Winter 266B: Homework 3. Due Jan. 30th.

1. Uniqueness theorems in general only applies to bounded domains, unless additional assumptions are imposed. Let $g : \partial \mathbb{R}^n_+ \to \mathbb{R}$ be a continuous function. Show that the *bounded* solution $u \in C^2(\mathbb{R}^n_+) \cap C^1(\{x : x_n \geq 0\})$ of the Dirichlet problem

$$\begin{cases} -\Delta u(x) = 0 & \text{in} \quad \mathbb{R}^n_+ \\ u = g & \text{on} \quad \partial \mathbb{R}^n_+ \end{cases}$$

is unique. Give unbounded counterexamples.

2. Let U be an open, bounded domain in $\mathbb{R}^n$ with exterior ball property and let $g \in C(\partial U)$. Let $w(x)$ be the function constructed from the Perron's method in class, i.e.

$$w(x) := \sup\{v(x), v : \text{ subharmonic in } U, v \in C(\bar{U}) \text{ and } v \leq g \text{ on } \partial U.\}$$

Show that $w(x) \to g(x_0)$ as $x \in U$ converges to $x_0 \in \partial U$.

3. Suppose $u \in C^2(\bar{U})$ minimizes the energy

$$I[w] := \int_U \frac{1}{2}|Dw|^2 - wf dx,$$

among all functions $\omega \in C^2(\bar{U})$. Show that u is harmonic in U with the Neumann (or no-flux) boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U,$$

where ν is the outer normal.

4. Evans p.89, Problem 14.