Math 251A Spring 2024: Homework 2. Due: May 3rd

In all of the problems, suppose that U is a bounded domain in $\mathbb{R}^d$ with smooth boundary, and $\nu = \nu_x$ denotes the outward normal vector of U at $x \in \partial U$.

1.

(a) Show that there exists a minimizer u of $E(w) = \int_U \frac{1}{2}|Dw|^2 + \int_U \frac{1}{2}|x|^2 w^2 dx$ in $M = \{w \in H^1(U) : \int_U w = 1\}$.

(b) Show that u is the unique minimizer in M.

(c) Show that there is $\lambda > 0$ such that u is a weak solution of

$$\begin{cases} -\Delta u + |x|^2 u = \lambda & \text{in } U; \\ Du \cdot \nu = 0 & \text{on } \partial U, \end{cases}$$

2. [Elliptic regularity] Show that for $f \in H^{-1}(U)$, show that there exists a unique weak solution in $H^1(U)$ for the oblique boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } U\ ; \\ Du \cdot \nu + u = 0 & \text{on } \partial U, \end{cases}$$

with

$$\|u\|_{H^1(U)} \leq C\|f\|_{H^{-1}(U)}, \text{ with } C = C(d, U).$$

Here $H^{-1}(U)$ is as given in class (or see section 5.9 of Evans, Chapter 5).

3. Let $\vec{f} \in H^1(U : \mathbb{R}^d)$ and let $(\vec{v}_p, p)$ solve the Stokes problem in $H^1_0(U : \mathbb{R}^d) \times L^2(U)$, namely

$$\int (D\vec{v} : D\vec{\phi} - \vec{f} \cdot \vec{\phi} + p\nabla \cdot \vec{\phi})\, dx = 0 \quad \text{for any } \vec{\phi} \in H^1_0(U : \mathbb{R}^d).$$

(a) Show that $\vec{v}_p$ minimizes $E(\vec{v}) = \int (\frac{1}{2}|D\vec{v}|^2 - \vec{f} \cdot \vec{v})dx$ over divergence-free vector fields $\vec{v} \in H^1_0(U : \mathbb{R}^d)$.

(b) Show that $\vec{v}_p$ is unique a.e. regardless of the choice of p.

(c) Show that p can be uniquely chosen up to a constant.

4. Given $1 < p < \infty$ and $f \in L^p(U)$, one can find $\vec{v} \in L^q(U; I\!R^d)$, $1/q + 1/p = 1$, such that

$$f = \nabla \cdot \vec{v} \text{ in the distribution sense and } \|\vec{v}\|_{L^q(U)} \leq C\|f\|_{L^p(U)}, \text{ with } C = C(p, U).$$

Hint: Use energy method. This is a much easier exercise than the one by Dacorogna-Moser discussed in class, since we do not impose a boundary condition on $\vec{v}$.

5. Let $H \in C^1(I\!R^{2n})$, strictly convex and superlinear. Let A be the symplectic matrix given in the class, and let us denite $< a, b >$ as the inner product for vectors in $I\!R^{2n}$. Consider the energy

$$E(x) := \frac{1}{2}\int_0^1 < x, A\dot{x} > dx$$

in the space

$$M := \{x \in C^1(I\!R; I\!R^{2n} : x(t+1) = x(t), \int_0^1 H(x(t))dt = \alpha\}.$$

Show that, if there exists a minimizer in z in M and if $\alpha > \min H(x)$, then z satisfies the Hamilton's ODE $\dot{z} = \lambda ADH(z)$ for some $\lambda \neq 0$. Is the energy bounded from below in M?

6. This problem is to go through the derivation of Euler-Lagrange equation more carefully, completing the details in Theorem 4, section 8.3.2 of Evans. Let us assume the growth assumptions on L as in section 8.2.3 of Evans, and let us complete the details of the proof of Theorem 4 by answering the following questions:

(a) Please explain why (41) holds a.e.

(b) In the argument below (41), please explain where each of Fubini's theorem and Domniated convergence theorem is used.