$ due to the proof of Theorem 1.2.

Now we show that $h(x, t)$ is a supersolution of (HS) in $B_r(0) \times [0, 1]$. Note that $h(x, t) \leq |x|^{2+\epsilon'}$. Hence due to the exterior ball property of $\Omega'_t \cap B_r(0)$ and by Lemma 4.5,

$$|\nabla_x h| \leq C|x|^{1+\epsilon'} \text{ on } \Gamma_t(h) \cap B_r(0)$$

where C depends on L, n and a .

On the other hand the normal velocity of $\Gamma_t(h)$ is given as $a|x|^{1+\epsilon'/2}$ by its construction. Therefore for sufficiently small $r > 0$ it follows that h is a supersolution of (HS) in $B_r(0) \times [0, 1]$ with property (i)-(iv).

5. Recall that $\beta_1 \leq \beta_2$ is given as respectively the minimal and maximal decay rate of $h(x, 0)$ near the origin and $\beta_2 - \beta_1 < \delta/2$ by our construction of h . We will first prove the theorem for the case $2 < \alpha_1 \leq \alpha_2 < 2 + \delta$. Note that in this case we have $\beta_2 < \alpha_2 < 2 + \delta$.

Let us fix $0 < l \ll r$ and let $\alpha(t)$ be given as

$$\alpha(t) = t^{\delta/2 - \beta_2/2}$$

on $[0, t_0]$, where t_0 is chosen such that $\int_0^{t_0} \alpha(t) dt = 1$. Note that $\alpha(t)$ is integrable since $\delta - \beta_2 > -2$ by our hypothesis.

With above choice of $\alpha(t)$

$$(4.2) \quad \tilde{h}(x, t) := \alpha(t)h(x - le_n, \int_0^t \alpha(t) dt).$$

is also a supersolution of (HS) in $B_r(0) \times [0, t_0]$. Observe that \tilde{h} has a waiting time at the vertex le_n .

6. Now we construct a supersolution \tilde{v} of (ST) such that $u \leq \tilde{v}$ for a small time and \tilde{v} has a waiting time. Let v solve the heat equation in $\Omega(\tilde{h})$ with a proper initial data $v_0(x)$ to be chosen. We would like to construct the initial data $v_0(x)$ whose support lies inside $\Omega_0(\tilde{h})$ such that v_0 and v satisfies

(i) $v(x, 0) \in C^\delta(\mathbb{R}^n)$,

(ii) $u \prec v$ on the parabolic boundary of the following set:

$$\mathcal{O} := \{(x, t) : |x - le_n| \leq t^{1/2}, 0 \leq t \leq t_0\}.$$

Without loss of generality we may assume that

$$(4.3) \quad u_0(x) \leq c_0|x|^\delta$$

where $c_0 = c_0(\delta, L, n)$ is sufficiently small constant which will be chosen later. Since $u_0 \in C^\delta(\mathbb{R}^n)$ and $u_0(0) = 0$. For simplicity we may assume $C = 1$. Let $\phi(x) = \phi(|x|)$ to be a smooth function such that $0 \leq \phi \leq 1$, $\phi = 1$ on $|x| \leq 1/2$ and $\phi = 0$ if $|x| \geq 1$. We then let

$$v_0(x) = \int_{\Omega_0} |y|^\delta \phi\left(\frac{x-y}{g(x)}\right) dy$$

where $g(x) = a|x| + l$. Observe that, since $l \ll r$, the support of v_0 is contained in $\Omega(\tilde{h}) \cap \mathcal{O}$.

Note that $u_0 \prec c_0 v_0$ by properties of ϕ and g . Moreover from a straightforward computation it follows that $v_0(x) \in C^\delta(\mathbb{R}^n)$.

7. First we show that $u \prec c_1 v$ on the lateral boundary of \mathcal{O} , where $c_1 = Cc_0$. Let (y, s) be on the lateral boundary of \mathcal{O} , i.e., $s = a^2|y - le_n|^2$. By (4.3) and Theorem 4.1

$$u(y, s) \leq Cc_0|y|^\delta.$$

On the other hand due to Theorem 4.1,

$$v(y, \tau) \geq C|y|^\delta \text{ if } 0 \leq \tau \leq |y|^2 \text{ and if } d(y, \Omega_0) \leq (a/2)|y|.$$

We claim that for any $m > 0$ and $0 \leq s \leq a^2 m^2$,

$$(4.4) \quad d(p, \Omega_0) \leq am/2 \text{ if } p \in \Gamma_s(u) \cap \partial B_m(0).$$

If (4.4) holds, then from above estimates it would follow that $u(y, s) < Cc_0v(y, s)$ if $(y, s) \in \Omega(u)$.

