

Exercises from week 7

In all of the problems, suppose that U is a bounded domain in \mathbb{R}^d with smooth boundary, and $\nu = \nu_x$ denotes the outward normal vector of U at $x \in \partial U$.

1.

(a) Show that there exists a minimizer u of $E(w) = \int_U \frac{1}{2} |Dw|^2 + \int_U \frac{1}{2} |x|^2 w^2 dx$ in $M = \{w \in H^1(U) : \int_U w = 1\}$.

(b) Show that u is the unique minimizer in M .

(c) Show that there is $\lambda > 0$ such that u is a weak solution of

$$\begin{cases} -\Delta u + |x|^2 u = \lambda & \text{in } U; \\ Du \cdot \nu = 0 & \text{on } \partial U, \end{cases}$$

2. Let $\vec{f} \in H^1(U; \mathbb{R}^d)$ and let $(\vec{v}_p, p) \in H_0^1(U; \mathbb{R}^d) \times L^2(U)$ solve

$$\int (D\vec{v} : D\vec{\phi} - \vec{f} \cdot \vec{\phi} + p \nabla \cdot \vec{\phi}) dx = 0 \quad \text{for any } \vec{\phi} \in H_0^1(U; \mathbb{R}^d),$$

with $\nabla \cdot \vec{v}_p = 0$.

(a) Show that \vec{v}_p minimizes $E(\vec{v}) = \int (\frac{1}{2} |D\vec{v}|^2 - \vec{f} \cdot \vec{v}) dx$ over divergence-free vector fields $\vec{v} \in H_0^1(U; \mathbb{R}^d)$.

(b) Show that \vec{v}_p is unique a.e. regardless of the choice of p .

(c) Show that p can be uniquely chosen up to a constant.

3. Given $1 < p < \infty$ and $f \in L^p(U)$, show that one can find $\vec{v} \in L^p(U; \mathbb{R}^d)$ such that

$$f = \nabla \cdot \vec{v} \text{ in the distribution sense and } \|\vec{v}\|_{L^p(U)} \leq C \|f\|_{L^p(U)}, \text{ with } C = C(p, U).$$

Hint: Use energy method. This is a much easier exercise than the one by Dacorogna-Moser discussed in class, since we do not impose a boundary condition on \vec{v} .