Fall 2009 266A: Homework 8. Due Dec. 4th

1. Find the Green's function for $Lu = u'' + \lambda^2 u$ on $[0, L]$ with boundary conditions $u(0) = u(L)$ and $u'(0) = u'(L)$. Do they exist for all λ? If not, for which λ does the Green's function does not exist and why?

2. Let $G(x, t)$ be the Green's function for $Lu = u''$ with boundary conditions $u(0) = u'(1) = 0$. Find the boundary value problem whose Green's function is

$$\tilde{G}(x, t) := \int_0^1 G(x, s)G(s, t)ds.$$

3. Show that, when $a_n(a)$ and $a_n(b)$ are both nonzero, the matrix M in the form $Q(\tilde{u}, \tilde{v})$ in the notes "Adjoints" is nonsingular.

4. Let $Lu = -(pu')' + qu$, where $p(x) \in C^1([a, b])$, $q(x) \in C([a, b])$ and $p \neq 0$ on $[1, b]$.

(a) Show that the solutions to $Lu = 0$ with given boundary condition $B\tilde{u} = 0$ minimizes

$$\int_a^b (pv')^2 + qv^2 dx$$

among $v \in C^2([a, b])$ with $B\tilde{v} = 0$.

(b) Let the boundary conditions be given by

$$\alpha u(b) - \alpha u(a) - \beta u'(a) = 0, \quad u'(b) - \gamma u(a) - \delta u'(a) = 0.$$

Show that the problem is self-adjoint if and only if $\alpha = c_1 e^{\imath theta}$, $\beta = c_2 e^{i\theta}$, $\gamma = c_3 e^{i\theta}$, $\delta = c_4 e^{i\theta}$, where c_i and θ are real and $p(b)(c_1 c_4 - c_2 c_3) = p(a)$.

5. Consider the boundary value problem

$$u^{(4)}(x) = g(x) \text{ with } u(0) = u''(0) = u(1) = u''(1) = 0.$$

Find a necessary and sufficient condition on $g(x)$ to make $u'(0) = u'(1)$.

6. Let u and λ be the eigenfunction and eigenvalue of the boundary value problems on $0 \leq x \leq L$:

$$\begin{aligned} u''(x) - a(x)u(x) &= -\lambda u(x) \\ u(0) = u(L) &= 0. \end{aligned}$$

Here λ and L are constand and $a(x)$ is continuous. Assume that λ is the lowest eigenvalue for this problem.

(a) Show that $a > 0$ implies $\lambda > 0$.

(b) Find an example showing that $a < 0$ does not imply $\lambda < 0$.

(c) Show that λ is a decreasing function of L.

7. Consider the integral operator

$$[Tf](x) = -\frac{1}{2}\int_0^1 |x-t| f(t)dt.$$

Find a boundary value problem for which $[Tf](x) = \int_0^1 G(x,t)f(t)dt$.