To show (4.4), first note that by our hypothesis for any $m > 0$ and $p \in \Gamma_0(u) \cap \partial B_m(0)$, $B_{C_1am}(p + ame_n) \subset \Omega_0^C$ for C_1 depending on the Lipschitz constant of Γ_0 . For $m > 0$ let

$$t(m) = \sup\{t > 0 : u(p + am/2e_n, t) = 0.\}$$

By a barrier argument we will show that

$$(4.5) \quad t(m) \geq a^2m^2,$$

which proves (4.4).

Note that due to Theorem 4.1 $u(y, \tau) \leq Cc_0|y|^\delta$ for $0 \leq \tau \leq |y|^2$, and in particular

$$(4.6) \quad u(\cdot, \tau) \leq Cc_0m^\delta \text{ on } B_{3m}(p + ame_n) \text{ for } 0 \leq \tau \leq m^2$$

Let ϕ be a supersolution of (ST) in $\Sigma := (\mathbb{R}^n - B_{C_1am}(p + ame_n)) \times (0, m^2)$ whose initial and fixed boundary data are given as

$$\begin{cases} -\Delta\phi(x, 0) = 0 & \text{in } B_{3m}(p + ame_n) - B_{C_1am}(p + ame_n), \\ \phi(\cdot, 0) = 0 & \text{on } \partial B_{C_1am/2}(p + ame_n), \\ \phi(\cdot, t) = Cc_0m^\delta & \text{on } \partial B_{3m}(p + ame_n). \end{cases}$$

Due to (4.6) u is then less than ϕ on the lateral boundary of Σ . Moreover since $u_0 \in C^\delta(\mathbb{R}^n)$, $u_0 \leq C_3\phi$ for sufficiently large C_3 . Hence we can compare u and $C_3\phi$ in Σ and it follows that

$$t(m) \geq \min[m^2, C_4am^{2-\delta}] \geq a^2m^2.$$

for sufficiently small m .

8. Next we show that for some $0 < c_2 < 1$, c_2 only depends on $Landn$, $\tilde{v}(x, t) := c_2v(x, t)$ is a supersolution of (ST) in \mathcal{O} for a proper choice of $\alpha(t)$. Since $\Gamma(v) = \Gamma(\tilde{h})$ and \tilde{h} is a supersolution of (ST) in \mathcal{O} , we only need to show that

$$|Dv(x, t)| \leq C|D\tilde{h}(x, t)| \text{ on } \Gamma(v).$$

Observe that due to Theorem 4.1

$$v((-s + l)e_n, t) \leq Cs^\delta$$

for $0 \leq t \leq s^2$. Thus we can apply Lemma 4.3 and obtain $\epsilon > 0$ depending only on L, a and n such that $v_1 = v + v^{1+\epsilon}$ is subharmonic in $B_s(le_n)$ for any $z \in \Gamma_{a^2s^2}(v)$. Moreover by definition of v_1 we have

$$v_1(x, a^2s^2) \leq Cs^\delta \text{ if } d(x, \Gamma_{a^2s^2}(v)) \leq as.$$

On the other hand, since h increases in time,

$$h(-se_n, t) \geq h(-se_n, 0) \geq s^{\beta_2}$$

where β_2 is the maximal decay rate of $h(x, 0)$. By definition of $\alpha(t)$ and by previous estimates, for a given constant $C_2 > 0$ we can choose $C_1 > 0$ in the definition of $\alpha(t)$ such that we have

$$C_2\tilde{h}(x, t) \geq v_1(x, t) \text{ at } (x, t) = (-se_n, a^2s^2), 0 < s < t_0.$$

Now due to Lemma 4.4, we have $C\tilde{h} \geq v_1$ in \mathcal{O} , and therefore $C\tilde{h} \geq v$ in \mathcal{O} . Since $C\tilde{h} \geq v$ and $\Gamma(v) = \Gamma(\tilde{h})$, we obtain $C|D\tilde{h}| \geq |Dv|$ on $\Gamma(v)$. Hence it follows that there exists a positive constant $c_2 = c_2(\delta, L, n)$ such that $\tilde{v} := c_2v$ is a supersolution of (ST) in \mathcal{O} .

9. Now we choose c_0 in (4.3) small enough such that $c_1 \leq c_2$. Then by previous arguments it follows that $u \leq \tilde{v}$ on the parabolic boundary of \mathcal{O} . Now Theorem 2.7 yields that $u \leq \tilde{v}$ in \mathcal{O} , yielding that $u(le_n, s) = 0$ for $0 \leq s \leq t_1$, where t_1 is independent of l . Lastly we use the fact that l is arbitrarily small to conclude that u has a waiting time at 0.

