Fall 2009 266A: Homework 5. Due Nov. 4th

1. B and N p 175 Exercise 2

2. B and N p 175 Exercise 3

3. For A and g given as in Theorem 4.6., find the formula for the unstable manifold $\mathcal{C}_2$ with which corresponding statements in Theorem 4.6. holds for $t \to -\infty$.

4. Consider the system of equations

$$\begin{aligned} y_1' &= -y_1, \\ y_2' &= -y_2 + y_1^2, \\ y_3' &= y_3 + y_2^2. \end{aligned}$$

Compute the first four approximations $\theta^{(j)}(t,a)$ for the functions dening the stable manifold. Show that $\theta^{(3)}(t,a) = u^{(4)}(t,a)$ and thus $\theta(t,a) = \theta^{(3)}(t,a)$. Determine then the stable and unstable manifolds W^s and W^u .

* Problem 5 proves Theorem 4.8 in the textbook. Try first to solve the problem without studying the proof of Theorem 4.7.

5. Let all solutions of the linear system $y' = Ay$ be bounded for $t \geq 0$. Let f be a continuous function, and suppose that there exists a constant a and a continuous function $g(t)$ such that $|f(t,y)| \leq g(t)|y|$ for $|y| \leq a$ and $t \geq 0$, with $\int_0^\infty g(t)dt < \infty$.

(a) Show that there exists a constant M such that any solution $y(t)$ of the system $y' = Ay + f(t,y)$ satisfies $|y(t)| \leq M|y(0)|$ if $|y(0)| \leq a/M$.

(b) Observe that $e^{tA} = U_1(t) + U_2(t)$, where $U_1(t)$ contains elements of the form $\Sigma_i e^{-i\mu_i t}$, $\mu_i \in \mathbb{R}$ and $|U_2(t)| \leq Ke^{-\lambda t}$ for some $\lambda > 0$. Using this information, show that, for a given solution $y(t)$ there exists a constant vector p such that $y(t) - U_1(t)p \to 0$ as $t \to \infty$.

(Hint: First write out $y(t)$ using the variation of constants formula. Now use (a) and the fact that $\int_t^\infty g(s)ds \to 0$ as $t \to \infty$.)

6 (Optional). Consider the FitzHugh-Nagumo equation

$$\begin{aligned} y_1' &= f_1(y_1, y_2) = g(y_1) - y_2, \\ y_2' &= f_2(y_1, y_2) = \sigma y_1 - \gamma y_2 \end{aligned}$$

where σ and γ are positive constants and the function g is given by

$$g(y) = -y(y - 1/2)(y - 1).$$

Show that as the ratio σ/γ decreases the system undergoes a bifurcation from one equilibrium state to three equilibrium states. Compute the critical points and determine their stability properties. Some of the computations are lengthy and you might want to use a geometric argument to determine stability: just look at the directions of the vector field! It is also good idea to make a graph of the orbits before and after the bifurcation, using matlab or mathematica.