Math 251A Spring 2024: Homework 1. Due: Apr. 17th

1. Show that the energy

$$E(u) = \lambda \int_0^1 |u'(t)|dt + \int_0^1 |u(t) - g(t)|^2 dt, \qquad \text{with } g(t) = \chi_{[0,1/2]},$$

does not have a minimizer in $W^{1,1}((0,1))$ for $0 < \lambda < \lambda_0$, and has a smooth minimizer in $\lambda_0 < \lambda < 1$. Find the value of λ_0.

2. Let U be a bounded open domain in $I\!R^d$, and let $f : I\!R \to I\!R$ be a uniformly convex, C^2 function. Let $\{u_k\}$ be a sequence in $L^2(U)$ such that u_k and $f(u_k)$ respectively weakly converge to u and $f(u)$ in $L^2(U)$. Show that then u_k strongly converges to u in $L^2(U)$.

3. Let us define $u : I\!R^2 \to I\!R$ as below:

$$u(x,y) = \begin{cases} 1 & \text{if } x \geq |y|, \\ x/|y| & \text{if } 0 < x < |y|, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Show that u has a weak derivative.

4. Suppose that and $u \in W^{1,p}((0,1))$ for some $1 \leq p < \infty$. Show that u is a.e. equal to an absolutely continuous function.

5. Suppose that $F : I\!R \to I\!R$ is C^1 and uniformly Lipschitz. Suppose U is bounded and $u \in W^{1,p}(U)$ for some $1 \leq p \leq \infty$. Show that

$$v := F(u) \in W^{1,p}(U) \text{ and } Dv = F'(u)Du.$$

6. Show that if $u \in W^{1,p}(U)$, then $Du = 0$ in $\{u(x) = 0\}$. This is not a trivial statement, especially given that the set $\{u(x) = 0\}$ is merely measurable. To prove this, you may proceed as below:

(a) Prove that $u^+ = \max u, 0 \in W^{1,p}(U)$, by using problem 5. and a regularization of the function $F(s) := s_+$.

(b) Prove that $Du^+ = 0$ a.e. on $\{u = 0\}$.

(c) Conclude using that $u = u^+ - u^-$.

7. Let U be a bounded open domain in $\mathbb{R}^d$ with C^1 boundary. Given two exponents $\alpha, \beta > 0$, consider the following energy

$$E(u) := \int_U (-|u|^\alpha + |\nabla u|^p)dx + \int_{\partial U} |T(u)|^\beta dS.$$

(a) Show that $E(u)$ has a minimizer in $W^{1,p}(U)$ if $\alpha < \min[\beta, p]$.

(b) Find the Euler-Lagrange equation that the minimizer solve, including the boundary condition.