10. It remains to prove the theorem when $2 + \delta \leq \alpha_2$. We claim that in this case one can always have a bigger Lipschitz domain Σ which contains Ω with $0 \in \Sigma$ and the maximal and minimal decay rate α'_2 and α'_1 of the corresponding harmonic function satisfies that

$$2 < \alpha'_1 \leq \alpha'_2 < 2 + \delta.$$

Then we can proceed as in previous step to conclude. Hence our last step is to prove our claim.

11. Proof of the claim:

Suppose that $2 + \delta \leq \alpha_2$. Since $\alpha_2 - \alpha_1 < \delta/2$ by our hypothesis, it follows that $\alpha_1, \alpha_2 > 2 + \frac{\delta}{2}$. Let h be a constant such that $0 < h < \frac{\delta}{2}$ and

$$\alpha_2 = \alpha_1 + \frac{\delta}{2} - h; \quad r^{\alpha_2} \leq \sup_{|x|=r} u(x) \leq r^{\alpha_1}.$$

Let $\delta/2 < \delta' < \delta$ and let

$$\Omega_1 = \Omega_0 \cup (W(\frac{\pi}{2(2+\delta')}, -e_n) \cap B_1(0))$$

where

$$W(\theta, \nu) = \{x \in \mathbb{R}^n : \langle x, \nu \rangle \geq |x| \cos(\theta/2)\}.$$

We also let $v_1(x)$ be the harmonic function in $\Omega_1 - K$ with boundary data 1 on ∂K and 0 on $\partial\Omega_1$. Then

$$\sup_{|x|=r} v_1(x) \geq r^{2+\delta'} \text{ for } 0 < r < 1 \text{ and } \delta/2 < \delta' < \delta.$$

If $\sup_{|x|=r} v_1(x) \leq r^{2+k}$ for some $k > 0$, then let $\Sigma = \Omega_1$. If not then we inductively construct Σ as follows. Let z_i be a point in the middle of $A_i \cap W(\frac{\pi}{2(2+\delta')}, -e_n)$ where $A_i = B_{2^{-i}}(0) - B_{2^{-i-1}}(0)$. Let $i_1 > 0$ be the smallest number such that $|z_{i_1}|^{2+\frac{\delta}{2}} \leq v(z_{i_1}) \leq 10|z_{i_1}|^{2+\frac{\delta}{2}}$. Let

$$\Omega_2 = (\Omega_1 - B_{2^{-i_1}}(0)) \cup \Omega_0$$

and let v_2 be the corresponding harmonic function in $\Omega_2 - K$ with boundary value 1 on ∂K and 0 on $\partial\Omega_2$. Then for $2^{-i_1} < r < 1$

$$cr^{2+\delta'} \leq \sup_{|x|=r} v_1(x) \leq Cr^{2+(\delta/2)}.$$

Since Ω_0 is a Lipschitz domain, for

$$x \in \Omega_2 \cap B_{2^{-i_1}}(0) = \Omega_0 \cap B_{2^{-i_1}}(0)$$

we have

$$\begin{aligned} v_2(x) &\approx u(x) \cdot \frac{v_2(z_{i_1})}{u(z_{i_1})} \approx u(x) \cdot \frac{v_2(z_{i_1})}{u(z_{i_1})} \\ &\leq C|x|^{\alpha_1} |z_{i_1}|^{2+\frac{\delta}{2}-\alpha_2} = C|x|^{\alpha_1} |z_{i_1}|^{2-\alpha_1+h} \leq C|x|^{2+h}. \end{aligned}$$

If $v_2(x) \geq |x|^{2+\delta'}$, then let $\Sigma = \Omega_2$. If not, we repeat above argument, i.e. let i_2 be the smallest number such that $i_2 > i_1$ and $v_2(z_{i_2}) \approx |z_{i_2}|^{2+\delta/2}$ and let

$$\Omega_3 = \Omega_2 \cup (W(\frac{\pi}{2(2+\delta')}, -e_n) \cap B_{2^{-i_2}}(0)).$$

Let Σ be a region obtained inductively as above. At each steps Ω_i may not be Lipschitz, but with a slight modification of Ω_i we can get a Lipschitz domain without changing the minimal and maximal decay rates. \square

5 A counterexample for Theorem 1.8

As in section 4, let $u_0 \geq 0$, $\Omega_0(u) = \{u_0 > 0\}$ be a Lipschitz domain and let u be the viscosity solution of (ST) with initial data u_0 . The proofs of Theorems 1.7 and 1.8 suggests that the initial positive phase Ω_0 changes into a new phase while the initial data transforms into (almost) harmonic function, and the waiting time phenomena occurs when this new phase, associated with the corresponding harmonic initial data, has a waiting time at the vertex. Similar observation has been also made in the example of [K] mentioned in the introduction.

