

## Exercises from week 8

In all of the problems, suppose that  $U$  is a bounded domain in  $\mathbb{R}^d$  with smooth boundary.

1. Let  $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex, l.s.c., and proper function. Let  $L_f$  be the set of affine functions  $L(x)$  such that  $L \leq f$  in  $\mathbb{R}^d$ . Show that

$$f = \sup\{L(x) : L \in L_f\}.$$

2. Let  $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{+\infty\}$  be a convex, l.s.c., and proper function. Let us define its *subdifferential* at  $v$  as the set

$$\partial f(v) := \{p \in \mathbb{R}^d : f(w) \geq f(v) + (p, w - v) \text{ for any } w \in \mathbb{R}^d\}.$$

Note that when  $f$  is differentiable at  $v_0$ ,  $\partial f(v) = \{Df(v)\}$ . Show the following properties of the subdifferentials:

(a) Let  $v_0$  be in the interior of  $\{f < \infty\}$ . Then  $\partial f(v_0)$  is non-empty.

(b)  $p \in \partial f(v)$  if and only if  $v \in \partial f^*(p)$  if and only if  $f(v) + f^*(p) = (v, p)$ .

3. Let  $H \in C^1(\mathbb{R}^{2d})$ , strictly convex and superlinear. Let  $A$  be the symplectic matrix given in the class. Show that the energy  $E(x) := \frac{1}{2} \int_0^1 (x, A\dot{x}) dx$  in the space

$$M := \{x \in C^1(\mathbb{R}, \mathbb{R}^{2d}) : x(t+1) = x(t), \int_0^1 H(x(t)) dt = \alpha\}.$$

Show that if there is a minimizer  $z$  in  $M$ , and if  $\alpha > \min H(x)$ , then  $z$  satisfies  $\dot{z} = \lambda ADH(z)$  for some  $\lambda \neq 0$ .