Exercises from week 6

1. Show that the energy

$$E(u) = \lambda \int_0^1 |u'(t)| dt + \int_0^1 |u(t) - g(t)|^2 dt \text{ with } g(t) = \chi_{[0,1/2]}(t)$$

does not have a minimizer in $W^{1,1}((0,1))$ for $0 < \lambda < \lambda_0$, and has a smooth minimizer in $\lambda_0 < \lambda < 1$. Find the value of λ_0.

2. Let U be a bounded open domain in $I\!R^d$, and let $f \in I\!R \to I\!R$ be a uniformly convex, C^2 function. Let $\{u_k\}$ be a sequence in $L^2(U)$ such that u_k and $f(u_k)$ respectively weakly converge to u and $f(u)$ in $L^2(U)$. Show that then u_k strongly converges to u in $L^2(U)$.

3. Let U be a bounded open domain in $I\!R^d$ with C^1 boundary. Given two exponents $\alpha, \beta > 0$, consider the following energy

$$E(u) := \int_U (-|u|^\alpha + |Du|^p) dx + \int_{\partial U} |T(u)|^\beta dS.$$

(a) Show that $E(u)$ has a minimizer in $W^{1,p}(U)$ if $\alpha < min[\beta, p]$.

(b) Find the Euler-Lagrange equation that the minimizer solve, including the boundary condition.