Based on above observation we will construct a counterexample for Theorem 1.8 where the harmonic function h_0 associated with Ω_0 has more than quadratic decay at the origin and $u_0 \in C^\delta(\mathbb{R}^n)$ for some $\delta > 0$ but u has no waiting time. The key idea is to construct Ω_0 such that the change of the geometry of the positive phase caused by the initial heat u_0 is big enough to change the decay rate of the harmonic initial data in a drastic way.

Let a be sufficiently small, let A be sufficiently large. Define a sequence

$$1 > t_1 > s_1 > t_2 > s_2 \dots$$

by $s_m = at_m^{1+\delta}$ and $t_{m+1} = s_m \cdot \left(\frac{s_m}{t_m}\right)^A$.

Let $E_r(x) = x + B_r(0) \cap \{y \in \mathbb{R}^n : y_n := \langle y, e_n \rangle < 0\}$ be a half-ball with center x and radius r and let

$$\Omega_0 = (B_1(0) \cap W(\frac{\pi}{10}, -e_n)) \cup_{m=1,2,\dots} E_{2t_m}(-\frac{s_m}{2}e_n).$$

(see Figure 5.) Here $\pi/10$ is an arbitrarily chosen small number such that the harmonic function associated with $W(\pi/10, -e_n)$ decays strictly faster than quadratically. (Instead of $\pi/10$ one may choose any angle $\theta_0 < \theta_n$.)

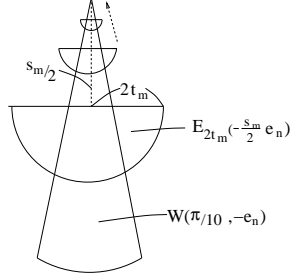
Since $\partial\Omega_0 \cap \{y_n = -\frac{s_m}{2}\}$ is a subset of a hyperplane, we may let $\beta_n = 1$ in Lemma 3.2 and obtain

$$(5.1) \quad h_0(-s_m e_n) \approx \frac{s_m}{t_m} h_0(-t_m e_n) > \frac{s_m^2}{t_m^{2+\delta}} h_0(-t_m e_n)$$

Also since $t_{m+1} = s_m \cdot \left(\frac{s_m}{t_m}\right)^A$, if A is sufficiently large and if $r < 1$, then

$$h_0(-r e_n) \leq r^{2+\epsilon} \text{ for some } \epsilon > 0.$$

Next let us consider a solution $v(x, t)$ of the heat equation in $\Omega_0 \times [0, \infty)$ with initial data $v(x, 0) \in C^\delta(\mathbb{R}^n)$ satisfying $\Omega_0 = \{v(x, 0) > 0\}$ and $v(-r e_n, 0) \geq r^\delta$ for sufficiently small r .



A counterexample for Theorem 1.8: constructing the initial domain

Figure 5.

Due to Theorem 4.1,

$$(5.2) \quad v(-re_n, t) \geq Cr^\delta \text{ for } 0 \leq t \leq r^2, r \ll 1.$$

On the other hand by Lemma 4.3 there is $\epsilon > 0$ such that $v - v^{1+\epsilon}$ is positive and superharmonic in $|x| \leq c_0 t^{1/2}$ for a universal constant $c_0 > 0$. Hence by Lemma 4.4 there is a universal constant $C > 0$ such that

$$(5.3) \quad v(-se_n, t) \geq C \frac{v(-t^{1/2}e_n, t)}{h(-t^{1/2}e_n, 0)} h(se_n, 0) \text{ for } s < c_0 t^{1/2},$$

By (5.1)-(5.2) we then obtain

$$(5.4) \quad v(-s_m e_n, t) \geq C 2^{k_0} \frac{s_m^2}{t_m^2} \text{ for } t \in (t_m^2, 2t_m^2)$$

Now consider a viscosity solution u of (ST) with initial data $v(x, 0)$. By the maximum principle for the heat equation, u is bigger than v . In particular one can replace v by u in (5.4).

Now let $\xi \in \Gamma_0 \in \{x_n = -s\}$. Then by (5.4) and a barrier argument with radially symmetric subsolution of (ST) as in the proof of Lemma 4.5, we obtain that

$$(5.5) \quad \text{dist}(\xi, \Gamma_{2t_m^2}(u)) \sim C \frac{u(-s_m e_n, t)}{s_m} \cdot t_m^2 \geq C 2^{k_0} s_m.$$

(5.5) and the fact that t_m and s_m is a decreasing sequence which converges to zero implies that for any small $t > 0$ there exists $r > 0$ such that

$$\Gamma_t(u) \cap \{x \in \mathbb{R}^n : |x| < r\} = \emptyset.$$

In other words there is no waiting time for u .

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