Math 251A Spring 2024: Homework 1. Due: Apr. 17th

7. Let U be a bounded open domain in $\mathbb{R}^d$ with C^1 boundary. Given two exponents $\alpha, \beta > 0$, consider the following energy

$$E(u) := \int_U (-|u|^\alpha + |\nabla u|^p)dx + \int_{\partial U} |T(u)|^\beta dS.$$

(a) Show that $E(u)$ has a minimizer in $W^{1,p}(U)$ if $\alpha < \min[\beta, p]$.

(b) Find the Euler-Lagrange equation that the minimizer solve, including the boundary condition.

Proof. I am only assuming that $p > 1$ (step 1.) To prove (a), we could go several ways, but I realize that the following may be the shortest way.

1. We claim that for $p > 1$ there is a constant $C = C(U, p)$ such that

$$\int_U |u|^p dx \leq C(\int_U |Du|^p dx + \int_{\partial U} |Tu|^\beta)dS.$$

If not proceed as in class for the proof of Poincare inequality, to obtain a sequence $\{u_k\}$ in $W^{1,p}(U)$ such that $\|u_k\|_{L^p(U)} = 1$ and $\int |Du_k|^p dx + \int_{\partial U} |T(u_k)|^\beta \to 0$. Hence we have u_k weakly converging to u $W^{1,p}(U)$. Now u satisfies $|Du| = 0$ a.e, and thus u is a constant, but since the trace of u is also zero, it follows that u is the zero function. But this contradicts the fact that u_k strongly converges to u in $L^p(U)$ and thus it must be that $\|u\|_{L^p(U)} = 1$. We conclude that the claim is true.

2. Due to step 1. and the fact that $\alpha < p$ and U being bounded, we have

$$\|u\|^\alpha_{L^\alpha(U)} dx \leq C\|u\|^\alpha_{L^p(U)} \leq C(\|Du\|_{L^p(U)} + \|Tu\|_{L^\beta(\partial U)})^{\alpha/p}, \text{ where } C = C(U, \alpha, p).$$

Since $\alpha/p < 1$, it follows that a large $\|u\|_{L^\alpha(U)}$ norm will result in a large energy $E(u)$. Hence it follows that the energy $E(u)$ is bounded from below.

3. (New argument that does not need restriction on α, β:) Since E is convex with ∇u-variable, E is lower semi-continuous in $W^{1,p}(U)$ along the minimizing sequence with respect to the weak convergence (we covered this in class, using Mazur's theorem). Hence we can conclude.

To prove (b), one perform first variation by all test functions $\phi \in C^\infty(U)$, not just those who vanish on the boundary of U, and perform the integration by parts only to the first to terms of energy with integral over U, to obtain, in the distrbution sense,

$$-\alpha u|u|^{\alpha-1} - p\nabla \cdot (Du|Du|^{p-2}) = 0 \text{ in } U$$

and

$$Tu|Tu|^{\beta-2} + p|Du|^{p-2}(Du \cdot \nu) = 0 \text{ on } \partial U,$$

where $\nu = \nu_x$ stands for the outward unit normal of ∂U at x.

$\square$