Exercises from week 2:

1. Let $u \in C^1(B_1)$ be a H^1-solution to $-\nabla \cdot (A(x)\nabla u) = f$ with $f \in L^d(B_1)$ and continuous a_{ij} with ellipticity constants λ, Λ.

(a) Let u and f be as given in Problem 4. Then

$$\|\nabla u\|_{L^2(B_{1/2})} \leq C(\|u\|_{L^2(B_1)} + \|f\|_{L^2(B_1)}),$$

where $C = C(d, \lambda, \Lambda)$.

(b) Show that for any $0 < \alpha < 1$ we have

$$\|u\|_{C^\alpha(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^d(B_1)}),$$

where C only depends on $\alpha, d, \lambda, \Lambda$ and the mode of continuity for a_{ij}. Explain where (a) is used.

2. [Measure theory exercise] For $u \in H^1(B_1)$, show that $u_+ := \max(u, 0) \in H^1(B_1)$ with $D(u_+) = Du\chi_{\{u>0\}}$ a.e.

3. [Existence via regularity estimates]

We will assume the following (which can be checked by Green's functions): For each $f \in C^\alpha(B_1)$ with $0 < \alpha < 1$, there is a unique $u \in C^{2,\alpha}(\bar{B}_1)$ that solves

$$-\Delta u = f \quad \text{in } B_1, \quad u = 0 \quad \text{on } \partial B_1.$$

Given the above existence, we will show that there exists at least one solution $u \in C^2(\bar{B}_1)$ of the following problem

$$\Delta u = \sin u \text{ in } B_1, \quad u = g \text{ in } \partial B_1,$$

with $g \in C^3(\bar{B}_1)$. Consider the map $T : X \to X$ with $X = C^\alpha(\overline{B_1})$, where $T(w) = u$ with u being the unique $C^{2,\alpha}(\bar{B}_1)$ solution of

$$\Delta u = \sin w \text{ in } B_1, \quad u = g \text{ in } \partial B_1.$$

(a) Show that $T(X)$ is contained in a compact subset of X.

(b) Apply the Schauder fixed point theorem to show that T has a fixed point.