Math 266B: Homework 6. Due Feb. 26th

1. p. 88, Problem 17.

2. Use Fourier Transform to derive d'Alembert's formula for the solution of one-dimensional wave equation.

3. Consider the telegraph equation (where α, β are positive constants)

$$\begin{cases} u_{tt} + (\alpha+\beta)u_t + \alpha\beta u = u_{xx} & \text{in } \mathbb{R} \times (0,\infty) \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t=0\}. \end{cases}$$

(a) Using Fourier transform, find a formula for the solution of above equation.

(b) For which α and β does all high frequency waves travel with the same speed?

4. Let $n \geq 2$ and consider the Dirichlet problem in the half-space $x_n > 0$:

$$\begin{cases} \Delta u + a\frac{\partial u}{\partial x_n} + k^2 u = 0, & \text{in } \{x_n > 0\} \\ u(x', 0) = f(x'), & \text{where } x' := (x_1, \ldots, x_{n-1}). \end{cases}$$

Here a and k are constants. Use the Fourier transform to show that for any $f \in L^2(\mathbb{R}^{n-1})$, there exists a solution u of the above with

$$\int_{\mathbb{R}^{n-1}} |u(x', x_n)|^2 \, dx' \leq C$$

for all $0 < x_n < +\infty$, with C independent of x_n.

5. Find the characteristics for the initial value problem

$$w_t = \frac{1}{2}((w_x)^2 + x^2)$$

with the initial condition $w(x, 0) = x$ on $t = 0$.The solution will not be defined for $|t| > \pi/2$. Explain this from the behavior of characteristics.

6. Evans page 162 Problem 5. Please specify the boundary points where the non-characteristic boundary condition fails. Does your solution make sense at those points?

7. Consider the eikonal equation $|Du|^2 = 1$ in $\mathbb{R}^2$ with $u = g(s)$ on a hypersurface $\Gamma = \{x = f(y), y \in \mathbb{R}\}$. Describe the condition on g and $f : \mathbb{R} \to \mathbb{R}^2$ such that we can find a proper initial data for p to solve the characteristic ODEs near Γ.

8. Consider the initial value problem

$$\begin{cases} w_x + w_x w_y = 1, \\ w(x,0) = f(x). \end{cases}$$

When is this problem non-characteristic? When it is non-characteristic, show that the solution near $y = 0$ is given by

$$w(x,y) = f(t) - y + \frac{2y}{f'(t)},$$

where $t(x,y)$ is defined by the equation

$$y = (f'(t))^2(x - t).$$

Explain why the solvability of that equation for $t(x,y)$ with $t(x,0) = x$ is equivalent to the non-characteristic condition at $(x,0)$.