Math 266B: Homework 4. Due Feb. 5th

1. Let $f(x,t) \in C^2((0,1)\times(0,\infty))$. Give an explicit formula for the solution $u(x,t) : [0,1]\times[0,\infty) \to I\!R$, solving the equation

$$\begin{cases} u_t - \Delta u = f \text{ in } (0,1)\times(0,\infty) \\ u(0,t) = g(t), \quad u(1,t) = h(t). \end{cases}$$

2.

(a) Suppose $u(x,t)$ solves the heat equation in $I\!R^n \times [0,\infty)$, and $u(x,0) = g(x)$ is bounded and is Hölder continuous: $|g(x) - g(y)| \le |x-y|^\delta$ for some $0 < \delta \le 1$. Then there exists a constant C only depending on the dimension such that

$$|u_t(x,t)| + |u_{x_i x_j}|(x,t) \le C t^{\delta/2-1}.$$

Hint: First note that it is enough to consider the case $x = 0$. You may use the fact that $\int \Phi(x-y;t)dy = 1$ for all t, and thus

$$\frac{\partial^2}{\partial x_i \partial x_j}\int_{I\!R^n} \Phi(x-y;t)dy = \int_{I\!R^n} \frac{\partial^2}{\partial y_i \partial y_j}\Phi(x-y;t)dy = 0.$$

(b) Explain why (a) implies that Duhamel's formula is still valid for a source function $f(x,t)$ which is bounded and uniformly Höder continuous in space.

3. Let U be open and bounded in $I\!R^n$. Consider a solution $u \in C^{2,1}(U_T)\cap C(\bar{U}\times[0,T])$ of

$$(P) \qquad u_t - \Delta u + x\cdot\nabla_x u = 0 \quad \text{in } U_T$$

with $u = g$ on Γ_T, where $0 \le g \le 1$. Show that $u \le 1$ for $0 \le t \le T$.

4. [An energy method exercise] Suppose u is a smooth solution of the following problem:

$$u_{xxt} + u_{xx} - u^3 = 0 \text{ in } [0,1]\times(0,\infty), \quad u(0,t) = u(1,t) = 0 \text{ for all } t \ge 0$$

with initial data $u(x,0) = x(x-1)$. Using the energy $E(t) = \int_0^1 (u_x)^2(x,t)dx$, show that $u(x,t)$ uniformly tends to zero as $t \to \infty$.