Math 266B: Homework 4. Due Feb.6th.

1. Evans p86, Problem 13.

2.

(a) Suppose $u(x,t)$ solves the heat equation in $\mathbb{R}^n \times [0,\infty)$, and $u(x,0) = g(x)$ is bounded and is Hölder continuous: $|g(x) - g(y)| \leq |x - y|^\delta$ for some $0 < \delta \leq 1$. Then there exists a constant C only depending on the dimension such that

$$|u_t(x,t)| + |u_{x_i x_j}|(x,t) \leq Ct^{\delta/2 - 1}.$$

Hint: First note that it is enough to consider the case $x = 0$. You may use the fact that $\int \Phi(x - y; t)dy = 1$ for all t, and thus

$$\frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^n} \Phi(x - y; t)dy = \int_{\mathbb{R}^n} \frac{\partial^2}{\partial y_i \partial y_j} \Phi(x - y; t)dy = 0.$$

(b) Explain why (a) implies that Duhamel's formula is still valid for a source function $f(x,t)$ which is bounded and uniformly Höder continuous in space.

3. Let U be open and bounded in $\mathbb{R}^n$. Consider a solution $u \in C^{2,1}(U_T) \cap C(\bar{U} \times [0,T])$ of

(P) $$u_t - \Delta u = u^\alpha \text{ in } U_T$$

with $u = g$ on Γ_T, where $0 \leq g \leq 1$.

For which power of α does the solution u remain uniformly bounded for all $T > 0$? explain.

4. [An energy method exercise] Suppose u is a smooth solution of the following problem:

$$u_{xxt} + u_{xx} - u^3 = 0 \text{ in } [0,1] \times (0,\infty), \quad u(0,t) = u(1,t) = 0 \text{ for all } t \geq 0$$

with initial data $u(x,0) = x(x - 1)$. Using the energy $E(t) = \int_0^1 (u_x)^2(x,t)dx$, show that $u(x,t)$ uniformly tends to zero as $t \to \infty$.

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