Exercises from week 3:

1. Show that, for positive constants C and γ, if the sequence $\{V_k\}_{k\in\mathbb{N}}$ satisfies the relation

$$0 \leq V_{k+1} \leq C^k (V_k)^{1+\gamma},$$

then $V_k \to 0$ as $k \to \infty$ if V_1 is sufficiently small.

2. [Review of L^2 - L^∞ proof] Let $v \in H^1(B_1)$ solve $\mathcal{L}v = f \in L^\infty(B_1)$, where $\mathcal{L}$ is the operator from (3.10) of Ros-Oton, Chapter 3.3, with symmetric matrix A.

(a) Show that $w := v_+$ satisfies $\mathcal{L}w \leq f\chi_{\{w>0\}}$ (as before, in the sense of testing against non-negative H^1 test functions).

(b) Show that, for any $\phi \in C_c^\infty(B_1)$,

$$\int_{B_1} |\nabla(\varphi w)|^2 dx \leq C(d, \lambda, \Lambda)[\int_{B_1} (|w\nabla\varphi|^2 + f\varphi^2 w) dx].$$

(c) Does the proof of Proposition 3.11 go through if we replace the assumption $\mathcal{L}v \leq 0$ by $\mathcal{L}v \leq f$? Explain the differences in the proof.