Fall 2009 266A: Homework 2. Due Oct.14th

1. Let $f(y,t) : I\!R \times I\!R \to I\!R$ be Lipschitz continuous in y-variable and in t-variable with Lipschitz constant L. Further suppose that $|f| \leq M$ in $I\!R \times I\!R$. Then there exists $y(t) : [-\alpha, \alpha] \to I\!R$: a local solution of $y' = f(t,y)$ with $y(0) = y_0$. As a corollary to the stability theorem, show that

$$|y_k(t) - y(t)| \leq \frac{1+M}{k}(\exp^{L|t|} - 1),$$

where $y_k(t)$ is the solution of the Forward Euler Scheme for $y' = f(y,t)$ with $y(0) = y_0$ and time step size $1/k$.

2. (D'Alembert reduction method) Consider $y' = A(t)y$ where $A(t)$ is $n \times n$ matrix and assume that we know one non-trivial solution $x(t)$. Show that one can reduce the equation $y' = A(t)y$ to the problem $z' = B(t)z$ where $z \in I\!R^{n-1}$ and $B(t)$ is a $(n-1)\times(n-1)$ matrix.

(Hint: Without loss of generality we may assume that the n^{th} component of $x(t)$ is a nonzero function. Look for solutions of the form $y(t) = \phi(t)x(t) + z(t)$, where $\phi(t)$ is a scalar function and z has the form $z = (z_1, ..., z_{n-1}, 0)^T$.)

3. Show that if $A(t)$ is anti-symmetric, i.e., $A^T = -A$, then the resolvant of $y' = A(t)y$ is orthogonal.

4. Consider the two-dimensional linear differential system

$$y' = A(t)y, \quad A(t) = S(t)^{-1}BS(t)$$

where

$$B = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix}, \quad S(t) = \begin{pmatrix} \cos at & -\sin at \\ \sin at & \cos at \end{pmatrix}.$$

(a) Show that, for any t, all eigenvalues of $A(t)$ have negative real parts.

(b) Show, that for a suitable choice of a, above differential equation has solutions $y(t)$ which satisfy $\lim_{t\to\infty} |y(t)| = \infty$. (Hint: set $y(t) = Bx(t)$.)